František Tumajer

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THE FIXED POINT THEOREM AND THE BOUNDEDNESS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS IN THE BANACH SPACE

FRANTIŠEK TUMAJER, Liberec

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Summary. The properties of solutions of the nonlinear differential equation $x' = A(s)x + f(s,x)$ in a Banach space and of the special case of the homogeneous linear differential equation $x' = A(s)x$ are studied. Theorems and conditions guaranteeing boundedness of the solution of the nonlinear equation are given on the assumption that the solutions of the linear homogeneous equation have certain properties.

Keywords: Banach space, differential equation, bounded solution, derivative of the norm of a linear mapping, fixed point

Many problems related to the existence and unicity of solutions of differential equations in a Banach space can be transferred to the problem of existence and unicity of a fixed point of a certain mapping of the Banach space into itself. Among various criteria of the existence and unicity of a fixed point of a mapping the principle of contractive mappings can be considered as one of the simplest and simultaneously the most important criteria.

If $(B, \|.\|)$ is a Banach space then a mapping $Z\colon B \to B$ is called contractive if and only if there exists a constant $k \in \langle 0, 1)$ such that for any two points $x, y \in B$ the inequality $\|Zx - Zy\| \leqslant k\|x - y\|$ holds. Each point $x \in B$ for which $Zx = x$ is called a fixed point of the mapping Z. For these points the so-called Banach Fixed Point Theorem hold:

Every contractive mapping $Z\colon B \to B$ in a Banach space has exactly one fixed point.

We shall use this theorem for determining the bounded solutions of a differential equation

$$x' = A(s)x + f(s, x) \tag{1}$$

in a Banach space $(B, \|.\|)$, whose particular case is the equation

$$x' = A(s)x. \tag{2}$$

The symbol x' denotes the derivative dx/ds, $A\colon B \to B$ is a bounded linear mapping continuous on the interval $J = \langle 0, +\infty)$, $f\colon J \times B \to B$ is such a continuous mapping that for any $(s_0, x_0) \in J \times B$ there exists exactly one solution $x\colon J \to B$ of the differential equation with the property that for each $s \in J$ we have $x'(s) = A(s)x(s) + f(s, x(s))$ and $x(s_0) = x_0$.

In the paper [1] it is proved that for any $s_0 \in J$ there exists a bounded linear mapping $F(s)\colon B \to B$ for $s \in J$, the so-called fundamental mapping of the equation (2), and its inverse bounded mapping $F^{-1}(s)\colon B \to B$ such that $F(s_0)$ is the identical mapping $I\colon B \to B$ and for all $s \in J$ the following equalities hold:

$$F'(s) = A(s) \circ F(s),$$
$$x(s) = F(s)x_0 + F(s) \int_{s_0}^{s} F^{-1}(t) f(t, x(t))\, dt.$$

We say that a solution $x\colon J \to B$ of the equation (1) is *bounded* if and only if there exists a constant $k > 0$ such that for any $s \in J$ the inequality $\|x(s)\| \leqslant k$ holds.

The set of all solutions of the equation (2) consists of two disjoint sets: of the set M_1 of all bounded solutions and of the set M_2 of all unbounded solutions. The set M_1 is non-empty, because the equation (2) has always the zero solution which is bounded. Every linear combination of two bounded solutions of the equation (2) is again a bounded solution of this equation. This implies that for each $s_0 \in J$ the set

$$B_1(s_0) = \{x_0 \in B\colon \text{ there exists a solution } x \in M_1 \text{ for which } x(s_0) = x_0\} \tag{3}$$

is a vector subspace of the space B.

Therefore there exists an algebraic projection P_1 of the space B onto the space $B_1(s_0)$, i.e. a linear mapping $P_1\colon B \to B_1(s_0)$ with the set of values $P_1(B) = B_1(s_0)$ that $P_1 \circ P_1 = P_1$. The mapping $P_2\colon B \to B$ defined by

$$P_2 = I - P_1 \tag{4}$$

is also an algebraic projection with the set of values $P_2(B)$ which is called the direct complement of the vector space $P_1(B)$. In the case when the operators P_1, P_2 are continuous we call them projectors.

R e m a r k 1. If $P_1 : B \to B_1(s_0)$ is a non-zero algebraic projection onto the space $B_1(s_0)$, then the equation (2) has at least one non-zero bounded solution.

If the equation (2) has exactly one bounded solution, then there exists exactly one projector $P_1 : B \to B_1(s_0)$ onto the space $B_1(s_0)$ and this projector is the zero projector 0.

If we denote by h the derivative of the norm of the linear mapping by $h(A) = \lim(\|I + tA\| - 1)/t$ for $t \to 0+$, where $I: B \to B$ is the identical mapping, then the results of [2] imply the following propositions.

Proposition 2. *If x is a solution of the equation (2) such that $x(s_0) = x_0$, then for each $s \geqslant s_0$ we have*

$$\|x_0\| \exp\Big[-\int_{s_0}^{s} h(-A(\sigma))\,\mathrm{d}\sigma\Big] \leqslant \|x(s)\| \leqslant \|x_0\| \exp\Big[\int_{s_0}^{s} h(A(\sigma))\,\mathrm{d}\sigma\Big],$$

whenever the integrals are defined.

Proposition 3. *In the equation (2) let $A(s) = A_1 + A_2(s)$, where A_1 is a constant bounded linear mapping. If x is a solution of the equation (2) and $x(s_0) = x_0$, then the following implications hold:*

(i) $h(A_1) = 0$, $\int_{s_0}^{+\infty} h(A_2(\sigma))\,\mathrm{d}\sigma < +\infty \Rightarrow$ *the solution x is bounded;*

(ii) $h(A_1) < 0$, $\int_{s_0}^{+\infty} h(A_2(\sigma))\,\mathrm{d}\sigma < +\infty \Rightarrow \lim\limits_{s\to+\infty} \|x(s)\| = 0$;

(iii) $-h(-A_1) > 0$, $-\int_{s_0}^{+\infty} h(-A_2(\sigma))\,\mathrm{d}\sigma > -\infty$, $x_0 \neq o \Rightarrow \lim\limits_{s\to+\infty} \|x(s)\| = +\infty$;

(iv) $-h(-A_1) = 0$, $-\int_{s_0}^{+\infty} h(-A_2(\sigma))\,\mathrm{d}\sigma = +\infty$, $x_0 \neq o \Rightarrow \lim\limits_{s\to+\infty} \|x(s)\| = +\infty$.

On the set $C(s_0)$ of all continuous bounded mappings $g\colon J(s_0) = \langle s_0, +\infty) \to B$ let us define the norm $\|.\|_C$ by

$$\|g\|_C = \sup\{\|g(s)\| \colon s \in J(s_0)\}.$$

Then the vector space $C(s_0)$ with the norm $\|.\|_C$ is a Banach space.

If $G_1 : B \to B$, $G_2 \colon B \to B$ are linear continuous operators for which

$$G_1 + G_2 = I \tag{5}$$

holds and $s_0 \in J$, $J(s_0) = \langle s_0, +\infty)$, then the symbol $G(s_0, G_1)$ will denote the set of all continuous mappings $f\colon J \times B \to B$ having the following properties:

(i) *For each $f \in G(s_0, G_1)$ there exists a constant $k_f > 0$ such that for each $s \in J(s_0)$ we have*

$$\int_{s_0}^{s} \|F(s)\circ G_1 \circ F^{-1}(t)\|\cdot\|f(t,o)\|\,\mathrm{d}t + \int_{s}^{+\infty} \|F(s)\circ G_2\circ F^{-1}(t)\|\cdot\|f(t,o)\|\,\mathrm{d}t \leqslant k_f \tag{6}$$

where F is the fundamental mapping of the equation (2), $F(s_0) = I$ *and* G_1, G_2 *are operators from* (5).

(ii) *For each* $f \in G(s_0, G_1)$ *there exists a constant* $L_f > 0$, *the so-called Lipschitz constant, such that for all* $(s, x), (s, y) \in J \times B$ *we have*

$$\|f(s,x) - f(s,y)\| \leqslant L_f \|x - y\|. \tag{7}$$

Theorem 4. *If* $f \in G(s_0, G_1)$ *and there exists a constant* $k_1 > 0$ *such that* $L_f k_1 < 1$ *and for each* $s \in J(s_0)$ *we have*

$$\int_{s_0}^{s} \|F(s) \circ G_1 \circ F^{-1}(t)\| \, dt + \int_{s}^{+\infty} \|F(s) \circ G_2 \circ F^{-1}(t)\| \, dt \leqslant k_1, \tag{8}$$

then the equation (1) *has at least one bounded solution* x *for which* $\|x\|_C \leqslant k_f/(1 - L_f k_1)$, *where* k_f, L_f *are constants from* (6) *and* (7), *respectively.*

Proof. If $y \in (C(s_0), \|.\|_C)$, then for each $u > s \geqslant s_0$ the conditions (6), (7), (8) yield

$$\begin{aligned}
\Big\| \int_{s}^{u} F(s) \circ G_2 \circ F^{-1}(t) f(t, y(t)) \, dt \Big\| &\leqslant \int_{s}^{u} \big\| F(s) \circ G_2 \circ F^{-1}(t) f(t, y(t)) \big\| \, dt \\
&\leqslant \int_{s}^{u} \|F(s) \circ G_2 \circ F^{-1}(t)\| \cdot \big\| f(t, y(t)) - f(t, o) \big\| \, dt \\
&\quad + \int_{s}^{u} \|F(s) \circ G_2 \circ F^{-1}(t)\| \cdot \|f(t, o)\| \, dt \\
&\leqslant \int_{s}^{u} \|F(s) \circ G_2 \circ F^{-1}(t)\| L_f \|y\|_C dt + \int_{s}^{u} \|F(s) \circ G_2 \circ F^{-1}(t)\| \cdot \|f(t, o)\| \, dt \\
&\leqslant \|y\|_C L_f k_1 + k_f.
\end{aligned}$$

This means that $\int_s^{+\infty} F(s) \circ G_2 \circ F^{-1}(t) f(t, y(t)) \, dt$ exists for each $y \in (C(s_0), \|.\|_C)$. Differentiating we can verify that every solution $x \in (C(s_0), \|.\|_C)$ of the integral equation

$$x(s) = \int_{s_0}^{s} F(s) \circ G_1 \circ F^{-1}(t) f(t, x(t)) \, dt - \int_{s}^{+\infty} F(s) \circ G_2 \circ F^{-1}(t) f(t, x(t)) \, dt \tag{9}$$

is also a solution of the differential equation (1).

Now we shall prove that the equation (9) has a solution in the Banach space $(C(s_0), \|.\|_C)$ for each $f \in G(s_0, G_1)$. For this purpose we define the continuous mapping $Z \colon C(s_0) \to B$ by

$$Zx = \int_{s_0}^{s} F(s) \circ G_1 \circ F^{-1}(t) f(t, x(t)) \, dt - \int_{s}^{+\infty} F(s) \circ G_2 \circ F^{-1}(t) f(t, x(t)) \, dt. \tag{10}$$

This and the conditions (6), (7), (8) imply the inequality

$$\begin{aligned}
\|Zx\| &\leqslant \int_{s_0}^{s} \|F(s)\circ G_1\circ F^{-1}(t)\|\cdot\|f(t,x(t))-f(t,o)\|\,\mathrm{d}t\\
&\quad+\int_{s}^{+\infty} \|F(s)\circ G_2\circ F^{-1}(t)\|\cdot\|f(t,x(t))-f(t,o)\|\,\mathrm{d}t\\
&\quad+\int_{s_0}^{s} \|F(s)\circ G_1\circ F^{-1}(t)\|\cdot\|f(t,o)\|\,\mathrm{d}t\\
&\quad+\int_{s}^{+\infty} \|F(s)\circ G_2\circ F^{-1}(t)\|\cdot\|f(t,o)\|\,\mathrm{d}t\\
&\leqslant L_f\|x\|_C\Big[\int_{s_0}^{s} \|F(s)\circ G_1\circ F^{-1}(t)\|\,\mathrm{d}t\\
&\quad+\int_{s}^{+\infty} \|F(s)\circ G_2\circ F^{-1}(t)\|\,\mathrm{d}t\Big]+k_f\leqslant L_f\|x\|_C k_1+k_f,
\end{aligned}$$

and this implies also

$$\|Zx\|_C\leqslant L_f\|x\|_C k_1+k_f. \tag{11}$$

This means that $Zx\in C(s_0)$ and the mapping Z maps the Banach space $\big(C(s_0), \|.\|_C\big)$ into itself. Now we shall show that Z is a contraction operator. If $x,y\in\big(C(s_0),\|.\|_C\big)$, then after simple arrangements we obtain

$$\begin{aligned}
\|Zx-Zy\| &\leqslant \int_{s_0}^{s} \|F(s)\circ G_1\circ F^{-1}(t)\|L_f\|x(t)-y(t)\|\,\mathrm{d}t\\
&\quad+\int_{s}^{+\infty} \|F(s)\circ G_2\circ F^{-1}(t)\|L_f\|x(t)-y(t)\|\,\mathrm{d}t\\
&\leqslant L_f\|x-y\|_C\Big[\int_{s_0}^{s} \|F(s)\circ G_1\circ F^{-1}(t)\|\,\mathrm{d}t\\
&\quad+\int_{s}^{+\infty} \|F(s)\circ G_2\circ F^{-1}(t)\|\,\mathrm{d}t\Big]\leqslant L_f k_1\|x-y\|_C
\end{aligned}$$

and this implies also

$$\|Zx-Zy\|_C\leqslant L_f k_1\|x-y\|_C.$$

This and the inequality $L_f k_1<1$ imply that Z is a contraction operator on the Banach space $\big(C(s_0),\|.\|_C\big)$. From the Banach Fixed Point Theorem we can conclude that the operator Z has exactly one fixed point x in the space $\big(C(s_0),\|.\|_C\big)$, i.e. $Zx=x$, which is the required bounded solution of the differential equation (1). The inequality (11) implies that the solution x satisfies $\|x\|_C\leqslant L_f\|x\|_C k_1+k_f$, i.e. $\|x\|_C(1-L_f k_1)\leqslant k_f$. Therefore $\|x\|_C\leqslant k_f/(1-L_f k_1)$, which was to be proved. □

Theorem 5. *If there exists a constant $k_1 > 0$ such that*

$$\int_{s_0}^{s} \|F(s) \circ F^{-1}(t)\| \, \mathrm{d}t \leqslant k_1 \tag{12}$$

for all $s \in J(s_0)$, then each $f \in G(s_0, 0)$ all solutions of (1) are bounded.

Proof. Let the condition (12) be fulfilled. The equality

$$\int_{s_0}^{s} \|F(t)\|^{-1} \mathrm{d}t F(s) = \int_{s_0}^{s} F(s) \circ F^{-1}(t) \circ F(t) \|F(t)\|^{-1} \mathrm{d}t$$

and the inequality (12) imply the inequality

$$\begin{aligned} \|F(s)\| \int_{s_0}^{s} \|F(t)\|^{-1} \mathrm{d}t &\leqslant \int_{s_0}^{s} \|F(s) \circ F^{-1}(t)\| \cdot \|F(t)\| \cdot \|F(t)\|^{-1} \mathrm{d}t \\ &= \int_{s_0}^{s} \|F(s) \circ F^{-1}(t)\| \, \mathrm{d}t \leqslant k_1. \end{aligned}$$

Put $r(s) = \int_{s_0}^{s} \|F(t)\|^{-1} \mathrm{d}t$. Then $r'(s) = \|F(s)\|^{-1}$ and the preceding inequality implies

$$r(s) \leqslant k_1 r'(s) \tag{13}$$

for each $s \in J(s_0)$. The inequality (13) implies that for $s > s_0$ we have $r'(s)/r(s) > 1/k_1$; by integrating in the interval $\langle t_0, s \rangle$, $t_0 > s_0$, we obtain the inequality

$$[\ln r(t)]_{t_0}^{s} \geqslant (s - t_0)/k_1, \qquad \text{i.e.} \qquad r(s) \geqslant r(t_0) \exp\bigl(k_1^{-1}(s - t_0)\bigr)$$

for each $s \geqslant t_0$. This and the inequality (13) imply the inequality

$$r(t_0) \exp\bigl(k_1^{-1}(s - t_0)\bigr) \leqslant k_1 r'(s) = k_1 \|F(s)\|^{-1},$$

so that

$$\|F(s)\| \leqslant k_1 r^{-1}(t_0) \exp\bigl(-k_1^{-1}(s - t_0)\bigr) \leqslant k_1 r^{-1}(t_0) \exp\bigl(k_1^{-1}(t_0 - s_0)\bigr) = k_2$$

for each $s \in J(s_0)$. This and the fact that every solution x of the differential equation (1) is also a solution of the integral equation

$$x(s) = F(s)x(s_0) + \int_{s_0}^{s} F(s) \circ F^{-1}(t) f\bigl(t, x(t)\bigr) \, \mathrm{d}t$$

imply, by virtue of the conditions $f \in G(s_0, 0)$, (7), the inequality

$$\begin{aligned}\|x(s)\| &\leqslant \|F(s)\| \cdot \|x(s_0)\| + \int_{s_0}^{s} \|F(s) \circ F^{-1}(t)\| \cdot \big\|f(t, x(t)) - f(t, o)\big\| \, \mathrm{d}t \\ &\quad + \int_{s_0}^{s} \|F(s) \circ F^{-1}(t)\| \cdot \|f(t, o)\| \, \mathrm{d}t \\ &\leqslant k_2\|x(s_0)\| + k_f + L_f \int_{s_0}^{s} \|F(s) \circ F^{-1}(t)\| \cdot \|x(t)\| \, \mathrm{d}t\end{aligned}$$

for each $s \in J(s_0)$. If we apply Gronwall's Lemma to the preceding inequality, for all $s \in J(s_0)$ we obtain the inequality

$$\|x(s)\| \leqslant k_2\|x(s_0)\| + k_f \exp\Big[L_f \int_{s_0}^{s} \|F(s) \circ F^{-1}(t)\| \, \mathrm{d}t\Big],$$

which due to the condition (12) means the boundedness of the solution x. □

Theorem 6. *If there exists a constant $k_1 > 0$ such that $L_f k_1 < 1$ and*

$$\int_{s}^{+\infty} \|F(s) \circ F^{-1}(t)\| \, \mathrm{d}t \leqslant k_1 \tag{14}$$

for all $s \in J(s_0)$, then the equation (1) *has exactly one bounded solution for each $f \in G(s_0, 0)$.*

Proof. Let the condition (14) be satisfied. Choose $x_0 \in B$, $x_0 \neq o$, and put $v(s) = \|F(s)x_0\|^{-1}$. Then for each $s \in \langle s_0, u)$ we have

$$\int_{s}^{u} v(t) \, \mathrm{d}t F(s)x_0 = \int_{s}^{u} v(t)F(s) \circ F^{-1}(t) \circ F(t)x_0 \, \mathrm{d}t.$$

This implies the inequality

$$\begin{aligned}\int_{s}^{u} v(t) \, \mathrm{d}t \|F(s)x_0\| &\leqslant \int_{s}^{u} v(t)\|F(s) \circ F^{-1}(t)\| \cdot \|F(t)x_0\| \, \mathrm{d}t \\ &= \int_{s}^{u} \|F(s) \circ F^{-1}(t)\| \, \mathrm{d}t \leqslant k_1,\end{aligned}$$

so that

$$v^{-1}(s) \int_{s}^{u} v(t) \, \mathrm{d}t \leqslant k_1.$$

This means that $\int_{s}^{+\infty} v(t) \, \mathrm{d}t < +\infty$ and $\liminf v(s) = 0$ for $s \to +\infty$. Hence $\limsup \|F(s)x_0\| = +\infty$ for $s \to +\infty$ and for each $x_0 \in B$, $x_0 \neq o$. This implies that the equation (2) has exactly one bounded solution, namely the zero solution. According to Remark 1 we have $G_1 = P_1 = O$, $G_2 = P_2 = I$. According to Theorem 4 the equation (1) has at least one bounded solution x_1. It is easy to see that the mapping $y\colon J(s_0) \to B$ defined by

$$y(s) = x_1(s) + \int_s^\infty F(s) \circ F^{-1}(t) f\big(t, x_1(t)\big)\,\mathrm{d}t \tag{15}$$

is a solution of the equation (2), because

$$\begin{aligned} y'(s) &= x_1'(s) + F'(s) \int_s^\infty F^{-1}(t) f\big(t, x_1(t)\big)\,\mathrm{d}t - F(s) \circ F^{-1}(s) f\big(s, x_1(s)\big) \\ &= A(s)x_1(s) + f\big(s, x_1(s)\big) + A(s) \circ F(s) \int_s^\infty F^{-1}(t) f\big(t, x_1(t)\big)\,\mathrm{d}t - f\big(s, x_1(s)\big) \\ &= A(s)\Big[x_1(s) + \int_s^\infty F(s) \circ F^{-1}(t) f\big(t, x_1(t)\big)\,\mathrm{d}t\Big] = A(s)y(s). \end{aligned}$$

From the equality (15) we obtain that for each $s \in J(s_0)$ we have

$$\begin{aligned} \|y(s)\| &\leqslant \|x_1(s)\| + \int_s^\infty \|F(s) \circ F^{-1}(t)\| L_f \|x_1(t)\|\,\mathrm{d}t \\ &\quad + \int_s^\infty \|F(s) \circ F^{-1}(t)\| \cdot \|f(t, o)\|\,\mathrm{d}t \\ &\leqslant \|x_1\|_C + L_f \|x_1\|_C k_1 + k_f. \end{aligned}$$

This means that y is a bounded solution of the equation (2) and therefore the mapping y is its zero solution. This implies that every bounded solution x of the differential equation (1) is also a solution of the integral equation

$$x(s) = -\int_s^\infty F(s) \circ F^{-1}(t) f\big(t, x(t)\big)\,\mathrm{d}t,$$

and according to the Banach Fixed Point Theorem this equation has exactly one solution. Thus the theorem is proved. □

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Souhrn

VĚTA O PEVNÉM BODU A OMEZENOST ŘEŠENÍ DIFERENCIÁLNÍCH ROVNIC V BANACHOVĚ PROSTORU

FRANTIŠEK TUMAJER

V práci jsou studovány vlastnosti řešení nelineární diferenciální rovnice $x' = A(s)x + f(s,x)$ v Banachově prostoru a jejího speciálního případu lineární homogenní diferenciální rovnice $x' = A(s)x$. Jsou formulovány věty a uvedeny podmínky, které na základě určitých vlastností řešení lineární homogenní rovnice zajišťují omezenost řešení nelineární rovnice.

Author's address: Katedra matematiky VŠST, Sokolská 8, 460 01 Liberec.