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ISOTOPIC INVARIANTS OF NATURAL PLANAR TERNARY RINGS

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Summary. In the paper the invariant (geometrical) character of some properties of natural planar ternary rings is shown by using isotopic transformations.

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1. Natural planar ternary rings

By a natural planar ternary ring (abb. NPTR) we mean an ordered pair (M, t) consisting of a set M having at least two different elements and of ternary operation t on M fulfilling the following axioms:

(A) $\forall x, y, m \in M \ \ \exists! b \in M \colon t(x, m, b) = y$,

(B) $\forall m, \bar{m}, b, \bar{b} \in M, m \neq \bar{m} \ \ \exists! x \in M \colon t(x, m, b) = t(x, \bar{m}, \bar{b})$,

(C) $\forall x, \bar{x}, y, \bar{y} \in M, x \neq \bar{x} \ \ \exists!(m, b) \in M \times M \colon t(x, m, b) = y \ \wedge \ t(\bar{x}, m, b) = \bar{y}$,

(D) there exist elements $0_L, 0_R \in M$ and a regular transformation $b \mapsto b^*$ of M (permutation of M) such that
$\forall m, b \in M \colon t(0_L, m, b^*) = b$,
$\forall x, b \in M \colon t(x, 0_R, b^*) = b$.

The element $0_L (0_R)$ will be called the left quasizero (the right quasizero) of (M, t), the regular transformation $b \mapsto b^*$ is the characteristic of (M, t).

Theorem 1.1. *Any NPTR (M, t) has exactly one left and exactly one right quasizero.*

P r o o f. 1. Let 0_L and $\bar{0}_L$ be two distinct left quasizeros of (M,t). Let m, $\overline{m}$, r be elements of M, $m \neq \overline{m}$. Then we have $t(0_L, m, r) = t(0_L, \overline{m}, r)$ and simultaneously $t(\bar{0}_L, m, r) = t(\bar{0}_L, \overline{m}, r)$, a contradiction with the axiom (B).

2. Let 0_R abd $\bar{0}_R$ be two different right quasizeros of (M,t). Let us choose an arbitrary element $q \in M$. Then there exists a unique $x \in M$ such that

$$t(x, 0_R, q) = t(x, \bar{0}_R, q). \tag{1}$$

For any $\bar{x} \in M$, especially for any $\bar{x} \neq x$, we have $t(\bar{x}, 0_R, q) = t(x, 0_R, q)$, $t(x, \bar{0}_R, q) = t(\bar{x}, \bar{0}_R, q)$, hence

$$t(\bar{x}, 0_R, q) = t(\bar{x}, \bar{0}_R, q). \tag{2}$$

The simultaneous validity of (1) and (2) contradicts the axiom (C). □

2. Multiplication and addition in NPTR

For any $a, b \in M$ let us put

$$a \cdot b = t(a, b, 0_L^*). \tag{3}$$

Then we get a binary operation $(a,b) \mapsto a \cdot b$ on M, the so called multiplication in NPTR (M,t). The element $a \cdot b$ is the product of a, b. The multiplication in (M,t) has the following properties:

(a) $\forall a, b \in M, a \neq 0_L \ \ \exists! x \in M : a \cdot x = b$,

(b) $\forall a, b \in M, a \neq 0_R \ \ \exists! y \in M : y \cdot a = b$,

(c) $\forall a, b \in M : a \cdot b = 0_L \Leftrightarrow a = 0_L \vee b = 0_R$.

It follows from (a) that for any element $x \in M$ different from 0_L there exists a uniquely determined $e_x \in M$ such that

$$x \cdot e_x = x.$$

Further, let us put $e_x = 0_R$ if $x = 0_L$. Putting for any $a, b \in M$

$$a + b = t(a, e_a, b^*)$$

we get another binary operation $(a,b) \mapsto a + b$ on M, the addition in NPTR (M,t). The element $a + b$ is the sum of a, b. The addition in (M,t) has the following properties:

(d) $\forall a \in M: a + 0_L = 0_L + a = a$,

(e) $\forall a, b \in M\ \exists! x \in M: a + x = b$.

We will denote by $-a$ the element fulfilling $a + (-a) = 0_L$, and we put $a - b = a + (-b)$.

3. Special cases of NPTR

Replacing the axiom (D) of NPTR by

(E) there exists $0 \in M$ such that
$\forall m, b \in M: t(0, m, b) = b$,
$\forall x, b \in M: t(x, 0, b) = b$

we get the axiomatical definition of a planar ternary ring with zero (abb. ZPTR). The zero is just the element 0 from the axiom (E). Evidently, any ZPTR (M, t) is a NPTR with both the left and right quasizero equal to 0. Its characteristic is the identical transformation of M.

Let (M, t) be a ZPTR. If (M, t) fulfils also

(F) there exists $e \in M$ such that
$\forall u \in M: t(e, u, 0) = t(u, e, 0) = u$,

then (M, t) is the Hall planar ternary ring (abb. HPTR). The element e from (F) is the unit element of (M, t). In this case, for any $x \in M, x \neq 0$ we have $e_x = e$.

4. Isotopy of planar ternary rings

If we delete from the axioms of NPTR the axiom (D) we get the definition a more general structure, a planar ternary ring (abb. PTR).

Let (M, t) and (M, T) be two planar ternary rings with the same support M. The ordered quadruple

$$(\alpha, \beta, \gamma, \delta) \tag{4}$$

of regular transformations of M will be called the isotopic transformation (isotopy) of PTR (M, t) onto PTR (M, T) if

$$\forall x, y, m, b \in M: y = T(x, m, b) \Leftrightarrow \delta(y) = t\big(\alpha(x), \beta(m), \gamma(b)\big) \tag{5}$$

or equivalently

$$\forall x, m, b \in M: \delta(T(x, m, b)) = t\big(\alpha(x), \beta(m), \gamma(b)\big). \tag{6}$$

Theorem 4.1. *Let (M,t) be a PTR and let (4) be quadruple of regular transformations of M. If we introduce a ternary operation T on M by (5), then (M,T) is also a PTR and (4) is an isotropy of (M,t) onto (M,T).*

The proof is obvious.

Theorem 4.2. *Let (4) be an isotopic transformation of the PTR (M,T) onto PTR (M,T). If (M,t) is an NPTR then (M,T) is an NPTR, too. Moreover, let 0_L and 0_R be the elements of M for which $\alpha(0_L)$ and $\beta(0_R)$ are respectively left and right quasizeros of (M,t). Then 0_L and 0_R are the left and right quasizeros of (M,T), respectively.*

Proof. Let $b \mapsto b^*$ be the characteristic of (M,t). Let us define a permutation $b \mapsto b^\times$ of M by

$$\forall b \in M : \gamma(b^\times) = \delta(b)^*. \tag{7}$$

Now, for any $m, b \in M$ we have $\delta\big(T(0_L, m, b^\times)\big) = t\big(\alpha(0_L), \beta(m), \gamma(b^\times)\big) = t\big(\alpha(o_L), \beta(m), \delta(b)^*\big) = \delta(b)$, hence $T(0_L, m, b^\times) = b$. Similarly for any $x, b \in M$ we obtain $\delta\big(T(x, 0_R, b^\times)\big) = t\big(\alpha(x), \beta(0_R), \gamma(b^\times)\big) = t\big(\alpha(x), \beta(0_R), \delta(b)^*\big) = \delta(b)$, therefore $T(x, 0_R, b^\times) = b$. □

Corollary 4.3. *The class of natural planar ternary rings is closed with respect to isotopic transformations.*

Theorem 4.4. *Let (M,t) be an NPTR. Then there exists an HPTR (M,T) isotopic to (M,t).*

Proof. Let we denote, as usual, by 0_L and 0_R the left and right quasizeros of (M,t). Let us choose an element e of M different from 0_L. Let f be the element of M for which

$$e \cdot f = e$$

is true. Let us define the quadruple (4) by conditions

(i) $\forall x \in M : x = \alpha(x) \cdot f$,
(ii) $\forall m \in M : m = e \cdot \beta(m)$,
(iii) $\forall b \in M : b^* = \gamma(b)$,
(iv) $\forall y \in M : y = \delta(y)$.

If we introduce a new ternary operation T on M by of (5) we obtain that (M,T) is an HPTR with the zero $0 = 0_L$ and the unity element e. □

Corollary 4.5. *Any class mutually isotopic NPTR contains at least one HPTR. The PTR (M,t) is an NPTR if and only if it is isotopic to some HPTR.*

5. Important types of NPTR

Let a, b be two elements of a given NPTR (M,t). If $a \neq 0_L$ then the uniquely determined element $x \in M$ satisfying the relation $a \cdot x = b$ will be denoted by $x = a|b$. Now, we will say that (M,t) is

(a) additively associative, if
$\forall a,b,c \in M\colon a + (b + c) = (a + b) + c$;

(b) linear, if
$\forall a,b,c \in M\colon t(a,b,c^*) = a \cdot b + c$;

(c) right distributive, if for any $a,b,c \in M$ the equation
$a \cdot m + b \cdot m = c \cdot m$
either has only the trivial solution ($m = 0_R$) or is fulfilled identically;

(d) left distributive, if for any $a,b,c \in M$ the equation
$m \cdot a + m \cdot b = m \cdot c$
either has only the trivial solution ($m = 0_L$) or is fulfilled identically.

(e) associative, if for any $x, \bar{x}, u, \bar{u} \in M$ different from 0_L the equation
$x \mid (u \cdot m) = \bar{x} \mid (\bar{u} \cdot m)$
either has only the trivial solution ($m = 0_R$) or is fulfilled identically.

(f) commutative, if
$\forall a,b,c,d \in M,\ c \neq 0_L\colon a \cdot (c \mid (b \cdot d)) = b \cdot (c \mid (a \cdot d))$.

Lemma 5.1. *Let (M,t) be an additively associative NPTR. Then $(M,+)$ is a group.*

P r o o f. The associative groupoid $(M,+)$ has the neutral element 0_L. Moreover, for any $a \in M$ there exists a unique $-a \in M$ such that $a + (-a) = 0_L$. Let x be the element of M for which $(-a) + x = 0_L$ is true. Then $x = 0_L + x = (a + (-a)) + x = a + ((-a) + x) = a + 0_L = a$. □

Lemma 5.2. *Let (M,t) be an additively associative, linear and right distributive NPTR. Then the group $(M,+)$ is abelian.*

P r o o f. Let us suppose that there exist elements $a,b \in M$ for which $a+b \neq b+a$. Then obviously $a \neq 0_L$ and consequently there exists exactly one $x \in M$ such that

$$a \cdot x = b + a + (-b). \tag{8}$$

Moreover, $x \neq e_a$. Now, we again have a unique $y \in M$ for which

$$y \cdot x + b = y \cdot e_a. \tag{9}$$

(In fact, (9) is equivalent to $t(y, x, b^*) = t(y, e_a, 0_L^*)$.) Finally, there exists a unique $z \in M$ fulfilling

$$z \cdot e_a = y \cdot e_a + a \cdot e_a. \tag{10}$$

As (M, t) is right distributive, (10) gives

$$z \cdot x = y \cdot x + a \cdot x, \tag{11}$$

henceforth $z \cdot x + b = y \cdot x + a \cdot x + b = y \cdot x + b + a + (-b) + b = y \cdot x + b + a = y \cdot e_a + a \cdot e_a = z \cdot e_a$. Thus we get

$$z \cdot x + b = z \cdot e_a. \tag{12}$$

This means that the equation (9) also has the solution z. We have $y = z$ and according to (10) $a = 0_L$, a contradiction. □

Lemma 5.3. *Let (M, t) be an additively associative and left distributive NPTR. Then*

$$\forall a, b \in M, \forall c \in M, c \neq 0_L \colon a \cdot (a \mid (-b)) = -(a \cdot (c \mid b)). \tag{13}$$

Proof. Putting $k = c \mid (-b)$, $h = c \mid b$, we get

$$c \cdot k + c \cdot h = c \cdot 0_R,$$

hence

$$a \cdot k + a \cdot h = a \cdot 0_R,$$

which yields (13). □

6. Isotopic invariance

Let us consider two NPTR (M,t) and (M,T) with left quasizeros $\bar{0}_L$, 0_L and with right quasizeros, $\bar{0}_R$, 0_R, respectively. Let $+$ and $\cdot$ denote the addition and multiplication in (M,t), let $\oplus$ and $\circ$ mean the addition and multiplication in (M,T). Let $-a$ and $\ominus a$ denote the opposite elements to a in the group $(M,+)$ and $(M,\oplus)$, respectively. Finally, let a, b be two elements of M. If $a \neq \bar{0}_L$, then the element $x \in M$ fulfilling $a \cdot x = b$ will be denoted (as above) $x = a \,|\, b$. If $a \neq 0_L$, the element y for which $a \cdot \circ y = b$ is true will be denoted $y = a \,\big|\, b$. Let $b \mapsto b^*$, $b \mapsto b^\times$ denote the characteristics of (M,t) and (M,T), respectively.

Let us suppose that there exists an isotopic transformation (4) of (M,t) onto (M,T). It follows from 4.2 that $\alpha(0_L) = \bar{0}_L$, $\beta(0_R) = \bar{0}_R$. Moreover, the relation (7) is true.

Theorem 6.1. *If NPTR (M,t) is additively associative and linear, then (M,T) is additively associative and linear, too. Further, we have*

(14a) $$\forall x,m,b \in M\colon \delta(T(x,m,b^\times)) = \alpha(x)\cdot\beta(m) + \delta(b),$$

(14b) $$\forall x,m \in M\colon \delta(x \circ m) = \alpha(x)\cdot\beta(m) + \delta(0_L),$$

(14c) $$\forall x,b \in M\colon \delta(x \oplus b) = \delta(x) - \delta(0_L) + \delta(b).$$

Proof. $\delta\big(T(x,m,b^\times)\big) = t\big(\alpha(x),\beta(m),\delta(b)^*\big) = \alpha(x)\cdot\beta(m) + \delta(b)$. Putting $b = 0_L$ in (14 a) we obtain (14 b). Let e_x denote the element for which $x \circ e_x = x$ if $x \neq 0_L$ and $e_x = 0_R$ if $x = 0_L$. Then $\delta(x \oplus b) = \delta\big(T(x,e_x,b^\times)\big) = \alpha(x)\cdot\beta(e_x) + \delta(b) = \delta(x \circ e_x) - \delta(0_L) + \delta(b) = \delta(x) - \delta(0_L) + \delta(b)$.

Using (14 c) repeatedly we get without trouble that

$$\forall a,b,c \in M\colon \delta\big(a \oplus (b \oplus c)\big) = \delta\big((a \oplus b) \oplus c\big),$$

therefore

$$\forall a,b,c \in M\colon a \oplus (b \oplus c) = (a \oplus b) \oplus c.$$

Finally, let $x,m,b \in M$. Then $\delta(T(x,m,b^\times)) = \alpha(x)\cdot\beta(m) + \delta(b) = \delta(x \circ m) - \delta(0_L) + \delta(b) = \delta(x \circ m \oplus b)$. Hence $\forall x,m,b \in M\colon T(x,m,b^\times) = x \circ m \oplus b$. □

Combining formulas (14 b) and (14 c) we obtain

(15a) $$\forall a, b, c, d \in M\colon a \circ b = c \circ d \Leftrightarrow \alpha(a) \cdot \beta(b) = \alpha(c) \cdot \beta(d),$$

(15b) $$\forall a, b, c, d, u, v \in M\colon a \circ b = c \circ d \oplus u \circ v \Leftrightarrow \alpha(a) \cdot \beta(b) = \alpha(c) \cdot \beta(d) + \alpha(u) \cdot \beta(v),$$

Theorem 6.2. *Let NPTR (M, t) be additively associative and linear (so that (M, T) is also additively associative and linear). If (M, t) is right (left) distributive then (M, T) is right (left) distributive, too.*

Proof. Let $a, b, c \in M$ be given, let m be another element of M. Then

$$a \circ m \oplus b \circ m = c \circ m$$

if and only if

$$\alpha(a) \cdot \beta(m) + \alpha(b) \cdot \beta(m) = \alpha(c) \cdot \beta(m).$$

As α and β are regular transformations of M and $m = 0_R$ if and only if $\beta(m) = \bar{0}_R$ then the right distributivity of (M, t) obviously implies the right distributivity of (M, T). For the left distributivity the proof is quite analogous. □

Theorem 6.3. *Let NPTR (M, t) be additively associative, linear, right and left distributive (so that (M, T) is also additively associative, linear, right and left distributive). If (M, t) is associative, then (M, T) is associative, too.*

Proof. Let x, $\bar{x}$, u, $\bar{u}$ be elements of M all different from 0_L. It satisfies to prove that for any $m \in M$ the relation

(18) $$x \mid (u \circ m) = \bar{x} \mid (\bar{u} \circ m)$$

is equivalent to the relation

(19) $$\alpha(x) \mid (\alpha(u) \cdot \beta(m)) = \alpha(\bar{x}) \mid (\alpha(\bar{u}) \cdot \beta(m)).$$

Let us suppose that (18) is true. Putting $w = x \mid (u \circ m)$ we get

(20) $$x \circ w = u \circ m \quad \text{and} \quad \bar{x} \circ w = \bar{u} \circ m.$$

According to (15a) we have

(21) $$\alpha(x) \cdot \beta(w) = \alpha(u) \cdot \beta(m) \quad \text{and} \quad \alpha(\bar{x}) \cdot \beta(w) = \alpha(\bar{u}) \cdot \beta(m).$$

It follows from (21) that both sides of (19) are equal to $\beta(w)$, which means that (19) is true. Conversely, let (19) be true. Then there exists exactly one $w \in M$ such that $\beta(w) = \alpha(x) \mid (\alpha(u) \cdot \beta(m))$. Now we obtain (21), hence (20) and consequently (18). □

Theorem 6.4. *Let NPTR (M,t) be additively associative, linear, right and left distributive and associative (so that (M,T) is also additively associative, linear, right and left distributive and associative). If (M,t) is commutative, then (M,T) is commutative, too.*

Proof. Let a, b, c, d be elements of M, $c \neq 0_L$ ($\Rightarrow \alpha(c) \neq \bar{0}_L$). We have uniquely determined elements $u, w \in M$ fulfilling

$$\alpha(c) \cdot \beta(u) = \alpha(b) \cdot \beta(d), \quad \alpha(c) \cdot \beta(w) = \alpha(a) \cdot \beta(d). \tag{21}$$

As (M,t) is commutative we have

$$\alpha(a) \cdot (\alpha(c) \mid (\alpha(b) \cdot \beta(d))) = \alpha(b) \cdot (\alpha(c) \mid (\alpha(a) \cdot \beta(d))). \tag{22}$$

Determining $\beta(u)$ and $\beta(w)$ from (21) and substituting into (22) we obtain

$$\alpha(a) \cdot \beta(u) = \alpha(b) \cdot \beta(w). \tag{23}$$

According to (15a), (21) and (23) give

$$c \circ u = b \circ d, \quad c \circ w = a \circ d, \tag{24}$$

$$a \circ u = b \circ w. \tag{25}$$

Now, determining u and w from (24) and substituting into (25) we get

$$a \circ (c \,|\, (b \circ d)) = b \circ (c \,|\, (a \circ d)).$$

□

7. The case of HPTR

In this section we will show that the significant properties (a)–(e) of NPTR introduced in Section 5 have the usual meaning if the considered NPTR (M,t) is Hall PTR.

Evidently, we have

Theorem 7.1. *A HPTR (M,t) is additively associative and linear if and only if*
(i) $\forall a,b,c \in M\colon a + (b+c) = (a+b) + c$,

(ii) $\forall a, b, c \in M\colon t(a, b, c) = a \cdot b + c$.

Theorem 7.2. *A HPTR (M, t) is right or left distributive if and only if the right or left distributive law is true, i.e. if*
(iii) $\forall a, b, m \in M\colon (a + b) \cdot m = a \cdot m + b \cdot m$, *or*
(iv) $\forall a, b, m \in M\colon m \cdot (a + b) = m \cdot a + m \cdot b$, *respectively.*

Proof (for the right distributivity only). Let the right distributive law (iii) in the HPTR (M, t) be true. Then $a \cdot m + b \cdot m = c \cdot m$ gives $(a + b) \cdot m = c \cdot m$. If $m \neq 0$ we have $c = a + b$, consequently $a \cdot m + b \cdot m = c \cdot m$ for any $m \in M$. Conversely, let (M, t) be a right distributive HPTR. Then $a \cdot m + b \cdot m = (a + b) \cdot m$ is fulfilled for $m = e$. Hence the same relation holds for any $m \in M$. □

Theorem 7.3. *A HPTR (M, t) is associative if and only if the associative law for multiplication holds, i.e.*
(v) $\forall a, b, m \in M\colon a \cdot (b \cdot m) = (a \cdot b) \cdot m$.

Proof. Let the associative law (v) for multiplication in the HPTR (M, t) be true. Let nonzero elements u, $\bar{u}$, x, $\bar{x}$ from M be given. Then the equation $x \,|\, (u \cdot m) = \bar{x} \,|\, (\bar{u} \cdot m)$ may be rewritten in the form

$$(x^{-1} \cdot u) \cdot m = (\bar{x}^{-1} \cdot \bar{u}) \cdot m. \tag{26}$$

If (26) is fulfilled for some $m \neq 0$, then (26) as well as the original equation is fulfilled identically.

Conversely, let (M, t) be an associative HPTR. We may assume that the given elements $a, b \in M$ are different from zero. The equation $a \,|\, ((a \cdot b) \cdot m) = e \,|\, (b \cdot m)$ (whose right hand side equals $b \cdot m$) has the nontrivial solution $m = e$. Consequently, it is fulfilled identically, (v) is true. □

Theorem 7.4. *An associative HPTR (M, t) is commutative if and only if the commutative law for multiplication is valid, i.e.*
(vi) $\forall a, b \in M\colon a \cdot b = b \cdot a$.

Proof. As the HPTR (M, t) is associative we have $(M^*, \cdot)$, where M^* is the set of nonzero elements of M, is a group. Now, let the commutative law (vi) for multiplication be true. Let $a, b, c, \in M$ be given, let $c \neq 0$. Then $a \cdot \big(c \,|\, (b \cdot d)\big) = a \cdot c^{-1} \cdot b \cdot d = b \cdot c^{-1} \cdot a \cdot d = b \cdot \big(c \,|\, (a \cdot d)\big)$.

Conversely, let (M, t) be commutative. Then

$$a \cdot b = a \cdot \big(e \,|\, (b \cdot e)\big) = b \cdot \big(e \,|\, (a \cdot e)\big) = b \cdot a.$$

□

8. Geometrical meaning of significant properties

The process of the coordinatization of a given projective plane P by an NPTR (M,t) is well known. Let V be a point and let n be a line without coordinates (or having the same coordinate $\infty \Rightarrow V \in n$. Let v be a line having the unique coordinate $0_L \Rightarrow V \in v$. Finally, let H be a point having the unique coordinate $0_R \Rightarrow H \in n$. If we replace the NPTR (M,t) by a HPTR (M,T) isotopic to (M,t), then V, n, v, H remain without change. Combining the results introduced in [1] and Theorems 6.1–6.4 we obtain:

(A) (M,t) is additively associative and linear if and only if P is (V,n)-transitive;

(B) $M,t)$ is additively associative, linear and right distributive if and only if P is n-transitive;

(C) (M,t) is additively associative, linear and left transitive if and only if P is V-transitive;

(D) (M,t) is additively associative, linear, right and left distributive and associative if and only if P is desargueasian;

(E) (M,t) is additively associative, linear, right and left distributive, associative and commutative if and only if P is pappian.

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