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A Note on Fuzzy Cardinals

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We compare different notions of fuzzy cardinals and discuss which is the most appropriate one.

In last years, a variety of papers on fuzzy sets and other fuzzy topics was concerned with set-algebraic operations for and properties of fuzzy sets. However, only few remarks are devoted to fuzzy cardinals.

In classical set theory the cardinality of a set is a measure of its size or "power". In the fuzzy case one has to differentiate: there are measures of fuzziness and measures of power.

Here measures of fuzziness are not our main concern. The interested reader may consult e.g. [1], [2], [5], [9].

Fuzzy cardinals as measures of power of fuzzy sets are considered e.g. in [2], [3], and [6]. To describe and compare these definitions needs some notation.

A fuzzy set A over some universe of discourse X is a function $A : X \to [0, 1]$. Instead of $A(x)$ for $x \in X$ we write also $x \varepsilon A$ for this membership value of x in A. The universe of discourse X shall be fixed throughout the paper. By $\mathscr{F}(X)$ we denote the class of all fuzzy sets over X; for every $A \in \mathscr{F}(X)$, the support $|A|$ of A is the classical set

$$|A| = \{x \in X \mid (x \varepsilon A) \neq 0\} .$$

As a first, but very rough measure of power for fuzzy sets one can consider for each $A \in \mathscr{F}(X)$

$$\text{card}_0 A =_{\text{df}} \overline{\overline{|A|}} ,$$

with $\overline{\overline{M}}$ for the classical cardinality of the classical set M.

For fuzzy sets A with finite support $|A|$ one has in the book [6] of A. Kaufmann as further cardinalities for fuzzy sets

$$\text{card}_1 A =_{\text{df}} \sum_{x\in|A|} A(x) = \sum_{x\in|A|} (x \varepsilon A) ,$$

$$\text{card}_2 A =_{\text{df}} \sum_{x\in|A|} A^2(x) = \sum_{x\in|A|} (x \varepsilon A)^2 .$$

A. DeLuca and S. Termini [2] consider $\text{card}_1 A$ also for fuzzy sets A with denumerable support, in which case $\sum_{x\in|A|} A(x)$ can be a divergent series in the sense of analysis; but in case of convergence it is absolutely convergent.

To explain also the essential points of the definition of fuzzy cardinals in the authors paper [3], we introduce for every $A \in \mathcal{F}(X)$ and every $0 \neq i \in [0, 1]$ the level sets

$$A^i =_{\text{df}} \{x \in X \mid (x \varepsilon A) = i\} ,$$

which themselves are classical sets. Furthermore, put $W^+ = (0, 1]$. Obviously, every fuzzy set A can be characterized by the family $(A^i)_{i\in W^+}$ of its level sets.

Now, [3] leads to the definition

$$\text{card}_W A =_{\text{df}} (\overline{\overline{A^i}})_{i\in W^+} ,$$

which is independent of the cardinality of $|A|$. Hence, $\text{card}_W A$ is a family of usual cardinals of usual sets.

It is easy to see that, given $\text{card}_W A$, one can get any one of $\text{card}_k A$ for $k = 0, 1, 2$. Put always $a_i = \overline{\overline{A^i}}$. Then clearly

$$\text{card}_0 A = \sum_{i\in W^+} a_i$$

with summation understood as usual addition of cardinals. In case of a finite support $|A|$ there is a finite subset $I = \{i_1, \ldots, i_n\} \subseteq W^+$ such that: $a_i \neq 0$ iff $i \in I$. Furthermore, with the finite cardinals as the natural numbers, in this case each of a_i is a natural number. Hence now

$$\text{card}_1 A = \sum_{i\in I} i \cdot a_i ,$$

$$\text{card}_2 A = \sum_{i\in I} i^2 \cdot a_i$$

for Kaufmann's [6] notions of fuzzy cardinals. Because of $a_i = 0$ if $i \in W^+ \setminus I$, we write by abuse of language

$$\text{card}_j A = \sum_{i\in W^+} i^j \cdot a_i$$

for $j = 1, 2$. To do the same thing with denumerable supports as deLuca/Termini [2], we have to add ∞ as a "real", which can be done e.g. as sketched in [4] (giving ∞ already as an "integer"). Now, there exists a countable subset $I = \{i_1, i_2, i_3, \ldots\} \subseteq W^+$ such that $a_i = 0$ for $i \in W^+ \setminus I$, and [2] leads to

$$\text{card}_1\, A = \sum_{i \in I} i \,.\, a_i$$

(a_i always a natural number or ∞). Again by abuse of language we can write:

$$\text{card}_1\, A = \sum_{i \in W^+} i \,.\, a_i \,.$$

In the same way it is possible to understand the entropy $d(A)$ of a fuzzy set A (cf. [2]), and also other measures of fuzziness (cf. [5]). In general, the structure of such definitions is

$$f(A) = O(\text{card}_W\, A)\,,$$

A any fuzzy set, O some operator.

Hence, to choose $\text{card}_W\, A$ as *the* fuzzy cardinality of a fuzzy set $A \in \mathscr{F}(X)$ seems to be the most promising variant. The essential idea behind that definition is also independent of the choice of the set $[0, 1]$ as set of generalized membership grades — it does work equally well also in the case of L-fuzzy sets (cf. e.g. [8]). Furthermore, almost the same idea applied to the set $W = \{0, 1/2, 1\}$ as set of membership grades was used by D. Klaua [7] to give a set-theoretical construction of interval numbers.

As a further advantage, from the set-theoretical point of view adopted in [3], $\text{card}_W A$ is the result of a fuzzification of the usual definition of cardinals in any one of the standard systems of set theory.

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