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SUMS OF QUASICONTINUOUS FUNCTIONS

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Summary. It is proved that every real cliquish function defined on a separable metrizable space is the sum of three quasicontinuous functions.

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In this paper I show that every cliquish function $f\colon X \to \mathbf{R}$, where X is a separable metrizable space, is the sum of three quasicontinuous functions.

In what follows X denotes a topological space. For a subset A of a topological space denote by $\operatorname{Cl} A$ and $\operatorname{Int} A$ the closure and the interior of A, respectively. The letters $\mathbf{N}$, $\mathbf{Q}$ and $\mathbf{R}$ stand for the set of natural, rational and real numbers, respectively. C_f denotes the set of all continuity points of $f\colon X \to \mathbf{R}$. The terminology concerning topology comes from [3].

Recall (e.g. [4]) that a function $f\colon X \to \mathbf{R}$ is cliquish at a point $x \in X$ if for each $\varepsilon > 0$ and each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$. A function $f\colon X \to \mathbf{R}$ is said to be cliquish if it is cliquish at each point $x \in X$.

A function $f\colon X \to \mathbf{R}$ is quasicontinuous at a point $x \in X$ if for each neighbourhood U of x and each neighbourhood V of $f(x)$ there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. Denote by Q_f the set of all points at which f is quasicontinuous. If $Q_f = X$, then f is said to be quasicontinuous.

It is easy to see that if $f, g\colon X \to \mathbf{R}$ are cliquish, then $f + g$ is cliquish ([6]).

In [2] it is shown that every cliquish function $f\colon \mathbf{R} \to \mathbf{R}$ is the sum of four quasicontinuous functions. In [5] it is proved that every cliquish function $f\colon \mathbf{R}^m \to \mathbf{R}$ is the

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sum of six quasicontinuous functions. And in [6] it is shown that every cliquish function $f\colon X \to \mathbf{R}$ is the sum of four quasicontinuous functions provided X is a Baire separable metrizable space without isolated points. In this paper I show that such a function is the sum of three quasicontinuous functions. Moreover, the assumption "X is Baire without isolated points" may be omitted.

Lemma 1. ([6; Theorem 3]) *Let X be a Baire separable metrizable space without isolated points. Let $w\colon X \to \mathbf{R}$ be a cliquish function such that $w^{-1}(0)$ is dense in X. Then there exist quasicontinuous functions $s, t\colon X \to \mathbf{R}$ such that $w = s + t$.*

Lemma 2. *Let X be a Baire separable metrizable space without isolated points. Then every cliquish function $f\colon X \to \mathbf{R}$ is the sum of three quasicontinuous functions.*

Proof. Denote $A = \{x \in X : \omega_f(x) \geqslant 1\}$ (ω_f is the oscillation of f). The cliquishness of f yields that A is nowhere dense. Since C_f is dense ([1]) in X we may define $g\colon X \to \mathbf{R}$ as

$$g(x) = \begin{cases} \limsup\limits_{u \to x, u \in C_f} f(u), & \text{for } x \in X - A, \\ f(x) & \text{for } x \in A. \end{cases}$$

Evidently

(1) $$f(x) = g(x) \quad \text{for each } x \in C_f.$$

Let $x \in X - A$. Let U be a neighbourhood of x and $\varepsilon > 0$. Then there is $u \in C_f \cap U$ such that $|f(u) - g(x)| < \frac{\varepsilon}{2}$. There is an open neighbourhood $G \subset U$ of u such that $|f(u) - f(y)| < \frac{\varepsilon}{2}$ for each $y \in G$. Hence for each $y \in G$ we have $|f(u) - g(y)| \leqslant \frac{\varepsilon}{2}$ and therefore $|g(x) - g(y)| \leqslant |g(x) - f(u)| + |f(u) - g(y)| < \varepsilon$. This yields $X - A \subset Q_g$ and

(2) $$X - Q_g \text{ is nowhere dense.}$$

Since $X - Q_g$ is nowhere dense, C_g is dense and hence g is cliquish ([1]). Then $h = f - g$ is cliquish and by (1) the set $h^{-1}(0)$ is dense in X. According to Lemma 1 there are quasicontinuous functions $s, t\colon X \to \mathbf{R}$ such that $h = s + t$.

Let $\mathscr{B}$ be a countable base in X. Put $\mathscr{A} = \{B \in \mathscr{B} : \operatorname{Cl} B \subset \operatorname{Int} Q_g\}$. Then $\mathscr{A} = \{A_1, A_2, \ldots\}$. Let $W \subset X - \operatorname{Int} Q_g$ be a countable dense subset of $X - \operatorname{Int} Q_g$. Then $W = \{w_i\}_{i \in M}$, where $w_i \neq w_j$ for $i \neq j$ and $M = \emptyset$ or $M = \{1, 2, \ldots, n\}$ or $M = \mathbf{N}$.

Since s and g are cliquish, the set $C_s \cap C_g$ is dense in X and by virtue of (2) also $\operatorname{Int} Q_g \cap C_s \cap C_g$ is dense in X.

Let $i \in M$. Since $X - \bigcup_{k=1}^{i} \operatorname{Cl} A_k$ is an open neighbourhood of w_i, there is a sequence $(v_j^i)_j$ of points such that $v_j^i \in (\operatorname{Int} Q_s \cap C_t \cap C_s) - \bigcup_{k=1}^{i} \operatorname{Cl} A_k$ and $(v_j^i)_j$ converges to w_i. Put

$$E = \{v_j^i : i \in M, j \in \mathbb{N}\}.$$

Since $E \cap A_k$ is finite for each $k \in \mathbb{N}$, $E \subset \bigcup_{k=1}^{\infty} A_k$ and X is Hausdorff, the set E is discrete. Let $E = \{a_1, a_2, \ldots\}$ (where $a_r \neq a_s$ for $r \neq s$).

Let $(D_n)_n$ be a sequence of open sets in X such that $\operatorname{Cl} E = \bigcap_{n=1}^{\infty} D_n$ and $\operatorname{Cl} D_{n+1} \subset D_n$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Since E is discrete, there is an open neighbourhood V_n of a_n such that $V_n \cap E = \{a_n\}$. Then also $V_n \cap \operatorname{Cl} E = \{a_n\}$. (Indeed, if $d \in V_n \cap \operatorname{Cl} E$ and $d \neq a_n$, then $V_n - \{a_n\}$ is a neighbourhood of d and hence $(V_n - \{a_n\}) \cap E \neq \emptyset$, a contradiction.) Let W_n be a neighbourhood of a_n such that $\operatorname{Cl} W_n \subset V_n \cap D_n$. Then $H_n = W_n - \bigcup_{j=1}^{n-1} \operatorname{Cl} W_j$ is a neighbourhood of a_n.

Denote $G_n = H_n - \{a_n\}$. Then $G_n = H_n - \operatorname{Cl} E$. There is a one-to-one sequence $(b_k^n)_k$ of points in G_n converging to a_n. Denote

$$F = \{b_k^n : n, k \in \mathbb{N}\}.$$

It is easy to see that $b_k^n \neq b_s^r$ for $(n, k) \neq (r, s)$ and that F is discrete. We shall show that

$$\operatorname{Cl} F = F \cup \operatorname{Cl} E.$$

Evidently $F \subset \operatorname{Cl} F, \operatorname{Cl} E \subset \operatorname{Cl} F$. Let $x \in \operatorname{Cl} F$. If $x \notin \operatorname{Cl} E$, then there is $n \in \mathbb{N}$ such that $x \notin \operatorname{Cl} D_{n+1}$. Then $X - \operatorname{Cl} D_{n+1}$ is a neighbourhood of x and there is a sequence $(x_k)_k$ in $F - \operatorname{Cl} D_{n+1}$ converging to x. Then, with respect to the construction of F, for each $k \in \mathbb{N}$ there are $p(k), r(k) \in \mathbb{N}$ such that $p(k) < n + 1$ and $x_k = b_{r(k)}^{p(k)}$. Hence there is $p < n + 1$ such that $x_k = b_{r(k)}^{p}$ for infinitely many k. Thus we obtain a sequence in $F \cap G_p$ converging to x. However, the set $F \cap G_p$ has a unique accumulation point $a_p \in E$ and $x \notin E$, hence this sequence is constant except for finitely many members. This yields $x \in F$ and $\operatorname{Cl} F = F \cup \operatorname{Cl} E$.

Hence we get $\operatorname{Cl} F \cap (X - \operatorname{Cl} E) = F \cap (X - \operatorname{Cl} E)$. Therefore the set F is closed in $X - \operatorname{Cl} E$. Let $\mathbb{Q} = \{q_1, q_2, \ldots\}$ (one-to-one sequence). Let $\pi : \mathbb{N} \to \mathbb{Q} \times \mathbb{N}$ be a bijection (i.e. $\pi(n) = (q_r, s)$) and let $\kappa : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}$, $\kappa(q_r, s) = q_r$.

Define a function $p : F \to \mathbb{R}$ by:

$$p(b_k^n) = \kappa(\pi(k)).$$

Since F is dicrete, p is continuous on F. Since F is closed in $X - \mathrm{Cl}\, E$, there is a continuous function $k\colon X - \mathrm{Cl}\, E \to \mathbf{R}$ such that $k(x) = p(x)$ for each $x \in F$.

Now define a function $m\colon X \to \mathbf{R}$ by:

$$m(x) = \begin{cases} k(x), & \text{if } x \in X - Cl\, E, \\ 0, & \text{if } x \in Cl\, E. \end{cases}$$

Further, define functions $f_1, f_2, f_3\colon X \to \mathbf{R}$ as:

$$\begin{aligned} f_1 &= g - m, \\ f_2 &= s + m, \\ f_3 &= t. \end{aligned}$$

Then $f_1 + f_2 + f_3 = f$. We shall show that f_i ($i = 1, 2, 3$) are quasicontinuous. Since m is continuous on $X - \mathrm{Cl}\, E$ and g is quasicontinuous on $X - \mathrm{Cl}\, E$, f_1 is quasicontinuous on $X - \mathrm{Cl}\, E$.

Let $x \in \mathrm{Cl}\, E$. Let U be a neighbourhood of x and let $\varepsilon > 0$. Then there is $n \in \mathbf{N}$ such that $a_n \in U$. Since $a_n \in C_g$, there is an open neighbourhood V of a_n such that $|g(t) - g(a_n)| < \frac{\varepsilon}{4}$ for each $t \in V$. Let $j \in \mathbf{N}$ be such that $|g(a_n) - g(x) - q_j| < \frac{\varepsilon}{4}$. Then there is $k_0 \in \mathbf{N}$ such that $b_k^n \in V$ for each $k \geqslant k_0$.

Let $r > k_0$ be such that $\kappa(\pi(r)) = q_j$. Since $b_r^n \in X - \mathrm{Cl}\, E$, there is an open neighbourhood $H \subset V$ of b_r^n such that $|m(t) - m(b_r^n)| < \frac{\varepsilon}{4}$ for each $t \in H$. Therefore for each $t \in H$ we have

$$\begin{aligned} |f_1(t) - f_1(x)| &= |g(t) - m(t) - g(x)| \leqslant \\ &|g(t) - g(a_n)| + |g(a_n) - g(x) - q_j| + |q_j - m(b_r^n)| + |m(b_r^n) - m(t)| < \varepsilon. \end{aligned}$$

Hence f_1 is quasicontinuous at x. Similarly we can prove that f_2 is quasicontinuous. □

Lemma 3. *Let X be a Baire separable metrizable space. Then every cliquish $f\colon X \to \mathbf{R}$ is the sum of three quasicontinuous functions.*

Proof. Let D be the set of all isolated points of X and let $B = X - \mathrm{Cl}\, D$. Then $g = f_{|B}$ is cliquish and according to Lemma 2 there are quasicontinuous functions $g_1, g_2, g_3\colon B \to \mathbf{R}$ such that $g = g_1 + g_2 + g_3$. Let $W \subset \mathrm{Cl}\, D - D$ be a countable dense subset of $\mathrm{Cl}\, D - D$. Then $W = \{w_i\colon i \in M\}$, where $w_r \neq w_s$ for $r \neq s$ and $M \subset \mathbf{N}$. For each $i \in M$ there is a sequence $(v_j^i)_j$ in D converging to w_i such that $v_j^i \neq v_s^r$ for $(i,j) \neq (r,s)$. Let $\mathbf{Q} = \{q_1, q_2, \dots\}$ (one-to-one sequence) and $L = \{2, 4, 6, \dots, 2j, \dots\}$.

Let $\pi\colon L \to \mathbf{Q} \times \mathbf{N}$ be a bijection (i.e. $\pi(2j) = (q_r, s)$) and let $\kappa\colon \mathbf{Q} \times \mathbf{N} \to \mathbf{Q}$, $\kappa(q_r, s) = q_r$. Define functions $f_1, f_2, f_3\colon X \to \mathbf{R}$ by:

$$f_1(x) = \begin{cases} \kappa(\pi(2j)), & \text{if } x = v^i_{2j}, \\ g_1(x), & \text{if } x \in B, \\ f(x), & \text{otherwise,} \end{cases}$$

$$f_2(x) = \begin{cases} f(x) - \kappa(\pi(2j)), & \text{if } x = v^i_{2j}, \\ g_2(x), & \text{if } x \in B, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_3(x) = \begin{cases} g_3(x), & \text{if } x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f = f_1 + f_2 + f_3$.

We shall show that f_i $(i = 1, 2, 3)$ are quasicontinuous. It suffices to verify that f_1 is quasicontinuous at $x \in \operatorname{Cl} D - D$. Let $x \in \operatorname{Cl} D - D$, let U be an open neighbourhood of x and $\varepsilon > 0$. Then there is $m \in \mathbf{N}$ such that $|q_m - f(x)| < \varepsilon$. Let $i \in M$ be such that $w_i \in U$ and $j \in \mathbf{N}$ such that $v^i_{2j} \in U$ and $\kappa(\pi(2j)) = q_m$. Then $\{v^i_{2j}\}$ is a nonempty open subset of U and hence f_1 is quasicontinuous at x. □

Lemma 4. *Let X be a topological space, let D be a dense subset of X. Let $f\colon D \to \mathbf{R}$ be a cliquish function. Then there is a cliquish function $g\colon X \to \mathbf{R}$ such that $g_{|D} = f$.*

Proof. Denote $A = \{x \in X\colon \limsup\limits_{u \to x, u \in D} f(u) \in \{-\infty, \infty\}\}$.

Let B be an open nonempty set in X. Then there is $z \in B \cap D$ and the cliquishness of f at z yields that there is an open nonempty set G in X such that f is bounded on $G \cap D$. Then $G \cap A = \emptyset$ and A is nowhere dense.

Define $g\colon X \to \mathbf{R}$ by:

$$g(x) = \begin{cases} \limsup\limits_{u \to x, u \in D} f(u), & \text{for } x \in (X - A) - D, \\ f(x), & \text{for } x \in D, \\ 0, & \text{for } x \in A - D. \end{cases}$$

Then $g_{|D} = f$. We shall show that g is cliquish . Let $x \in X - A$, let U be an open neighbourhood of x and $\varepsilon > 0$. Then there is $z \in U \cap D$ and the cliquishness of f at z implies that there is an open nonempty set H such that $H \subset U$ and $|f(t) - f(s)| < \frac{\varepsilon}{3}$ for each $s, t \in H \cap D$. Thus there is $a \in \mathbf{R}$ such that $f(t) \in (a - \frac{\varepsilon}{3}, a + \frac{\varepsilon}{3})$ for each

$t \in H \cap D$. Then $\limsup_{t \to y, t \in D} f(t) \in [a - \frac{\varepsilon}{3}, a + \frac{\varepsilon}{3}]$ for each $y \in H$ and hence $|g(y) - a| \leqslant \frac{\varepsilon}{3}$ for each $y \in H - D$. Evidently $|g(y) - a| \leqslant \frac{\varepsilon}{3}$ also for $y \in D \cap H$.

Let $s, t \in H$. Then $|g(s) - g(t)| \leqslant |g(s) - a| + |g(t) - a| < \varepsilon$. Hence g is cliquish at x. Since A is nowhere dense and the set of all cliquishness points of g is closed ([4]), g is cliquish on X. □

Remark 1. If X is a Baire separable metrizable space and $f: X \to \mathbf{R}$ is a cliquish function in the Baire class α, then it is the sum of three quasicontinuous functions in the Baire class α.

Proof. If f is a cliquish function in the Baire class α, then by [6; Corollary 1] the functions s, t in Lemma 1 are in the Baire class α. Since the function g is in the Baire class α as well, the functions f_1, f_2, f_3 in Lemma 2 are in the Baire class α. It is easy to see that then also the functions f_1, f_2, f_3 in Lemma 3 are ih the Baire class α. □

Theorem. *Let X be a separable metrizable (= T_3 second countable) space. Then every cliquish $f: X \to \mathbf{R}$ is the sum of three quasicontinuous functions.*

Proof. Let d be a metric which metrizes the space X and let $(\tilde{X}, \tilde{d})$ be the completion of (X, d). Then $\tilde{X}$ is a Baire separable metrizable space. According to Lemma 4 there is a cliquish function $g: \tilde{X} \to \mathbf{R}$ such that $g_{|X} = f$. According to Lemma 3 there are quasicontinuous functions $g_1, g_2, g_3: \tilde{X} \to \mathbf{R}$ such that $g = g_1 + g_2 + g_3$. Denote $f_i = (g_i)_{|X}$ $(i = 1, 2, 3)$. Since the restriction of a quasicontinuous function on a dense subset is quasicontinuous, f_i are quasicontinuous functions. Evidently $f = f_1 + f_2 + f_3$. □

Remark 2. The assumption "X is T_3 second countable" cannot be replaced by "X is normal second countable". The space $X = \mathbf{R}$ with the topology $\mathcal{T}$, where $A \in \mathcal{T}$ iff $A = \emptyset$ or $A = (a, \infty)$ (where $a \in \mathbf{R}$) is normal second countable, every quasicontinuous function on X is constant, however there are nonconstant cliquish functions.

Problem. Is every cliquish function $f: X \to \mathbf{R}$ (X as in Theorem) the sum of two quasicontinuous functions?

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Súhrn

SÚČTY KVÁZISPOJITÝCH FUNKCIÍ

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V práci je dokázané, že každá reálna kľukatá funkcia definovaná na separabilnom metrizovateľnom priestore je súčtom troch kvázispojitých funkcií.

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