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DECOUPLING NORMALIZING TRANSFORMATIONS AND LOCAL STABILIZATION OF NONLINEAR SYSTEMS

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Summary. The existence of the normalizing transformation completely decoupling the stable dynamic from the center manifold dynamic is proved. A numerical procedure for the calculation of the asymptotic series for the decoupling normalizing transformation is proposed. The developed method is especially important for the perturbation theory of center manifold and, in particular, for the local stabilization theory. In the paper some sufficient conditions for local stabilization are given.

Keywords: nonlinear system, stabilization, center manifold, normalizing transformation, smooth feedback

AMS classification: 34A34, 34D05, 34D35, 93C10, 93D15

1. Introduction

Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{x} = A\bar{x} + \bar{\Phi}(\bar{x},\bar{y}),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{y} = B\bar{y} + \bar{\Psi}(\bar{x},\bar{y}), \tag{1}$$

where

- $(\bar{x},\bar{y}) \in \mathbb{R}^m \times \mathbb{R}^n$, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space,
- $A \in \mathbb{R}^{m\times m}$ and $A = -A^T$,
- the eigenvalues of $B \in \mathbb{R}^{n\times n}$ have negative real parts,
- $\bar{\Phi}$ and $\bar{\Psi}$ are at least C^2 functions which vanish together with their derivatives at the origin, i.e.,

$$\bar{\Phi} \in C^k(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m), \qquad \bar{\Phi}(0,0) = 0, \quad \mathrm{d}\bar{\Phi}(0,0) = 0,$$
$$\bar{\Psi} \in C^k(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^n), \qquad \bar{\Psi}(0,0) = 0, \quad \mathrm{d}\bar{\Psi}(0,0) = 0, \tag{2}$$

where $k \geqslant 2$, $\mathrm{d}\Phi = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}\right)$ and $C^k(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^\ell)$ is the class of all functions

$$\zeta\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^\ell$$

which have continuous derivatives of order k.

To investigate the dynamic of the system (1) in a neighborhood of the origin we apply the center manifold theory which mainly consists of the following three theorems.

Theorem 1.1 [4, 8]. *Given the conditions* (2), *there exists a center manifold*

$$M_c = \{(\bar{x}, \bar{y}) \in B_\delta(0) \times \mathbb{R}^n\,;\, \bar{y} = h(\bar{x})\},$$

where $B_\delta(0) = \{x \in \mathbb{R}^m\,;\, |\bar{x}| < \delta\}$, $|\bar{x}|^2 = \langle \bar{x}, \bar{x}\rangle$ *and* $\langle x, z\rangle = \sum_{i=1}^{\ell} x_i z_i$ *for* $x, z \in \mathbb{R}^\ell$, $h \in C^{k-1}(\mathbb{R}^m, \mathbb{R}^n)$, $h(0) = 0$ *and* δ *is a sufficiently small real positive number.*

It is convenient to use the following notation:

$$\bar{f}(\bar{x}, \bar{y}) = \left(A\bar{x} + \bar{\Phi}(\bar{x}, \bar{y}), B\bar{y} + \bar{\Psi}(\bar{x}, \bar{y})\right)^T,$$

$\mathrm{e}^{t\bar{f}}$ denotes the flow generated by the vector field $\bar{f}$; $\mathrm{e}^{t\bar{f}}(x, y)$ is the point drifted by the flow $\mathrm{e}^{t\bar{f}}$ at time t from the point (x, y).

The zero solution is said to be stable, iff for every neighborhood W of the origin there exists a neighborhood V of the origin such that

$$\mathrm{e}^{t\bar{f}} V \subset W \qquad \forall t \geqslant 0,$$

where $\mathrm{e}^{t\bar{f}} V = \{\mathrm{e}^{t\bar{f}}(x, y)\,;\, (x, y) \in V\}$. The zero solution is asymptotically stable, iff it is stable and there exists a neighborhood Ξ such that

$$\lim_{t \to +\infty} \mathrm{e}^{t\bar{f}}(x, y) = 0$$

for all $(x, y) \in \Xi$.

The flow on the center manifold M_c is governed by the system

$$\dot{z} = Az + \bar{\Phi}(z, h(z)). \tag{3}$$

The next theorem tells us that (3) possesses all the necessary information needed to determine the asymptotic behavior of (1) in a neighborhood of the origin.

Theorem 1.2 [4].

(a) *If the zero solution of* (3) *is stable (asymptotically stable) (unstable), then the zero solution of* (1) *is stable (asymptotically stable) (unstable).*

(b) *If the zero solution of* (3) *is stable, then there exists a neighborhood* V *of the origin such that for every* $(x_0, y_0) \in V$ *one can find* z_0 *such that*

$$e^{t\bar{f}}(x_0, y_0) = \big(z(t, z_0), h(z(t, z_0))\big) + O(e^{-\gamma t}),$$

where $\gamma > 0$ *is a constant,* $z(t, z_0)$ *is the solution of* (3) *with the initial condition* $z(0, z_0) = z_0$.

The center manifold can be approximated to any degree of accuracy. For C^1 functions $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ define the nonlinear operator

$$(M\varphi)(\bar{x}) = d\varphi(\bar{x})\big[A\bar{x} + \bar{\Phi}(\bar{x}, \varphi(\bar{x}))\big] - B\varphi(\bar{x}) - \bar{\Psi}(\bar{x}, \varphi(\bar{x})).$$

For the function $h(\bar{x})$ defining the center manifold M_c we have $(Mh)(\bar{x}) = 0$.

Theorem 1.3 [4]. *Let* φ *be a* C^1 *mapping of a neighborhood of the origin in* $\mathbb{R}^m$ *into* $\mathbb{R}^n$ *with* $\varphi(0) = 0$, $d\varphi(0) = 0$. *Suppose that* $(M\varphi)(x) = O(|x|^q)$ *as* $x \to 0$, *where* $q > 1$. *Then* $|h(x) - \varphi(x)| = O(|x|^q)$ *as* $x \to 0$.

The main results of this paper occupy the place of Theorem 1.2 among these three theorems. In fact, Theorem 1.2 can be replaced by two stronger theorems (Theorem 2.2 and Theorem 3.1), which are the core of the theory proposed here. At the same time, the method developed here together with Theorems 1.1, 1.3 give us a powerful tool for the investigation of stability and stabilizability of nonlinear systems.

For small $(\bar{x}, \bar{y})$ we prove the existence of the decoupling normalizing transformation

$$\begin{aligned} \tilde{x} &= \bar{x} + \nu(\bar{x}, \bar{y} - h(\bar{x})),\ \nu(\bar{x}, 0) = 0,\ d\nu(0,0) = 0, \\ \tilde{y} &= \bar{y} - h(\bar{x}), \end{aligned} \tag{4}$$

under which the system (1) has the form

$$\begin{aligned} \frac{d}{dt}\tilde{x} &= A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})), \\ \frac{d}{dt}\tilde{y} &= B\tilde{y} + \tilde{\Psi}(\tilde{x}, \tilde{y}), \end{aligned} \tag{5}$$

where $h(x)$ is the function from Theorem 1.1, $\bar{\Phi}(\tilde{x}, h(\tilde{x}))$ is from (3), $\tilde{\Psi}(\tilde{x}, 0) = 0$ for all $\tilde{x}$ sufficiently small and $d\tilde{\Psi}(0,0) = 0$. If $\bar{\Phi}, \bar{\Psi}$ are C^k functions, then $\nu(\bar{x}, \bar{y} - h(\bar{x}))$

is a C^{k-2} function. $\nu(x, y)$ can be approximated by some known function. We will show how to calculate this approximation in the third section of this paper. To know $\nu(x, y,)$ is important, both for the investigation of the stabilization and for the design of a stabilizing feedback. To illustrate that, we will prove several sufficient conditions for local stabilizability of nonlinear systems with noncontrollable linearizations and propose a stabilizer design procedure for a bilinear system.

2. Existence of decoupling normalizing transformation

Here we prove the existence of the decoupling normalizing transformation (4). The proof is analogous to the proof of Theorem 1.1 [8].

It is more convenient to rewrite the system (1) in the new coordinates·

$$x = \bar{x},$$
$$y = \bar{y} - h(\bar{x}),$$

where $h(\bar{x})$ is from Theorem 1.1. Under the coordinate transformation the system (1) assumes the form

$$\begin{aligned} \dot{x} &= Ax + \hat{\Phi}(x, y), \\ \dot{y} &= By + \hat{\Psi}(x, y), \end{aligned} \tag{6}$$

where

$$\hat{\Phi}(x, y) = \bar{\Phi}(x, y + h(x)),$$
$$\hat{\Psi}(x, y) = \mathrm{d}h(x)\big(\bar{\Phi}(x, h(x)) - \bar{\Phi}(x, y + h(x))\big) + \bar{\Psi}(x, y + h(x)) - \bar{\Psi}(x, h(x)).$$

Now for the system (6) we prove the existence of the function $\nu(x, y)$ such that under the transformation

$$\begin{aligned} \tilde{x} &= x + \nu(x, y) \\ \tilde{y} &= y \end{aligned} \tag{7}$$

the system (6) becomes (5).

Theorem 2.1. *Let $\hat{\Phi}(x, y)$, $\hat{\Psi}(x, y)$ be C^k functions ($k \geqslant 3$) which vanish together with their derivatives at the origin, i.e.,*

$$\hat{\Phi}(0, 0) = 0, \quad \mathrm{d}\hat{\Phi}(0, 0) = 0, \quad \mathrm{d}\hat{\Psi}(0, 0) = 0$$

and, in addition,

$$\hat{\Psi}(x,0) = 0 \text{ for all } (x,0) \in Q,$$

where Q is a neighborhood of the origin. Then there exist a neighborhood $\dot{Q} \subseteq Q$ of the origin and a C^{k-2} function $\nu(x,y)$ such that

$$\nu(x,0) = 0 \quad \forall\,(x,0) \in \dot{Q}, \quad \mathrm{d}\nu(0,0) = 0,$$

and under the normalizing transformation (7) *the system* (6) *assumes the form* (5).

Proof. Introducing the scalar change of variables $(x,y) \longrightarrow (\lambda x, \lambda y)$ and multiplying $\hat{\Phi}$, $\hat{\Psi}$ by $\omega(|x|^2+|y|^2+K\lambda^2)$ where K is a sufficiently large positive constant and $\omega(r)$ is a C^∞ real valued function satisfying

$$0 \leqslant \omega(r) \geqslant 1 \ \forall r \geqslant 0,$$

$$\omega(r) \equiv 1 \ \forall 0 \leqslant r \leqslant \tfrac{1}{2},$$

$$\omega(r) \equiv 0 \ \forall 1 \leqslant r < \infty,$$

we obtain

$$\begin{aligned} \dot{x} &= Ax + \Phi(x,y,\lambda), \\ \dot{y} &= By + \Psi(x,y,\lambda), \end{aligned} \tag{8}$$

where

$$\Phi(x,y,0) = \Psi(x,y,0) = 0,$$

and for $\lambda \neq 0$

$$\Phi(x,y,\lambda) = \frac{1}{\lambda}\omega(|x|^2+|y|^2+K\lambda^2)\hat{\Phi}(\lambda x, \lambda y),$$

$$\Psi(x,y,\lambda) = \frac{1}{\lambda}\omega(|x|^2+|y|^2+K\lambda^2)\hat{\Psi}(\lambda x, \lambda y),$$

and the following conditions hold:

(ai) $\Phi(x,y,\lambda)$, $\Psi(x,y,\lambda)$ exist and are continuous for all (x,y,λ) and for each fixed λ they are C^k functions in (x,y).

(aii) $\Phi(0,0,\lambda) = 0$, for any fixed λ we have $\mathrm{d}\Phi(0,0,\lambda) = 0$, $\mathrm{d}\Psi(0,0,\lambda) = 0$. There exists a real positive value $\delta > 0$ such that $\Psi(x,0,\lambda) = 0 \ \forall x \in \mathbb{R}^m$, $|\lambda| < \delta$.

(aiii) Φ, $\Psi \equiv 0 \ \forall |x|^2+|y|^2 \geqslant 1$, where $|\cdot|$ represents the Euclidean norm corresponding to the usual scalar product $\langle\cdot,\cdot\rangle$ on pairs of vectors.

(aiv) $\left(\frac{\partial}{\partial x}\right)^i\left(\frac{\partial}{\partial y}\right)^j(\Phi,\Psi) \longrightarrow 0$ uniformly in $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ as $\lambda \to 0$ for $|i|+|j| \leqslant k$;

$$\left(\frac{\partial}{\partial x}\right)^i\left(\frac{\partial}{\partial y}\right)^j = \left(\frac{\partial}{\partial x_1}\right)^{i_1}\cdots\left(\frac{\partial}{\partial x_m}\right)^{i_m}\left(\frac{\partial}{\partial y_1}\right)^{j_1}\cdots\left(\frac{\partial}{\partial y_n}\right)^{j_n},$$

where $i = (i_1, \ldots, i_m)$, $j = (j_1, \ldots, j_n)$ are an m-tuple and an n-tuple of non-negative integers respectively, $|i| = i_1 + \cdots + i_m$, $|j| = j_1 + \cdots + j_n$.

If $\lambda \neq 0$, then the systems (6) and (8) are locally (near the origin) related by a scalar change of variables. Therefore it is sufficient to prove Theorem 2.1 only for the system (8).

The function $\nu(x,y)$ is a solution of the following equation in partial derivatives.

$$(9)\ A\nu - \frac{\partial \nu}{\partial x}Ax - \frac{\partial \nu}{\partial y}By = \frac{\partial \nu}{\partial x}\Phi(x,y,\lambda) + \frac{\partial \nu}{\partial y}\Psi(x,y,\lambda) + \Phi(x,y,\lambda) - \Phi(x+\nu,0,\lambda),$$
$$\nu(x,0) = 0 \quad \forall x \in \mathbb{R}^m,$$
$$\mathrm{d}\nu(0,0) = 0.$$

To solve the equation (9) we take into account that

$$(10) \qquad \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{At}(\mathrm{e}^{-tf})^*\nu(x,y)] = \mathrm{e}^{At}(\mathrm{e}^{-tf})^*[\Phi(x,y,\lambda) - \Phi(x+\nu,0,\lambda)],$$

where $f = \big(Ax + \Phi(x,y,\lambda),\ By + \Psi(x,y,\lambda)\big)^T$, $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{At} = A\mathrm{e}^{At}$, $\mathrm{e}^{At}|_{t=0} = I$, I is the identity matrix,

$$(\mathrm{e}^{tf})^*\varphi(x,y) = \varphi(\mathrm{e}^{tf}(x,y)) \ \forall t \in \mathbb{R}.$$

Integrating (10) with respect to t we obtain

$$(11) \quad \mathrm{e}^{tA}(\mathrm{e}^{-tf})^*\nu(x,y) - \nu(x,y) = \int_0^t \mathrm{e}^{A\tau}(\mathrm{e}^{-\tau f})^*[\Phi(x,y,\lambda) - \Phi(x+\nu,0,\lambda)]\,\mathrm{d}\tau.$$

Since $A = -A^T$ and the eigenvalues of $B \in \mathbb{R}^{n\times n}$ have negative real parts, there exists a compact convex set $\Lambda \subset \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n;\ |x| + |y| \leqslant 2\} \subset \Lambda$$

and

$$\mathrm{e}^{tf}\Lambda \subset \Lambda \quad \forall\, t \geqslant 0.$$

For a proof see e.g. [11].

Consider the Banach space

$$\Upsilon^l = \{\nu = \nu(x,y) \text{ satisfying (bi)–(biii)}\}.$$

(bi) ν is a real vector-valued function such that $\nu\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\frac{\partial}{\partial y}\nu(x,y)$ is a C^l function.

(bii) $\nu(x,0) = 0\ \forall\, x \in \mathbb{R}^m$, $\mathrm{d}\nu(0,0) = 0$.

(biii)
$$\|\nu\| = \max_{|i|+|j|\leqslant l} \sup_{(x,y)\in\mathbb{R}^m\times\mathbb{R}^n} \Big|\Big(\frac{\partial}{\partial x}\Big)^i\Big(\frac{\partial}{\partial y}\Big)^{j+1}\nu(x,y)\Big| < \infty.$$

If $\nu \in \Upsilon^{k-2}$, then

$$|\nu(x,y)| = \Big|\int_0^y \frac{\partial}{\partial y}\nu(x,\theta)\,\mathrm{d}\theta\Big| \leqslant \|\nu\|\cdot|y| \quad \forall (x,y)\in\mathbb{R}^m\times\mathbb{R}^n. \tag{12}$$

In accordance with condition (aii) we have

$$\Psi(x,y,\lambda) = \Big(\int_0^1 \frac{\partial}{\partial y}\Psi(x,sy,\lambda)\,\mathrm{d}s\Big)\cdot y$$

and (aiv) yields

$$\int_0^1 \frac{\partial}{\partial y}\Psi(x,sy,\lambda)\,\mathrm{d}s \to 0$$

uniformly in $(x,y)\in\mathbb{R}^m\times\mathbb{R}^n$ as $\lambda\to 0$.

We can choose a positive real value δ such that, for $|\lambda|<\delta$,

$$\begin{aligned} &|P_y(\mathrm{e}^{tf}(x,y))| \leqslant \alpha(t)\cdot\mathrm{e}^{(-\mu+\beta(\lambda))t} \ \ \forall\, t>0,\ (x,y)\in\mathbb{R}^m\times\mathbb{R}^n,\\ &|\beta(\lambda)|<\mu, \end{aligned} \tag{13}$$

where

- $P_y\colon \mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^n$, $P_y(x,y)=y$;
- $\alpha(t)$ is a polynomial in t with positive coefficients;
- $\beta(\lambda)\geqslant 0$ is continuous in λ and $\beta(\lambda)\to 0$ as $\lambda\to 0$;
- $\mu=\frac{1}{2}\min\{|\operatorname{Re} z|;\ z$ is from the set of eigenvalues of $B\}$.

For a proof of (13) see Lemma 3 on page 552 of [8] or Lemma 1 on page 23 of [4].

Therefore (12), (13) imply, for $\nu\in\Upsilon^{k-2}$,

$$|\mathrm{e}^{-At}(\mathrm{e}^{tf})^*\nu(x,y)| \leqslant \overline{\alpha}(t)\mathrm{e}^{(-\mu+\beta(\lambda))t}\|\nu\| \ \ \forall\, t>0,\ (x,y)\in\mathbb{R}^m\times\mathbb{R}^n, \tag{14}$$

where $\overline{\alpha}(t)$ is a polynomial in t with positive coefficients which does not depend on $\nu\in\Upsilon^{k-2}$.

Thus if $\nu\in\Upsilon^{k-2}$, then it follows from (11), (14) that

$$\nu(x,y) = \int_{-\infty}^0 \mathrm{e}^{A\tau}(\mathrm{e}^{-\tau f})^*\Big[\Phi(x,y,\lambda)-\Phi(x+\nu(x,y),0,\lambda)\Big]\,\mathrm{d}\tau.$$

Consider the nonlinear operator

$$T_\lambda\nu(x,y) = \int_{-\infty}^0 \mathrm{e}^{A\tau}(\mathrm{e}^{-\tau f})^*\Big[\Phi(x,y,\lambda)-\Phi\big(x+\nu(x,y),0,\lambda\big)\Big]\,\mathrm{d}\tau$$

which is defined, for $|\lambda| < \delta$, on the Banach space Υ^{k-2}.

The conditions (aii)–(aiv) imply

$$\Phi(x, y, \lambda) - \Phi(x + \nu(x, y), 0, \lambda) \in \Upsilon^{k-2},$$

whenever Φ is a C^k function and $\nu \in \Upsilon^{k-2}$.

Since the eigenvalues of A all have zero real parts, (14) implies

$$\begin{aligned} &|a^{-At}(\mathrm{e}^{tf})^*[\Phi(x, y, \lambda) - \Phi(x + \nu(x, y), 0, \lambda)]| \\ &\qquad \leqslant \overline{\alpha}(t) \cdot \mathrm{e}^{(-\mu+\beta(\lambda))t} \cdot \|\Phi(P_x(\cdot), P_y(\cdot), \lambda) - \Phi(P_x(\cdot) + \nu(P_x(\cdot), P_y(\cdot)), 0, \lambda)\|, \end{aligned}$$

where $P_x\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $P_x(x, y) = x$.

In what follows,

$$|(T_\lambda \nu)(x, y)| < \infty \quad \forall (x, y) \in \mathbb{R}^m \times \mathbb{R}^n$$

for all $\nu \in \Upsilon^{k-2}$ and $|\lambda| < \delta$.

We now prove the existence of $\hat{\delta} > 0$ such that, for $|\lambda| < \hat{\delta}$,

$$\|(\mathrm{e}^{tf})^*\| \leqslant \hat{\alpha}(t) \cdot \mathrm{e}^{(-\mu+\hat{\beta}(\lambda))t} \quad \forall t \geqslant 0, \tag{15}$$

where $\|(\mathrm{e}^{tf})^*\|$ is the norm of the operator

$$(\mathrm{e}^{tf})^*\colon \Upsilon^{k-2} \to \Upsilon^{k-2}$$

and $\hat{\alpha}(t)$, $\hat{\beta}(\lambda)$ are of the same type as $\overline{\alpha}(t)$, $\alpha(t)$, $\beta(\lambda)$ from (13), (14).

Introduce the notation

$$X^{i,j}_{x,y}(t) = \Big(\frac{\partial}{\partial x}\Big)^i \Big(\frac{\partial}{\partial y}\Big)^j P_x(\mathrm{e}^{tf}(x, y)),$$

$$Y^{i,j}_{x,y}(t) = \Big(\frac{\partial}{\partial x}\Big)^i \Big(\frac{\partial}{\partial y}\Big)^j P_y(\mathrm{e}^{tf}(x, y)).$$

Then $\{(X^{i,j}_{x,y}(t), Y^{i,j}_{x,y}(t))\}_{|i|+|j|\leqslant k-1}$ is the solution of the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \Phi(x(t), y(t), \lambda), \\ \dot{y}(t) &= By(t) + \Psi(x(t), y(t), \lambda), \\ \frac{\mathrm{d}}{\mathrm{d}t} X^{i,j}_{x,y}(t) &= AX^{i,j}_{x,y}(t) + \Big(\frac{\partial}{\partial x}\Big)^i \Big(\frac{\partial}{\partial y}\Big)^j \Phi(x(t), y(t), \lambda), \\ \frac{\mathrm{d}}{\mathrm{d}t} Y^{i,j}_{x,y}(t) &= BY^{i,j}_{x,y}(t) + \Big(\frac{\partial}{\partial x}\Big)^i \Big(\frac{\partial}{\partial y}\Big)^j \Psi(x(t), y(t), \lambda), \end{aligned}$$

where $|i|+|j| \leqslant k-1$, $x(t) = P_x(\mathrm{e}^{tf}(x,y))$, $y(t) = P_y(\mathrm{e}^{tf}(x,y))$ and

$$X^{i,j}_{x,y}(0) = 0,\ Y^{i,j}_{x,y}(0) = 0 \quad \text{for} \quad |i|+|j| \geqslant 2,$$
$$\frac{\partial}{\partial y} P_x(\mathrm{e}^{tf}(x,y))\Big|_{t=0} = 0, \quad \frac{\partial}{\partial x} P_y(\mathrm{e}^{tf}(x,y))\Big|_{t=0} = 0,$$
$$\Big[\frac{\partial}{\partial x_1} P_x(\mathrm{e}^{tf}(x,y)), \ldots, \frac{\partial}{\partial x_m} P_x(\mathrm{e}^{tf}(x,y))\Big]\Big|_{t=0} = I_m,$$
$$\Big[\frac{\partial}{\partial y_1} P_y(\mathrm{e}^{tf}(x,y)), \ldots, \frac{\partial}{\partial y_n} P_y(\mathrm{e}^{tf}(x,y))\Big]\Big|_{t=0} = I_n,$$

where $I_m \in \mathbb{R}^{m\times m}$, $I_n \in \mathbb{R}^{n\times n}$ are identity matrices. Using induction with respect to $|i|+|j| = l$ we can prove the existence of $\tilde{\delta} > 0$ (which may depend on (i,j)) such that for $|\lambda| < \tilde{\delta}$

$$\sup_{(x,y)} \Big|\Big(\frac{\partial}{\partial x}\Big)^i \Big(\frac{\partial}{\partial y}\Big)^j P_y(\mathrm{e}^{tf}(x,y))\Big| \leqslant \tilde{\alpha}(t)\cdot \mathrm{e}^{(-\mu+\tilde{\beta}(\lambda))t}, \ 1 \leqslant |i|+|j| \leqslant k-1 \tag{16}$$

where $\tilde{\alpha}(t)$ is a polynomial in t with positive coefficients, $\tilde{\beta}(\lambda) \geqslant 0$ is continuous in λ and $\tilde{\beta}(\lambda) \to 0$ as $\lambda \to 0$, μ is defined in (13).

Step 1. Let $|i|+|j| = 0$. Then

$$\dot{y}(t) = By(t) + \int_0^1 \frac{\partial}{\partial y}\Psi(x(t), sy(t), \lambda)\,\mathrm{d}s \cdot y(t)$$

and the eigenvalues of B have negative real parts. Therefore there exists a positive real value $\delta > 0$ such that for $|\lambda| < \delta$ and $|i|+|j| = 1$ the inequality (16) holds. For a proof see Lemma 3 on page 552 of [8] or Lemma 1 on page 23 of [4].

Step 2. Let the inequality (16) hold for all $|i|+|j| < l$. Consider the case $|i|+|j| = l$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} Y^{i,j}_{x,y}(t) &= BY^{i,j}_{x,y}(t) + \frac{\partial}{\partial y}\Psi(x(t), y(t), \lambda)\cdot Y^{i,j}_{x,y}(t)\\ &\quad + \frac{\partial}{\partial x}\Psi(x(t), y(t), \lambda)\cdot X^{i,j}_{x,y}(t)\\ &\quad + \Xi(\{X^{i,j}_{x,y}(t)\}_{|i|+|j|<l}, \{Y^{i,j}_{x,y}(t)\}_{|i|+|j|<l}, \lambda),\end{aligned} \tag{17}$$

where $X^{0,0}_{x,y}(t) = x(t)$, $Y^{0,0}_{x,y}(t) = y(t)$ and the function $\Xi(\cdot,\cdot,\lambda)$ satisfies the conditions

$$\Xi(\{X^{i,j}_{x,y}(t)\}_{|i|+|j|<l}, 0, \lambda) = 0,$$
$$\Xi(\{X^{i,j}_{x,y}(t)\}_{|i|+|j|<l}, \{Y^{i,j}_{x,y}(t)\}_{|i|+|j|<l}, 0) = 0.$$

Due to (aiii), (aiv) and the induction hypothesis there exists $\bar{\delta} > 0$ such that

$$\sup_{(x,y)} \left\{ \left| \Xi(\{X^{i,j}_{x,y}(t)\}_{|i|+|j|<l}, \{Y^{i,j}_{x,y}(t)\}_{|i|+|j|<l}, \lambda) \right| + \left| \frac{\partial}{\partial x} \Psi(x(t), y(t), \lambda) \cdot X^{i,j}_{x,y}(t) \right| \right\}$$

$$\leqslant \bar{\alpha}(t) \cdot \mathrm{e}^{(-\mu + \bar{\beta}(\lambda))t} \ \forall t \geqslant 0 \ |\lambda| < \delta, \tag{18}$$

where $\bar{\alpha}(t)$ is a polynomial in t with positive coefficients, $\bar{\beta}(\lambda) \geqslant 0$ is continuous in λ and $\bar{\beta}(\lambda) \to 0$ as $\lambda \to 0$. The estimate for

$$\frac{\partial}{\partial x} \Psi(x(t), y(t), \lambda) \cdot X^{i,j}_{x,y}(t)$$

follows from the fact that $X^{i,j}_{x,y}(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} X^{i,j}_{x,y}(t) = A X^{i,j}_{x,y}(t) + \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j \Phi(x(t), y(t), \lambda)$$

and $y(t)$ is exponentially decreasing as $t \to \infty$.

Thus (17) and (18) imply (16) for $|i| + |j| = l$.

The inequality (16) yields (15).

Since the conditions (aii)–(aiv) imply

$$\Phi(x, y, \lambda) - \Phi(x, 0, \lambda) \in \Upsilon^{k-2}$$

we have

$$\Phi(x, y, \lambda) - \Phi(x + \nu(x, y), 0, \lambda) \in \Upsilon^{k-2}$$

whenever Φ is a C^k function and $\nu \in \Upsilon^{k-2}$. Thus we obtain

$$\begin{aligned}
&\|\Phi(P_x(\cdot), P_y(\cdot), \lambda) - \Phi(P_x(\cdot) + \nu(P_x(\cdot), P_y(\cdot)), 0, \lambda)\| \\
&\quad \leqslant \|\Phi(P_x(\cdot), P_y(\cdot), \lambda) - \Phi(P_x(\cdot), 0, \lambda)\| \\
&\qquad + \|\Phi(P_x(\cdot), 0, \lambda) - \Phi(P_x(\cdot) + \nu(P_x(\cdot), P_y(\cdot)), 0, \lambda)\| \\
&\quad \leqslant \|\Phi(P_x(\cdot), P_y(\cdot), \lambda) - \Phi(P_x(\cdot), 0, \lambda)\| \\
&\qquad + D_k \cdot \|\Phi(P_x(\cdot), 0, \lambda)\|_{C^k} \cdot (\|\nu\| + 1)^{k-1},
\end{aligned}$$

where $k \geqslant 3$, the constant D_k depends only on k and

$$\|\Phi(P_x(\cdot), 0, \lambda)\|_{C^k} = \max_{|i| \leqslant k} \sup_{x \in \mathbb{R}^m} \left| \left(\frac{\partial}{\partial x}\right)^i \Phi(x, 0, \lambda) \right|.$$

Thus, taking into account (15), we obtain, for $|\lambda| < \hat{\delta}$,

$$\|T_\lambda \nu\| \leqslant \int_0^\infty \hat{\alpha}(\tau) e^{(-\mu+\beta(\lambda))\tau}\, d\tau \cdot \big(\|\Phi(P_x(\cdot), P_y(\cdot), \lambda) - \Phi(P_x(\cdot), 0, \lambda)\| + D_k \cdot \|\Phi(P_x(\cdot), 0, \lambda)\|_{C^k} \cdot (1 + \|\nu\|)^{k-1}\big) \quad \forall\, \nu \in \Upsilon^{k-2},$$

where $\hat{\alpha}(t)$ is a polynomial in t with positive coefficients.

(aiv) implies

$$\lim_{\lambda \to 0} \left\{ \int_0^\infty \hat{\alpha}(\tau) e^{(-\mu+\beta(\lambda))\tau}\, d\tau \cdot (\|\Phi(P_x(\cdot), P_y(\cdot), \lambda) - \Phi(P_x(\cdot), 0, \lambda)\| + D_k \cdot \|\Phi(P_x(\cdot), 0, \lambda)\|_{C^k} \cdot (1 + r)^{k-1}) \right\} = 0$$

for any positive real value r. Hence, for any $r > 0$, there exists $\delta(r) > 0$ such that

$$T_\lambda \colon B_r \to B_r \text{ for } |\lambda| < \delta(r),$$

where $B_r = \{\nu \in \Upsilon^{k-2};\ \|\nu\| \leqslant r\}$.

We now prove the existence of $r > 0$ such that, for all $\nu_1, \nu_2 \in B_r$,

$$\|T_\lambda \nu_1 - T_\lambda \nu_2\| \leqslant \tfrac{1}{2} \cdot \|\nu_1 - \nu_2\| \text{ for } |\lambda| < \delta(r). \tag{19}$$

It follows from (15) and the definition of T_λ that

$$\|T_\lambda \nu_1 - T_\lambda \nu_2\| \leqslant \int_0^\infty \hat{\alpha}(\tau) e^{(-\mu+\beta(\lambda))\tau}\, d\tau \cdot \|\Phi(P_x(\cdot) + \nu_1, 0, \lambda) - \Phi(P_x(\cdot) + \nu_2, 0, \lambda)\|. \tag{20}$$

It is easy to see that

$$\begin{aligned} &\Phi(x + \nu_1(x,y), 0, \lambda) - \Phi(x + \nu_2(x,y), 0, \lambda) \\ &\quad = \int_0^1 \frac{\partial}{\partial x} \Phi(x + s\nu_1 + (1-s)\nu_2, 0, \lambda)\, ds \int_0^y \Big(\frac{\partial}{\partial y}\nu_1(x,\theta) - \frac{\partial}{\partial y}\nu_2(x,\theta)\Big)\, d\theta. \end{aligned} \tag{21}$$

Due to (aiii) we obtain from (21) that $\forall\, \nu_1,\, \nu_2 \in B_r$,

$$\|\Phi(P_x(\cdot) + \nu_1, 0, \lambda) - \Phi(P_x(\cdot) + \nu_2, 0, \lambda)\| \leqslant C(r) \cdot \|\Phi(P_x(\cdot), 0, \lambda)\|_{C^{k-1}} \cdot \|\nu_1 - \nu_2\| \tag{22}$$

where $C(r)$ is a constant depending only on r.

Thus (22) together with (20) and (aiv) yield (19). We have proved the existence of $r > 0$ and $\delta(r) > 0$ such that, for $|\lambda| < \delta(r)$, T_λ is a contraction mapping on $B_r \subset \Upsilon^{k-2}$. Therefore, according to Banach's contraction principle [6], there exists a single function $\nu(x,y) \in \Upsilon^{k-2}$ such that $\nu = T_\lambda \nu$. The function $\nu(x,y)$ was constructed by the following procedure: if $\nu(x,y)$ fulfils (9), then $\nu(x,y)$ fulfils $\nu = T_\lambda \nu$. The opposite implication follows from the fact that, according to Banach's contraction principle [6], $\nu = T_\lambda \nu$ has unique solution. □

Theorem 2.1 can be reformulated in terms of the original system (1).

Theorem 2.2. *Let $\bar{\Phi}(\bar{x},\bar{y})$, $\bar{\Psi}(\bar{x},\bar{y})$ be C^k functions ($k \geqslant 3$) which vanish together with their derivatives at the origin, i.e.,*

$$\bar{\Phi}(0,0) = 0, \ \bar{\Psi}(0,0) = 0, \ \mathrm{d}\bar{\Phi}(0,0) = 0, \ \mathrm{d}\bar{\Psi}(0,0) = 0.$$

Then there exist a neighborhood Q of the origin, a C^{k-2}function $\nu(x,y)$ and a C^k function $h(x)$ such that

$$\nu(x,0) = 0 \ \forall\, (x,0) \in Q, \ \mathrm{d}\nu(0,0) = 0, \ h(0) = 0, \ \mathrm{d}h(0) = 0,$$

and under the normalizing transformation

$$\begin{aligned}\tilde{x} &= \bar{x} + \nu(\bar{x}, \bar{y} - h(\bar{x})),\\ \tilde{y} &= \bar{y} - h(\bar{x}),\end{aligned}$$

the system (1) assumes the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} &= A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})),\\ \frac{\mathrm{d}}{\mathrm{d}t}\tilde{y} &= B\tilde{y} + \tilde{\Psi}(\tilde{x}, \tilde{y}),\end{aligned}$$

where $\tilde{\Psi}(\tilde{x},0) = 0 \ \forall\, (\tilde{x},0) \in Q, \ \mathrm{d}\tilde{\Psi}(0,0) = 0$.

Remark. A decoupling normalizing transformation is not unique because of non-uniqueness of the center manifold.

3. Approximation of the decoupling normalizing transformation

The function $\zeta(\bar{x},\bar{y}) = \nu(\bar{x},\bar{y}-h(\bar{x}))$ can be approximated to any degree of accuracy. To show that we introduce the nonlinear operator

$$\mathcal{J}(\mu) = A\mu - L_{\bar{f}}\mu + \bar{\Phi}(\bar{x}+\mu, h(\bar{x}+\mu)) - \bar{\Phi}(\bar{x},\bar{y}),$$

where $L_{\bar{f}}\mu$ is the Lie derivative, i.e.,

$$L_{\bar{f}}\mu = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{t\bar{f}})^*\mu\Big|_{t=0},$$

$h(x)$ is the function from Theorem 1.1. We recall that

$$g(\bar{x},\bar{y}) = O((|\bar{x}|+|\bar{y}|)^q \cdot |\bar{y}-h(\bar{x})|) \text{ as } (\bar{x},\bar{y}) \to 0,$$

iff there exists a neighborhood of the origin W such that

$$|g(\bar{x},\bar{y})| \leqslant C \cdot (|\bar{x}|+|\bar{y}|)^q \cdot |\bar{y}-h(\bar{x})| \quad \forall\, (\bar{x},\bar{y}) \in W,$$

where C is a positive real constant.

Theorem 3.1. *Suppose μ is a C^{k-2} ($k \geqslant 3$), function such that* $\mathrm{d}\mu(0,0) = 0$ *and, for some $\varrho > 0$, $\mu(\bar{x},h(\bar{x})) = 0$ $\forall\,|\bar{x}| < \varrho$. Moreover, assume*

$$\mathcal{J}(\mu) = O((|\bar{x}|+|\bar{y}|)^q \cdot |\bar{y}-h(\bar{x})|) \text{ as } (\bar{x},\bar{y}) \to 0$$

where $q \geqslant 1$. Then

$$\zeta(\bar{x},\bar{y}) - \mu(\bar{x},\bar{y}) = O((|\bar{x}|+|\bar{y}|)^q \cdot |\bar{y}-h(\bar{x})|) \text{ as } (\bar{x},\bar{y}) \to 0. \tag{23}$$

Proof. Following the proof of Theorem 2.1, it is sufficient to prove (23) only for the system (8) with λ sufficiently small. Take the function

$$\theta_\lambda(x,y) = \begin{cases} \frac{1}{\lambda}\mu(\lambda x, \lambda y) \cdot \omega(|x|^2+|y|^2+K\lambda^2) & \text{for } \lambda \neq 0, \\ 0 & \text{for } \lambda = 0, \end{cases} \tag{24}$$

where $x = \bar{x}$, $y = \bar{y} - h(\bar{x})$ and $\omega(r)$ is the truncated function introduced in the proof of Theorem 2.1. Then $\theta_\lambda \in \Upsilon^{k-2}$ and there exist $r > 0$ and $\bar{\lambda} > 0$ such that

$$\theta_\lambda \in \mathrm{Int}B_r = \{\nu \in \Upsilon^{k-2};\ \|\nu\| < r\} \ \ \forall|\lambda| < \bar{\lambda}.$$

define a mapping $S_\lambda \colon \Upsilon^{k-2} \to \Upsilon^{k-2}$ by

$$S_\lambda z = T_\lambda(z + \theta_\lambda) - \theta_\lambda.$$

Since there exists $\delta(r) > 0$ such that T_λ is a contraction mapping on B_r for $|\lambda| < \delta(r)$, S_λ is a contraction mapping on

$$\Xi(\lambda, q) = \{\varphi \in \Upsilon^{k-2};\ \|\varphi + \theta_\lambda\| \leqslant r, |\varphi(x,y)| \leqslant \tilde{K} \cdot ((|x| + |y|)^q \cdot |y|) \quad \forall\, (x,y) \in \mathbb{R}^m \times \mathbb{R}^n\},$$

where $\tilde{K}$ is a positive real constant. Indeed, it is sufficient only to show that

$$S_\lambda \colon \Xi(\lambda, q) \to \Xi(\lambda, q).$$

If $\varphi \in \Xi(\lambda, q)$, then

$$\|S_\lambda \varphi + \theta_\lambda\| = \|T_\lambda(\varphi + \theta_\lambda)\| \leqslant r,$$

where the last inequality follows from

$$T_\lambda \colon B_r \to B_r.$$

Thus it remains to prove that, for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$,

$$|\varphi(x,y)| \leqslant \tilde{K} \cdot ((|x| + |y|)^q \cdot |y|)$$

yields

$$|(S_\lambda \varphi)(x,y)| \leqslant \tilde{K} \cdot ((|x| + |y|)^q |y|)$$

for some positive $\tilde{K}$.

The function $\theta_\lambda(x,y)$ can be represented as

$$-\theta_\lambda(x,y) = -\int_{-\infty}^{0} \frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{e}^{A\tau}(\mathrm{e}^{-\tau f})^* \theta_\lambda(x,y))\, \mathrm{d}\tau = -\int_{-\infty}^{0} \mathrm{e}^{A\tau}(\mathrm{e}^{-\tau f})^*(A\theta_\lambda - L_f \theta_\lambda)\, \mathrm{d}\tau.$$

Since, for $q \geqslant 1$,

$$\lim_{\lambda \to 0} \frac{|A\theta_\lambda - L_f\theta_\lambda + \Phi(x + \theta_\lambda, 0, \lambda) - \Phi(x, y, \lambda)|}{(|x| + |y|)^q \cdot |y|} = 0$$

uniformly with respect to (x, y), we have

$$|A\theta_\lambda - L_f\theta_\lambda + \Phi(x + \theta_\lambda, 0, \lambda) - \Phi(x, y, \lambda)| \leqslant R(\lambda)((|x| + |y|)^q \cdot |y|),$$

where $R(\lambda) \geqslant 0$ and $R(\lambda) \to 0$ as $\lambda \to 0$. Therefore, we obtain

$$-\theta_\lambda(x,y) = -\int_{-\infty}^{0} \mathrm{e}^{A\tau}(\mathrm{e}^{-\tau f})^*\{[\Phi(x,y,\lambda) - \Phi(x+\theta_\lambda,0,\lambda)] + N(x,y)\}\,\mathrm{d}\tau,$$

where

$$N(x,y) = A\theta_\lambda - L_f\theta_\lambda + \Phi(x+\theta_\lambda,0,\lambda) - \Phi(x,y,\lambda) \quad \text{and}$$
$$|N(x,y)| \leqslant R(\lambda)\big((|x|+|y|)^q \cdot |y|\big).$$

Thus

$$(S_\lambda\varphi)(x,y) = \int_{-\infty}^{0} \mathrm{e}^{A\tau}(\mathrm{e}^{-\tau f})^*[\Phi(P_x(\cdot)+\theta_\lambda,0,\lambda) - \Phi(P_x(\cdot)+\theta_\lambda+\varphi,0,\lambda) - N(x,y)]\,\mathrm{d}\tau.$$

Since

$$|\Phi(x+\theta_\lambda,0,\lambda) - \Phi(x+\theta_\lambda+\varphi,0,\lambda)|$$
$$= \Big|\Big(\int_0^1 \frac{\partial}{\partial x}\Phi(x+\theta_\lambda+s\varphi,0,\lambda)\,\mathrm{d}s\Big)\cdot\varphi(x,y)\Big| \leqslant \|\Phi(P_x(\cdot),0,\lambda)\|_{C^1}\cdot\tilde{K}(|x|+|y|)^q\cdot|y|$$

and the eigenvalues of A all have zero real parts, we have

$$|\mathrm{e}^{-At}(\mathrm{e}^{tf})^*(\Phi(x+\theta_\lambda,0,\lambda) - \Phi(x+\theta_\lambda+\varphi,0,\lambda) - N(x,y))|$$
$$\leqslant \alpha(t)\cdot\Big(\|\Phi(P_x(\cdot),0,\lambda)\|_{C^1}\cdot\tilde{K} + R(\lambda)\Big)\cdot\Big|(\mathrm{e}^{tf})^*((|x|+|y|)^q \times |y|)\Big|$$
$$\text{for} \quad t>0, \quad (x,y)\in\mathbb{R}^m\times\mathbb{R}^n,$$

where $\alpha(t)$ is a polynomial in t with positive coefficients. Using (13) we obtain the existence of $C>0$ and $\tilde{\delta}>0$ such that

$$\alpha(t)\cdot\big|(\mathrm{e}^{tf})^*((|x|+|y|)^q\cdot|y|)\big| \leqslant C\cdot(|x|+|y|)^q\cdot|y|$$

for all $t>0$, $|\lambda|<\tilde{\delta}$ and $(x,y)\in\mathbb{R}^m\times\mathbb{R}^n$. Thus it follows from (aiv) that there exists $\hat{\delta}>0$ such that

$$C\cdot(\|\Phi(P_x(\cdot),0,\lambda)\|_{C^1}\cdot\tilde{K} + R(\lambda)) \leqslant \tilde{K}$$

for all $|\lambda|\leqslant\hat{\delta}$. Therefore

$$|(S_\lambda\varphi)(x,y)| \leqslant \tilde{K}\cdot(|x|+|y|)^q\cdot|y|$$

for all (x,y) and $|\lambda|\leqslant\hat{\delta}$. The proof is completed. □

Now using Theorem 1.3 and Theorem 3.1 we can approximate the decoupling normalizing transformation

$$\tilde{x} = \bar{x} + \nu(\bar{x}, \bar{y} - h(\bar{x})),$$
$$\tilde{y} = \bar{y} - h(\bar{x}),$$

to any degree of accuracy, where $\nu(\bar{x}, 0) = 0$, $h(0) = 0$, $\mathrm{d}\nu(0,0) = 0$, $\mathrm{d}h(0) = 0$.

Consider more thoroughly the numerical procedure for the calculation of asymptotic series for ν. For simplicity we suppose that the coordinate transformation

$$x = \bar{x},$$
$$y = \bar{y} - h(\bar{x})$$

has been already applied. Thus we deal with the system (6). Then the function $\nu(x, y)$ satisfies the equation

$$\Lambda\nu = -\,\mathrm{d}\nu\Omega - \{\hat{\Phi}(x,y) - \hat{\Phi}(x+\nu, 0)\},$$

where

$$\Lambda\nu = \mathrm{ad}_A\,\nu + \frac{\partial\nu}{\partial y}By, \;\; \mathrm{ad}_A\,\nu = \frac{\partial\nu}{\partial x}Ax - A\nu$$

and

$$\Omega(x,y) = \left(\hat{\Phi}(x,y), \hat{\Psi}(x,y)\right)^T.$$

Let $y \cdot \wp^i$ be a linear space of vector fields whose coefficients are homogeneous polynomials of degree $i+1$ and for every $g \in y \cdot \wp^i$ we have $g(x,0) = 0\ \forall x \in \mathbb{R}^m$. Suppose further that we have the asymptotic series

$$\nu = \sum_{i=1}^{\infty} \nu_i,$$

$$\Omega = \sum_{i\geqslant 2} \Omega_i,$$

$$\hat{\Phi}(x,y) - \hat{\Phi}(x+\nu,0) = \sum_{i=1}^{\infty} [\Phi(x,y) - \Phi(x+\nu,0)]_{i+1},$$

where ν_i, $[\Phi(x,y) - \Phi(x+\nu,0)]_{i+1} \in y \cdot \wp^i$ and $\Omega_i \in \wp^i$, $\wp^i$ is a linear space of vector fields whose coefficients are homogeneous polynomials of degree i. Then we have to solve for $\{\nu_i\}_{i=1}^{\infty}$ the following linear equations in the linear spaces $\{y \cdot \wp^i\}_{i=1}^{\infty}$:

(25) $$\Lambda\nu_l = -\sum_{\substack{i+j=l+1\\ l\geqslant 1,\, j\geqslant 2}} \mathrm{d}\nu_i\Omega_j - [\Phi(x,y) - \Phi(x+\nu,0)]_{l+1} \quad (l = 1, 2, \ldots)$$

The solution $\{\nu_l\}_{l=1}^{\infty}$ exists and is unique. Namely, the following statement is true.

Proposition 3.1. *There exists* $\Lambda^{-1}: y \cdot \wp^i \to y \cdot \wp^i$ *and*

$$\Lambda^{-1}h = -\int_0^\infty e^{-A\tau} h(e^{A\tau}x, e^{B\tau}y)\, d\tau$$

for $h \in y \cdot \wp^i$ $(i = 1, 2, \ldots)$.

Proof. Suppose there exists $g \neq 0$, $g \in y \cdot \wp^i$, such that $\Lambda g = 0$. Then

$$\frac{d}{dt}\{e^{-At} g(e^{At}x, e^{Bt}y)\} = 0.$$

Thus

$$e^{-At} g(e^{At}x, e^{Bt}y) = g(x, y)$$

for $t \geqslant 0$. But $g \in y \cdot \wp^i$ and consequently

$$\lim_{t\to\infty} e^{-At} g(e^{At}x, e^{Bt}y) = 0.$$

Hence $g(x, y) = 0$. Thus $\Lambda g = 0$ implies $g = 0$. This yields the existence of Λ^{-1}. □

Example 3.1. Consider the polynomial system

$$\dot{x} = Ax + (V_{11}x + V_{12}y) \cdot \langle k, y \rangle\,,$$

$$\dot{y} = By + (V_{21}x + V_{22}y) \cdot \langle k, y \rangle\,,$$

where the eigenvalues of $A \in \mathbb{R}^{m\times m}$ have zero real parts, the eigenvalues of $B \in \mathbb{R}^{n\times n}$ have negative real parts, $V_{11} \in \mathbb{R}^{m\times m}$, $V_{12} \in \mathbb{R}^{m\times n}$, $V_{21} \in \mathbb{R}^{n\times m}$, $V_{22} \in \mathbb{R}^{n\times n}$ and $k \in \mathbb{R}^n$. Then for $l = 1$ the equation (25) has the form

$$\Lambda\nu_1 = -(V_{11}x + V_{12}y) \cdot \langle k, y \rangle\,.$$

Using Proposition 3.1, we obtain

$$\nu_1 = \int_0^\infty e^{-A\tau}(V_{11}e^{A\tau}x + V_{12}e^{B\tau}y) \cdot \langle k, e^{B\tau}y \rangle\, d\tau$$

and

$$\nu = \nu_1 + O((|x| + |y|)^2|y|).$$

4. ADDITIONAL SMOOTHNESS

Smoothness and/or real analyticity of the decoupling normalizing transformation is completely determined by smoothness and/or real analyticity of a center manifold. Consider the sequence

$$\xi_0 = -\int_{-\infty}^{0} e^{A\tau}(e^{-\tau f})^*[\Phi(x,y,\lambda) - \Phi(x,0,\lambda)]\,d\tau,\ \xi_1 = T_\lambda \xi_0, \ldots, \xi_j = T_\lambda \xi_{j-1}, \ldots,$$

where $\Phi(x,y,\lambda)$ and T_λ are defined in the proof of Theorem 2.1. Then $\{\xi_j\}_{j=0}^{\infty}$ are C^k functions whenever f is a C^k vector field and $|\lambda| < \delta$, where δ is a sufficiently small positive real value. It has been proved in Section 2 that $\lim_{i\to\infty} \xi_i = \nu$ in the Υ^{k-2} topology. Thus a restriction of ν to any closed ball in $\mathbb{R}^m \times \mathbb{R}^n$ is the limit of $\{\xi_i\}_{i=0}^{\infty}$ in the C^{k-2} topology. Moreover for sufficiently small δ and $|\lambda| < \delta$ the $(k-2)$nd derivatives of ν are uniformly Lipschitzian. Using this fact and the method of proof of Theorem 4.2 from [5], one can show that, for λ sufficiently small, ν is a C^k function on a closed ball in $\mathbb{R}^m \times \mathbb{R}^n$.

In general real analyticity of the vector field f does not imply the existence of a real analytic center manifold [8]. But if the function $h(x)$ from Theorem 1.1 and the vector field $\bar{f}$ are real analytic and moreover

$$A = -A^T,$$

then the decoupling normalizing transformation is also real analytic. To prove that one defines the norm

$$\|g\|_i = \sup_{|x|+|y|=1} \left|\frac{\partial}{\partial y} g(x,y)\right| \text{ on } y \cdot \wp^i.$$

If $A = -A^T$, then there exists a constant $K > 0$ such that

$$\|\Lambda^{-1}\|_i \leqslant K^i \ \forall i = 1,2,\ldots. \tag{22}$$

Thus using (21) one can show that

$$\|\nu_i\|_i \leqslant M^i, \tag{23}$$

where the constant $M > 0$. (23) means real analyticity of ν. The details of this scenario are quite laborious so we do not present them here. It is necessary only to note that the condition $A = -A^T$ is quite important. In general, for an arbitrary matrix A, whose eigenvalues have zero real parts, there exists no constant $K > 0$ for which (22) holds.

5. Local stabilization of nonlinear system with noncontrollable linearization

Here we continue the work begun in [1, 3]. Namely, we apply the results obtained above in order to investigate the local stabilization of the single-input nonlinear system

$$(26)\qquad \begin{aligned} \dot{\bar{x}} &= A\bar{x} + \bar{\Phi}(\bar{x},\bar{y}) + \bar{G}(\bar{x},\bar{y})\cdot u,\\ \dot{\bar{y}} &= B\bar{y} + \bar{\Psi}(\bar{x},\bar{y}) + (q + \bar{Q}(\bar{x},\bar{y}))\cdot u, \end{aligned}$$

where the control value $u \in \mathbb{R}$ and A, B, $\bar{\Phi}$, $\bar{\Psi}$ have been defined in (1),

$$\begin{aligned} \bar{G}&\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m,\\ \bar{Q}&\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \end{aligned}$$

are C^∞ function which vanish at the origin, i.e., $\bar{G}(0,0) = 0$, $\bar{Q}(0,0) = 0$.

Definition 5.1. The system (26) is said to be locally stabilizable at the origin iff there exists a C^2 feedback $u = w(x,y)$ which vanishes together with its derivatives at the origin (i.e., $w(0,0) = 0$, $\mathrm{d}w(0,0) = 0$), such that the zero solution of the closed loop system (the system (26) with $u = w(x,y)$) is asymptotically stable.

Due to Theorem 2.2 there exists a decoupling normalizing transformation (4) under which the system (26) has the form

$$(27)\qquad \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} &= A\tilde{x} + \bar{\Phi}(\tilde{x},h(\tilde{x})) + \tilde{G}(\tilde{x},\tilde{y})\cdot u,\\ \frac{\mathrm{d}}{\mathrm{d}t}\tilde{y} &= B\tilde{y} + \tilde{\Psi}(\tilde{x},\tilde{y}) + (q + \tilde{Q}(\tilde{x},\tilde{y}))\cdot u, \end{aligned}$$

where

$$\begin{aligned} \tilde{G}(\tilde{x},\tilde{y}) &= \bar{G}(\bar{x},\bar{y}) + \frac{\partial}{\partial z}\nu(z,\bar{y}-h(\bar{x}))|_{z=\bar{x}}\bar{G}(\bar{x},\bar{y})\\ &\quad + \frac{\partial}{\partial \bar{y}}\nu(\bar{x},\xi)|_{\xi=\bar{y}-h(\bar{x})}(q + \bar{Q}(\bar{x},\bar{y}) - \frac{\partial}{\partial \bar{x}}h(\bar{x})\bar{G}(\bar{x},\bar{y})),\\ \tilde{Q}(\tilde{x},\tilde{y}) &= \bar{Q}(\bar{x},\bar{y}) - \frac{\partial}{\partial \bar{x}}h(\bar{x})\bar{G}(\bar{x},\bar{y}) \end{aligned}$$

and $(\tilde{x},\tilde{y})$, $(\bar{x},\bar{y})$ are connected by the decoupling normalizing transformation (4). It is easy to see that $\tilde{y} = 0$ yields $\tilde{x} = \bar{x}$ and $\bar{y} = h(\bar{x})$. Thus

$$\tilde{G}(\tilde{x},0) = \bar{G}(\tilde{x},h(\tilde{x})) + \frac{\partial}{\partial \bar{y}}\nu(\tilde{x},0)\Big(q + \bar{Q}(\tilde{x},h(\tilde{x})) - \frac{\partial}{\partial \bar{x}}h(\tilde{x})\bar{G}(\tilde{x},h(\tilde{x}))\Big).$$

The next theorem gives us some sufficient conditions for local stabilizability of the nonlinear system (26).

Theorem 5.1. *Let the system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})) \tag{28}$$

be stable, let $V(\tilde{x})$ be its C^∞ weak Liapunov's function, i.e., there exists $\delta > 0$ such that $V(\tilde{x}) > 0$ for all $0 < |\tilde{x}| < \delta$, $V(0) = 0$, and $\langle \mathrm{d}V(\tilde{x}), A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x}))\rangle \leqslant 0 \ \forall |\tilde{x}| < \delta$. Suppose further that for every complete trajectory $\tilde{x}(t, \tilde{x}(0)) = \{\tilde{x}(t); |\tilde{x}(0)| < \delta, 0 \leqslant t < \infty\}$ of (26) which satisfies

$$\langle \mathrm{d}V(\tilde{x}(t)), \tilde{G}(\tilde{x}(t), 0)\rangle = 0 \quad \forall t \geqslant 0 \tag{29}$$

we have $x(t) = 0$. Then the system (27) is locally stabilizable at the origin by the feedback $u = -\langle \mathrm{d}V(\tilde{x}), \tilde{G}(\tilde{x}, \tilde{y})\rangle$.

Proof. According to Theorem 1.1 the system (27) with $u = -\langle \mathrm{d}V(\tilde{x}), \tilde{G}(\tilde{x}, \tilde{y})\rangle$ has a center manifold $\tilde{y} = H(\tilde{x})$. Then due to Theorem 1.2 (and/or Theorem 2.2) the zero solution of the closed loop system is asymptotically stable iff the zero solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})) - \tilde{G}(\tilde{x}, H(\tilde{x}))\langle \mathrm{d}V(\tilde{x}), \tilde{G}(\tilde{x}, H(\tilde{x}))\rangle \tag{30}$$

is asymptotically stable. If there exists $\delta > 0$ such that $\lim_{t\to\infty} \tilde{x}(t, x^*) = 0 \ \forall |x^*| < \delta$, where $\tilde{x}(t, x^*)$ is the solution of (30) generated by the initial conditions $\tilde{x}(0, x^*) = x^*$, then the proof is completed. Otherwise for every $\delta > 0$ one can find $0 < |x^*| < \delta$ such that $\lim_{t\to\infty} \tilde{x}(t, x^*) \neq 0$ and $\tilde{x}(t, x^*)$ satisfies

$$\langle \mathrm{d}V(\tilde{x}(t, x^*)), \tilde{G}(\tilde{x}(t, x^*), H(\tilde{x}(t, x^*)))\rangle = 0 \quad \forall t \geqslant 0.$$

But $(\tilde{x}(t, x^*), H(\tilde{x}(t, x^*)))$ is a solution of the system (27) with $u = 0$. Hence, due to the stability of the zero solution of (28), $\lim_{t\to\infty} H(\tilde{x}(t, x^*)) = 0$. Thus there exists a nontrivial trajectory of (28) which satisfies (29). This contradicts the conditions of the theorem. The proof is completed. □

Using the sufficient conditions of stabilization obtained in [7] we can formulate the following corollary of Theorem 5.1.

Corollary 1. *Let $\bar{\Phi}(\tilde{x}, h(\tilde{x})) = 0$, $A^T = -A$, let $\tilde{G}(\tilde{x}, 0)$ be a C^∞ function and for δ sufficiently small let*

$$\operatorname{rank}\{\operatorname{ad}_A^i \tilde{G}(\tilde{x}, 0)\}_{i=0}^{\infty} = m, \ \forall\, 0 < |\tilde{x}| < \delta$$

where $\mathrm{ad}_A^0 \tilde{G}(\tilde{x},0) = \tilde{G}(\tilde{x},0)$, $\mathrm{ad}_A \tilde{G}(\tilde{x},0) = \frac{\partial}{\partial \tilde{x}}\tilde{G}(\tilde{x},0)A\tilde{x} - A\tilde{G}(\tilde{x},0)$ *and* $\mathrm{ad}_A^i \tilde{G}(\tilde{x},0) = \mathrm{ad}_A(\mathrm{ad}_A^{i-1} \tilde{G}(\tilde{x},0))$. *Then the system* (27) *is locally stabilizable at the origin by the feedback* $u = -\langle \tilde{x}, \tilde{G}(\tilde{x},\tilde{y})\rangle$.

Other corollaries of Theorem 5.1 can be formulated with the help of the sufficient conditions of stabilization obtained in [9, 10].

The next theorem follows from the sufficient conditions of stability of homogeneous polynomial systems [2].

Theorem 5.2. *Let* $A = -A^T$,

$$\bar{\Phi}(\tilde{x}, h(\tilde{x})) = \bar{\Phi}_\theta(\tilde{x}) + O(|\tilde{x}|^{\theta+1}),$$
$$\tilde{G}(\tilde{x},\tilde{y}) = \tilde{G}_\eta(\tilde{x},\tilde{y}) + O((|\tilde{x}| + |\tilde{y}|)^{\eta+1}),$$

where $\bar{\Phi}_\theta \in \wp^\theta$, $\tilde{G}_\eta \in \wp^\eta$ *and* $\wp^\theta$, $\wp^\eta$ *are defined in Section* 3. *Suppose further* $\theta \geqslant 2\eta + 1$ *and*

$$\{\tilde{x} \in S^{m-1};\ \langle \tilde{x}, \tilde{G}_\eta(\tilde{x},0)\rangle = 0\} \subset \{x \in S^{m-1};\ \langle x, \bar{\Phi}_\theta(\tilde{x})\rangle < 0\},$$

where S^{m-1} *is the* $(m-1)$*-dimensional unit sphere. Then there exists* $\gamma > 0$ *such that the feedback*

$$u(\tilde{x}) = -\gamma\langle \tilde{x}, \tilde{G}_\eta(\tilde{x},0)\rangle |\tilde{x}|^{\theta-2\eta-1}$$

stabilizes the system (27).

Proof. Consider the system (27) closed by $u(\tilde{x}) = -\gamma\langle \tilde{x}, \tilde{G}_\eta(\tilde{x},0)\rangle |\tilde{x}|^{\theta-2\eta-1}$. Having applied Theorem 1.1 we obtain the existence of the center manifold $y = H(\tilde{x})$ for the closed loop system. Hence the feedback stabilizes the system (27), iff the zero solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = A\tilde{x} + \bar{\Phi}(\tilde{x}, h(\tilde{x})) - \tilde{G}(\tilde{x}, H(\tilde{x})) \cdot \gamma\langle \tilde{x}, \tilde{G}_\eta(\tilde{x},0)\rangle |\tilde{x}|^{\theta-2\eta-1}$$

is asymptotically stable. Take Liapunov's function $V(\tilde{x}) = \frac{1}{2}|\tilde{x}|^2$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\tilde{x}) = \langle \tilde{x}, \bar{\Phi}_\theta(\tilde{x})\rangle - \gamma\big(\langle \tilde{x}, \tilde{G}_\eta(\tilde{x},0)\rangle\big)^2 \cdot |\tilde{x}|^{\theta-2\eta-1} + O(|\tilde{x}|^{\theta+2}). \tag{31}$$

According to the result of [2], there exists $\gamma > 0$ such that

$$\langle \tilde{x}, \bar{\Phi}_\theta(\tilde{x})\rangle < \gamma\big(\langle \tilde{x}, \tilde{G}_\eta(\tilde{x},0)\rangle\big)^2 |\tilde{x}|^{\theta-2\eta-1}\ \forall \tilde{x} \neq 0.$$

Thus the statement of the theorem follows from (31). □

Now we formulate sufficient conditions for local stabilizability of the bilinear system

$$\begin{aligned} \dot{x} &= Ax + (V_{11}x + V_{12}y)v, \\ \dot{y} &= \overline{B}y + (q + V_{21}x + V_{22}y)v, \end{aligned} \tag{32}$$

where the control value $v \in \mathbb{R}$, $q \in \mathbb{R}^n$, the system

$$\dot{y} = \overline{B}y + q \cdot v$$

is stabilizable and $A, \{V_{ij}\}_{i,j=1}^2$ are defined in Example 3.1.

We will design the stabilizing feedback in the form

$$v = \langle k, y \rangle + u(x, y) \tag{33}$$

with $u(0,0) = 0$, $\mathrm{d}u(0,0) = 0$ and with $k \in \mathbb{R}^n$ such that all eigenvalues of $B = \overline{B} + q \cdot k$ have negative real parts.

Substituting (33) in (32) we obtain

$$\begin{aligned} \dot{x} &= Ax + (V_{11}x + V_{12}y) \cdot \langle k, y \rangle + (V_{11}x + V_{12}y) \cdot u, \\ \dot{y} &= By + (V_{21}x + V_{22}y) \cdot \langle k, y \rangle + (q + V_{21}x + V_{22}y) \cdot u. \end{aligned} \tag{34}$$

Theorem 5.3. *If $A = -A^T$ and*

$$\langle x, V_{11}x \rangle + \int_0^\infty \langle \mathrm{e}^{A\tau}x, V_{11}\mathrm{e}^{A\tau}x \rangle \langle k, \mathrm{e}^{B\tau}q \rangle \, \mathrm{d}\tau = 0 \tag{35}$$

implies $x = 0$, then the system (32) is stabilized by the feedback

$$v = \langle k, y \rangle - \langle x, V_{11}x \rangle - \int_0^\infty \langle \mathrm{e}^{A\tau}x, V_{11}\mathrm{e}^{A\tau}x \rangle \langle k, \mathrm{e}^{B\tau}q \rangle \, \mathrm{d}\tau. \tag{36}$$

Proof. It is easy to see that for the system (34) with $u = 0$ we have $h(x) = 0$ and $\overline{\Phi}(x, h(x)) = 0$. The decoupling normalizing transformation is of the form

$$\begin{aligned} \tilde{x} &= x + \nu(x, y), \\ \tilde{y} &= y, \end{aligned} \tag{37}$$

where

$$\nu = \int_0^\infty \mathrm{e}^{-A\tau}(V_{11}\mathrm{e}^{A\tau}x + V_{12}\mathrm{e}^{b\tau}y)\langle k, \mathrm{e}^{B\tau}y \rangle \, \mathrm{d}\tau + O((|x| + |y|^2) \cdot |y|)$$

as was calculated in Example 3.1.

Under the normalizing transformation (37) the system (34) has the form

$$\dot{\tilde{x}} = A\tilde{x} + \tilde{G}(\tilde{x}, \tilde{y}) \cdot u, \qquad \dot{\tilde{y}} = B\tilde{y} + \tilde{\Psi}(\tilde{x}, \tilde{y}) + (q + \tilde{Q}(\tilde{x}, \tilde{y})) \cdot u, \tag{38}$$

where $\tilde{\Psi}$, $\tilde{Q}$ are analogous to the corresponding functions in (27).

Consider the system (38) closed by

$$u(x) = -\langle x, V_{11}x\rangle - \int_0^\infty \langle e^{A\tau}x, V_{11}e^{A\tau}x\rangle\langle k, e^{B\tau}q\rangle \,d\tau \tag{39}$$

where (x, y) and $(\tilde{x}, \tilde{y})$ are connected by the transformation (37). Then using Theorem 1.1 we obtain for the system (38) closed by (39) the center manifold $\tilde{y} = H(\tilde{x})$. Hence to prove the theorem we need to investigate the local behavior of the system

$$\frac{d}{dt}\tilde{x} = A\tilde{x} + \tilde{G}(\tilde{x}, H(\tilde{x})) \cdot u(x), \tag{40}$$

where $x = \tilde{x} - \nu(x, H(\tilde{x}))$. Take Liapunov's function $V(\tilde{x}) = \frac{1}{2}|\tilde{x}|^2$. Then

$$\frac{d}{dt}V(\tilde{x}) = \big\langle \tilde{x}, \tilde{G}(\tilde{x}, H(\tilde{x}))\big\rangle \cdot u(x).$$

However,

$$\big\langle \tilde{x}, \tilde{G}(\tilde{x}, H(\tilde{x}))\big\rangle = \langle \tilde{x}, V_{11}\tilde{x}\rangle + \int_0^\infty \langle e^{A\tau}\tilde{x}, V_{11}e^{A\tau}\tilde{x}\rangle \cdot \langle k, e^{B\tau}q\rangle \,d\tau + O(|\tilde{x}|^3),$$

$$u(x) = -\langle \tilde{x}, V_{11}\tilde{x}\rangle - \int_0^\infty \langle e^{A\tau}\tilde{x}, V_{11}e^{A\tau}\tilde{x}\rangle\langle k, e^{B\tau}q\rangle \,d\tau + O(|\tilde{x}|^3).$$

Therefore

$$\frac{d}{dt}V(\tilde{x}) = -\Big(\langle \tilde{x}, V_{11}\tilde{x}\rangle + \int_0^\infty \langle e^{A\tau}\tilde{x}, V_{11}e^{A\tau}\tilde{x}\rangle\langle k, e^{B\tau}q\rangle \,d\tau\Big)^2 + O(|\tilde{x}|^5)$$

and due to the condition (35) this means asymptotic stability of the zero solution of (40). Hence the zero solution of the system (32) which is closed by the feedback (36) is also asymptotically stable. □

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