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ON THE CHARACTERISTIC MODES OF A RIGID BODY UNDER FORCES[1]

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The characteristic modes of a rigid body under forces are determined from its analytical solution. This solution may be expressed in terms of eigenvalues and eigenvectors. The nonlinear feedback control law is deduced from the compatibility conditions of the differential equations.

1. INTRODUCTION

It is recently shown in [1, 2] that the exact analytical solution of a rigid body controlled by a linear feedback law may directly be written down in terms of eigenvalues and eigenvectors. However, this control has been computed in previous work [3] for the regulation of satellite angular momentum when the flywheels are at rest. Therefore, the method will here be extended to deal with the structure of the solution when the control is not linear in the state. Moreover, we show that, for the free motion, when the initial condition coincides with an eigenvector, the solution then remains in rotation about a parallel axis to this eigenvector. The eigenvectors thus define the characteristic directions. A preliminary result is given in [4].

The characteristic directions are also here determined for the nonlinear controlled system. In addition, the eigenvalues of the system under study are real or complex. It is claimed in [12] that, in general, it is not possible to evaluate closed form characteristic solutions for complex eigenvectors. This paper and the above author's references are devoted to show that it is possible to describe characteristic solutions with complex eigenvalues and complex eigenvectors.

Consider a rigid body in an inertial reference frame. Let $\omega_1, \omega_2, \omega_3$ as usual denote the angular velocity components and let I_1, I_2, I_3 be the moments of inertia of the body about the principal axes which are the body axes.

Set $x_1 = \omega_1 I_1$, $x_2 = \omega_2 I_2$ and $x_3 = \omega_3 I_3$. So the motion of the body under external forces is described by the Euler equations

$$\dot{x}_1 = k_1 x_2 x_3 + u_1$$

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$$\dot{x}_2 = k_2 x_3 x_1 + u_2 \qquad (1)$$
$$\dot{x}_3 = k_3 x_1 x_2 + u_3$$

where
$$u = (u_1, u_2, u_3)^T, \quad u \in R^3$$

represents the control vector and

$$k_1 = \frac{I_2 - I_3}{I_2 I_3}, \quad k_2 = \frac{I_3 - I_1}{I_3 I_1}, \quad k_3 = \frac{I_1 - I_3}{I_1 I_3}$$

are constant parameters such that $k_1.k_2.k_3 \neq 0$.

In other words, the system has no symmetries. Assume that $I_1 > I_2 > I_3$, we then have $k_1 > 0$, $k_2 < 0$, and $k_3 > 0$.

2. MOTION UNDER NO FORCES

2.1 Suppose that there are no forces acting on the body (i.e. $u_1 = u_2 = u_3 = 0$), then the total external torque is zero and the solutions of Euler equations are known. This classical problem of the force-free motion is treated extensively in standard literature. However, in order to determine the algebraic and the geometric spectra, let us express the solutions in terms of eigenvalues and eigenvectors. Such a form may obviously provide more insight into the problem. For this, it is more convenient to use Jacobi functions because the derivative of each function is proportional to the product of two copolar functions. The structure of the solutions has then the following form

$$\begin{aligned} x_1 &= ikv_1\,\omega_0\, cn(\omega_0\,\lambda t) \\ x_2 &= -kv_2\,\omega_0\, sn(\omega_0\,\lambda t) \\ x_3 &= iv_3\,\omega_0\, dn(\omega_0\,\lambda t) \end{aligned} \qquad (2)$$

where $v = (v_1\ v_2\ v_3)^T$ is an eigenvector of system (1) (with $u = 0$) and λ is the associated eigenvalue. The initial time is supposed to be $t_0 = 0$. k is the so-called modulus of the elliptic function ($k^2 < 1$) and $i^2 = -1$. It is also possible to use Weierstrass functions but in this case the eigenvalues appear implicitly through the invariants of the functions.

2.2. By using the derivative formulae of Jacobi elliptic functions and by entering the solutions into equations (1) we obtain the following algebraic equations satisfied by the eigenvalues and the eigenvectors

$$\begin{aligned} \lambda v_1 &= k_1\, v_2\, v_3 \\ \lambda v_2 &= k_2\, v_3\, v_1 \\ \lambda v_3 &= k_3\, v_1\, v_2 \end{aligned} \qquad (3)$$

These characteristic nonlinear equations may also be obtained from the theory of Section 6.

3. CHARACTERISTIC DIRECTIONS UNDER NO FORCES

3.1. When no forces are acting on the body, the angular momentum M and the kinetic energy T of rotation are constant. These equations follow from the principles of the conservation law,

$$\begin{aligned} M^2 &= x_1^2 + x_2^2 + x_3^2 \\ 2T &= x_1^2/I_1 + x_2^2/I_2 + x_3^2/I_3 \end{aligned} \tag{4}$$

From (4) and using (2), it immediately follows that

$$\omega_0 = \sqrt{\frac{(2TI_1 - M^2)I_3}{(I_1 - I_2)v_3^2}}$$

and

$$k = \sqrt{\frac{(2TI_3 - M^2)I_1 v_3^2}{(M^2 - 2TI_1)I_3 v_1^2}} \tag{5}$$

provided that $M^2 < 2TI_2$. The quantity $M^2/2T$ must lie between the minor and the major moment of inertia for the existence of a solution. Moreover, the condition $M^2 > 2TI_2$ holds when cn is replaced by dn in equation (2) and conversely.

3.2. Remark. One may notice that if $v_1 = 0$ (resp. $v_3 = 0$) then $M^2 = 2TI_3$ (resp. $M^2 = 2TI_1$).

3.3. Furthermore, let E and Δ be the geometric and algebraic spectra respectively, in view of Section 6, we have

$$E = \left\{ \begin{array}{l} V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, V_4 = \begin{pmatrix} a \\ 1 \\ b \end{pmatrix}, \\ V_5 = \begin{pmatrix} -a \\ 1 \\ b \end{pmatrix}, V_6 = \begin{pmatrix} a \\ 1 \\ -b \end{pmatrix}, V_7 = \begin{pmatrix} -a \\ 1 \\ -b \end{pmatrix} \end{array} \right\}, \tag{6}$$

where

$$a = \sqrt{\frac{k_1}{k_2}};\ b = \sqrt{\frac{k_3}{k_2}}, \tag{7}$$

and

$$\Lambda = \left\{ \begin{array}{l} \lambda_1 = \lambda_2 = \lambda_3 = 0;\ \lambda_4 = \lambda_7 = \sqrt{k_1 k_3}, \\ \lambda_5 = \lambda_6 = -\sqrt{k_1 k_3} \end{array} \right\}, \tag{8}$$

3.4. We now focus our attention on the role of these eigenvectors particularly whether or not any sense may be pointed out. Assume that the initial condition, for solution (2), coincides with the eigenvector $V_1 = (1\ 0\ 0)^T$. We have $x(0) = \alpha V_1$ where α is an arbitrary constant. From the third equation of (2), it therefore follows that $\omega_0 = 0$ which implies that $M^2 = 2TI_1$. One has to notice that $\omega_0 = 0$ while $\omega_0 k \neq 0$. We obtain $\alpha = M$ and hence

$$x(t) = \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

The rigid body is thus spinning around the axis of the greatest moment of inertia.

3.5. Similarly, if $x(0) = \alpha V_3$, α is again an arbitrary constant and $V_3 = (0\ 0\ 1)^T$. We find that $M^2 = 2TI_3$ and $\alpha = M$ and hence

$$x(t) = \begin{pmatrix} 0 \\ 0 \\ M \end{pmatrix} \tag{10}$$

The rigid body is thus spinning around the axis of the least moment of inertia.

3.6. However, if the initial condition coincides with the eigenvector $V_2 = (0\ 1\ 0)^T$, the rotation does not remain spinning around the intermediate axis, which corresponds to the fact that this rotation is known as being unstable. A similar procedure may be followed when the initial condition coincides with the remaining eigenvectors.

4. CHARACTERISTIC DIRECTIONS UNDER FORCES

4.1. Consider now that $u_1 \neq 0$, $u_2 \neq 0$ and $u_3 \neq 0$. As we are looking for a feedback control law, then $u = u(x)$. The problem has been reduced to the construction of an exact solution in terms of the eigenvalues and the eigenvectors of system (1).

Let $v = (v_1\ v_2\ v_3)^T$ be an eigenvector of system (1) and λ the associated eigenvalue. From the theory of Section 6, v and λ satisfy the following algebraic equations

$$\begin{aligned} \lambda v_1 &= k_1\, v_2\, v_3 + u_1(v) \\ \lambda v_2 &= k_2\, v_3\, v_1 + u_2(v) \\ \lambda v_3 &= k_3\, v_1\, v_2 + u_3(v) \end{aligned} \tag{11}$$

where $u(v)$ is the unknown function $u(x)$ evaluated in v.

4.2. If we want the closed loop system to have a linear behaviour, the solution should obviously be

$$x = \omega_0\, v e^{\lambda t} \tag{12}$$

where the eigenvalue and the eigenvector are possibly specified a priori.

By differentiating equations (12), using equations (11), and by entering the solutions into equations (1) lead to the linearizing feedback controls

$$\begin{aligned} u_1(x) &= k_1\, x_2\, x_3 + \beta_1 x_1 \\ u_2(x) &= k_2\, x_1\, x_3 + \beta_2 x_2 \\ u_3(x) &= k_3\, x_1\, x_2 + \beta_3 x_3 \end{aligned} \tag{13}$$

where β_1, β_2 and β_3 are constants. In this case, the algebraic and the geometric spectra and therefore the characteristic modes are easy to determine.

4.3. The solution with the closed loop $u = u(x)$, in terms of Jacobi elliptic functions of pole c namely, nc, sc and dc may have, for instance, the following desired form

$$\begin{aligned} x_1 &= \omega_0\, v_1\, (\alpha_1\, sc(\lambda\omega_0 t) + \alpha_2\, nc(\lambda\omega_0 t)) \\ x_2 &= \omega_0\, v_2\, (\beta_1\, sc(\lambda\omega_0 t) + \beta_2\, nc(\lambda\omega_0 t)) \\ x_3 &= \omega_0\, v_3\, (\gamma_1\, sc(\lambda\omega_0 t) + \gamma_2\, dc(\lambda\omega_0 t)) \end{aligned} \tag{14}$$

where α_i, β_i and γ_i are constants. Proceeding in a manner similar to the above manner, the compatibility conditions thus lead to the following expression of the feedback control law

$$\begin{aligned} u_1(x) &= -x_1^2 \\ u_2(x) &= -x_2^2 \\ u_3(x) &= -x_3^2 \end{aligned} \tag{15}$$

4.4. Recall that the derivatives of the functions sc, nc and dc with respect to a variable s are

$$\begin{aligned} nc'(s) &= sc(s)\, dc(s) \\ sc'(s) &= nc(s)\, dc(s) \\ dc'(s) &= k_c^2\, sc(s)\, nc(s) \end{aligned} \tag{16}$$

k_c is the complementary modulus of the Jacobi elliptic functions ($k_c^2 \leq 1$).

4.5. Remark. These solutions are obtained when the complementary modulus of the Jacobi functions is 1, this amounts to say that $nc = dc$. Moreover and for the sake of simplicity, the parameters α_i, β_i and γ_i are chosen such that the feedback gain is -1.

4.6. From equations (11) and (15), it yields

$$v_2 = \frac{v_3^2 - v_3}{k_3 - k_1 v_3^2} \tag{17}$$

and

$$v_3 = \frac{v_2^2 - v_2}{k_2 - k_1 v_2^2} \tag{18}$$

substituting this into (17)

$$v_2(v_2^4 + a_3 v_2^3 + a_2 v_2^2 + a_1 v_2 + a_0) = 0 \tag{19}$$

where

$$\begin{aligned} a_0 &= k_2(k_2 k_3 - 1) \,/\, k_1(k_1 k_3 - 1), \\ a_1 &= (k_2 - 1) \,/\, k_1(k_1 k_3 - 1), \\ a_2 &= -2 \,/\, k_1, \\ a_3 &= (k_1 - 1) \,/\, k_1(k_1 k_3 - 1) \end{aligned}$$

Equation (19) may be written as

$$v_2(v_2^2 + \eta_1 v_2 + \mu_1)(v_2^2 + \eta_2 v_2 + \mu_2) = 0 \tag{20}$$

where

$$\eta_{1,2} = \frac{a_3}{2} \mp \frac{\sqrt{a_3^2 a_0 - 4a_1}}{2\sqrt{a_0}}, \tag{21}$$

$$\mu_{1,2} = \frac{-a_1 + a_2 a_0 \mp \delta}{2a_0} \tag{22}$$

with

$$\delta = \sqrt{a_1^2 - 2a_0 a_1 a_2 + a_1^2 a_2^2 - 4a_0^3}$$

By setting

$$\Delta = \sqrt{\eta_i^2 - 4\mu_i} \tag{23}$$

the roots of equation (19) are thus

$$v_2^0 = 0, \quad v_2^j = (-\eta \mp \Delta)/2; \quad j = 1,2 \text{ for } i = 1 \text{ and } j = 3,4 \text{ for } i = 2. \tag{24}$$

4.8. The characteristic equation when combined with equation (15) gives the geometric and the algebraic spectra respectively,

$$E = \left\{ \begin{array}{l} V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, V_4 = \begin{pmatrix} 1 \\ v_2^1 \\ v_3^1 \end{pmatrix}, \\ V_5 = \begin{pmatrix} 1 \\ v_2^2 \\ v_3^2 \end{pmatrix}, V_6 = \begin{pmatrix} 1 \\ v_2^3 \\ v_3^3 \end{pmatrix}, V_7 = \begin{pmatrix} 1 \\ v_2^4 \\ v_3^4 \end{pmatrix} \end{array} \right\}, \tag{25}$$

and

$$\Lambda = \{\lambda_1 = -1;\ \lambda_2 = \lambda_3 = 0;\ \lambda_j = k_1 v_2^{j-3} v_3^{j-3} - 1\ (j = 4, \ldots, 7)\}. \tag{26}$$

4.9. Remark. The above calculations have been done by using the symbolic calculus package MATHEMATICA.

4.10. Remark. The number of non equivalent solutions according to the Definition given in the Section hereafter, is at most $(p^n - 1)/(p-1)$ where p is the degree of homogeneity. This is connected to the transcendence degree of a homogeneous polynomial dynamical system [11]. In our case, $p = 2$ and $n = 3$ so the set E is of dimension 7.

5. CONCLUSION

We pointed out the fact that a rigid body may possess characteristic directions and thus characteristic modes for both free motion and controlled motion. In linear systems, as known, the characteristic modes are determined from a general solution. However, they can also be determined from all the particular solutions and that is the way used all throughout this paper. A construction of a general solution, in terms of eigenvalues and eigenvectors, for a rigid body is being carried out by the author and will be published in some future work. When a given desired solution is linear, the method described in this paper leads to the linearizing feedback law. Of course, this feedback is the expecting feedback. Otherwise, it will mean that something is wrong in the theory. Only in this case we can trivially write the general solution.

6. APPENDIX. CHARACTERISTIC NONLINEAR EQUATION

6.1. Let us consider the differential equation

$$\dot{x} = f(x) \tag{27}$$

where $f(x)$ is a homogeneous polynomial vector field of degree p, and each component $f_i(x)$ is homogeneous of degree p. x denotes the state vector of components $(x_1, x_2, \ldots, x_n)$, $x \in R^n$. So,

$$\dot{x}_i = f_i(x); \quad i = 1, \ldots, n. \tag{28}$$

Recall that a polynomial field P is called homogeneous of degree p if, for any $a \in R$ $P(ax) = a^p P(x)$, for all $x \in R^n$.

Let z be a new vector $z = (z_1, z_2, \ldots, z_n) \in C^n$ such that for any non vanishing x_j

$$x = x_j z$$

clearly, when $i = j$, $z_j = 1$. By differentiating with respect to time , we obtain

$$\dot{x} = \dot{x}_j z + x_j \dot{z} \tag{29}$$

and by using the fact that f is homogeneous of degree p , it yields

$$\dot{z} = x^{p-1} (f(z) - f_j(z)\, z) \tag{30}$$

Hence, the non-trivial singular solutions satisfy the vector equation

$$f(z) - f_j(z)\, z = 0 \tag{31}$$

which is called the nonlinear characteristic equation.

6.2. In order to consider all non-trivial solutions of (30), the components of z must belong to an algebraically closed field. Let us denote by $v = (v_1, v_2, \dots, v_n)^T$ a solution of equation (30) for a given j, $v_j \in C$. So,

$$f(v) - f_j(v)v = 0 \tag{32}$$

Post multiplying the left-hand side of (31) by v^T, we get

$$f_j(v) = \frac{v^T f(v)}{||v||^2} \tag{33}$$

Let $\lambda = f_j(v)$, the vector v and the value λ are called the characteristic vector and its associated characteristic value, respectively.

6.3. Let R_j denote, for a given j, the representation (31). It is clear that if the state vector x has no zero component, n representations R_j may then be obtained.

6.4. Definition. Two vectors V_1 and V_2, belonging to the set E of all solutions of the algebraic nonlinear equation $f(v) - f_j(v)v = 0$, $j = 1, \dots, n$, are said to be equivalent if and only if there exists a non zero element c of the complex field C such that $V_1 = cV_2$.

6.5. Consequence. When an eigenvector is multiplied by a nonzero element $c \in C$, the associated eigenvalue is therefore multiplied by c^{p-1}.

6.6. Remark. These definitions are identical to those obtained by non-associative algebras in [6, 9, 10, 11]. Equation (30) is observed in [5, 12] and in [7] where a transformation which projects the trajectories of equation (27) onto a unit sphere is described. For real characteristic vectors solution of equation (27) in terms of eigenvalues and eigenvectors is given [5, 12].

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