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GENERALIZED SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS IN THE COLOMBEAU ALGEBRA

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Summary. In this paper first order systems of linear of ODEs are considered. It is shown that these systems admit unique solutions in the Colombeau algebra $\mathcal{G}(\mathbf{R}^1)$.

Keywords: generalized ordinary differential equation, Cauchy problem generalized function, distribution, Colombeau algebra

AMS classification: 34A10, 46F99

1. INTRODUCTION

We consider the linear Cauchy problem

$$\begin{cases} x_k'(t) = \sum\limits_{j=1}^{n} A_{kj}(t)x_j(t) + f_k(t) & (1.0)\\ x_k(t_0) = x_{0k}, \qquad t_0 \in \mathbf{R}^1,\ k = 1,\dots,n, & (1.1)\end{cases}$$

where A_{kj}, x_j and f_k are elements of the Colombeau algebra $\mathcal{G}(\mathbf{R}^1)$, x_{0k} are known elements of the Colombeau algebra $\overline{\mathbf{C}}$ of generalized complex numbers, $x_k(t_0)$ is understood as the value of the generalized function x_k at the point t_0 and $k = 1, \dots, n$ (see [4]). Elements A_{kj} and f_k are given, elements x_k are unknown (for $k, j = 1, \dots, n$). Multiplication, derivative, sum and equality is meant in the Colombeau algebra sense. We prove theorems on existence and uniqueness of solutions of the Cauchy problem for the system (1.0).

In the paper [4] some differential equations with coefficients from the Colombeau algebra were examined. Certain problems for the quantum field theory lead to such equations. However, these equations cannot be considered in the theory of distributions, due to difficulties in defining a multiplication of distributions. The algebra $\mathcal{G}(\mathbf{R}^1)$, constructed by Colombeau in [4], contains the space of distributions $\mathcal{D}'(\mathbf{R}^1)$, has $C^\infty(\mathbf{R}^1)$ as a subalgebra, and admits a derivation operator which extends differentiation in $\mathcal{D}'(\mathbf{R}^1)$.

2. Notation

Let $\mathcal{D}(\mathbf{R}^1)$ be the space of all C^∞ functions $\mathbf{R}^1 \to \mathbf{C}$ with compact support. For $q = 1, 2, \ldots$ we denote by $\tilde{\mathcal{A}_q}$ the set of all functions $\varphi \in \mathcal{D}(\mathbf{R}^1)$ such that the relations

$$\int_{-\infty}^{\infty} \varphi(t)\,\mathrm{d}t = 1, \quad \int_{-\infty}^{\infty} t^k \Phi(t)\,\mathrm{d}t = 0, \quad 1 \leqslant k \leqslant q \tag{2.1}$$

hold while be set of all C^∞ functions $\mathbf{R}^1 \to \mathbf{R}^1$ with compact support satisfying (2.1) will be denoted by $\mathcal{A}_q$.

Next, $\mathcal{E}[\mathbf{R}^1]$ is the set of functions $R\colon \mathcal{A}_1 X \mathbf{R}^1 \to \mathbf{C}$ such that $R(\Phi, .) \in C^\infty$ for each fixed $\Phi \in \mathcal{A}_1$.

If $R \in \mathcal{E}[\mathbf{R}^1]$, then $D_k R(\Phi, t)$ for any fixed Φ denotes a differential operator in t (i.e. $D_k R(\Phi, t) = \frac{\partial R}{\partial t^k}(R(\Phi, t))$).

For given $\Phi \in \mathcal{D}(\mathbf{R}^1)$ and $\varepsilon > 0$, we define Φ_ε by

$$\Phi_\varepsilon(t) = \varepsilon^{-1}\Phi\left(\frac{t}{\varepsilon}\right). \tag{2.2}$$

An element R of $\mathcal{E}[\mathbf{R}^1]$ is moderate if for every compact set K of $\mathbf{R}^1$ and every differential operator D_k there is $N \in \mathbf{N}$ such that the following condition holds:
for every $\Phi \in \mathcal{A}_N$ there are $c > 0$ and $\eta_0 > 0$ such that

$$\sup_{t \in K} |D_k R(\Phi_\varepsilon, t)| \leqslant c\varepsilon^{-N} \quad \text{for} \quad 0 < \varepsilon < \eta_0. \tag{2.3}$$

We denote by $\mathcal{E}_M[\mathbf{R}^1]$ the set of all moderate elements of $\mathcal{E}[\mathbf{R}^1]$.

By Γ we denote the set of all increasing functions α from $\mathbf{N}$ into $\mathbf{R}^+$ such that $\alpha(q)$ tends to ∞ if $q \to \infty$.

We define an ideal $\mathcal{N}[\mathbf{R}^1]$ in $\mathcal{E}[\mathbf{R}^1]$ as follows: $u \in \mathcal{N}[\mathbf{R}^1]$ if for every compact subset K of $\mathbf{R}^1$ and every differential operator D_k there are $N \in \mathbf{N}$ and $\alpha \in \Gamma$ such that the following condition holds: for every $q \geqslant N$ and $\Phi \in \mathcal{A}_q$ there are $c > 0$ and $\eta_0 > 0$ such that

$$\sup_{t \in K} |D_k R(\Phi\varepsilon, t)| \leqslant c\varepsilon^{\alpha(q)-N} \quad \text{for} \quad 0 < \varepsilon < \eta_0. \tag{2.4}$$

The algebra $\mathscr{G}(\mathbf{R}^1)$ (the Colombeau algebra) is defined as the quotient algebra of $\mathscr{E}_M[\mathbf{R}^1]$ with respect to $\mathscr{N}[\mathbf{R}^1]$ (see [4]).

We denote by $\mathscr{E}_0$ the set of all functions from $\mathscr{A}_1$ into $\mathbf{C}$. Next, we denote by $\mathscr{E}_M$ the set of all the so—called moderate elements of $\mathscr{E}_0$ defined by

$$(2.5)\qquad \mathscr{E}_M = \big\{R \in \mathscr{E}_0\colon \text{ there is } N \in \mathbf{N} \text{ such that for every } \Phi \in \mathscr{A}_N \text{ there are } c > 0 \text{ and } \eta_0 > 0 \text{ such that } |R(\Phi\varepsilon)| \leqslant c\varepsilon^{-N} \text{ for } 0 < \varepsilon < \eta_0\big\}.$$

We define $\mathscr{J}$ of $\mathscr{E}_M$ by

$$(2.6)\qquad \mathscr{J} = \big\{R \in \mathscr{E}_0\colon \text{ there are } N \in \mathbf{N} \text{ and } \alpha \in \Gamma \text{ such that for every } q \geqslant \mathbf{N} \text{ and } \Phi \in \mathscr{A}_q \text{ there are } c > 0 \text{ and } \eta_0 > 0 \text{ such that } |R(\Phi\varepsilon)| \leqslant c\varepsilon^{\alpha(q)-N} \text{ for } 0 < \varepsilon < \eta_0\big\}.$$

We define an algebra $\bar{\mathbf{C}}$ by setting

$$\bar{\mathbf{C}} = \mathscr{E}_M/\mathscr{J} \quad \text{(see [4])}.$$

If $R \in \mathscr{E}_M[\mathbf{R}^1]$ is a representative of $G \in \mathscr{G}(\mathbf{R}^1)$, then for a fixed t the map $Y\colon \Phi \to R(\Phi, t) \in \mathbf{C}$ is defined on $\mathscr{A}_1$ and $Y \in \mathscr{E}_M$. The class of Y in $\mathbf{C}$ depends only on G and t. This class is denoted by $G(t)$ and is called the value of generalized function G at the point t (see [4]).

We say that $G \in \mathscr{G}(\mathbf{R}^1)$ is a constant generalized function on $\mathbf{R}^1$ if it admits a representative $R(\Phi, t)$ which is independent of $t \in \mathbf{R}^1$. With any $Z \in \bar{\mathbf{C}}$ we associate a constant generalized function which admits $R(\Phi, t) = Z(\Phi)$ as its representative, provided we denote by Z a representative of Z (see [4]).

Throughout in the paper K denotes a compact set in $\mathbf{R}^1$. We denote by $R_{A_{kj}}(\Phi, t)$, $R_{f_k}(\Phi, t)$, $R_{x_{0j}}(\Phi)$, $R_{x_j(t_0)}(\Phi)$ and $R_{x_j}(\Phi, t)$ representatives of elements A_{kj}, f_k, x_{0j}, $x_j(t)$ and x_j for k, $j = 1, \ldots, n$. Let $A(t) = \big(A_{kj}(t)\big)$, $f(t) = \big(f_1(t), \ldots, f_n(t)\big)^T$, $x(t) = \big(x_1(t), \ldots, x_n(t)\big)^T$, $x'(t) = \big(x'_1(t), \ldots, x'_n(t)\big)^T$, $x_0 = (x_{01}, \ldots, x_{0n})^T$, where

T denotes the transpose. We put

$$
\begin{aligned}
R_A(\Phi,t) &= \left(R_{A_{kj}}(\Phi,t)\right),\\
D_k R_A(\Phi,t) &= \left(D_k R_{A_{kj}}(\Phi,t)\right),\\
R_f(\Phi,t) &= \left(R_{f_1}(\Phi,t),\ldots,R_{f_n}(\Phi,t)\right)^T,\\
R_x(\Phi,t) &= \left(R_{x_1}(\Phi,t),\ldots,R_{x_n}(\Phi,t)\right)^T,\\
R_{x'}(\Phi,t) &= \left(R_{x'_1}(\Phi,t),\ldots,R_{x'_n}(\Phi,t)\right)^T,\\
R_{x_0}(\Phi) &= \left(R_{x_{01}},\ldots,R_{x_{0n}}(\Phi)\right)^T,\\
R_{x(t_0)}(\Phi) &= \left(R_{x_1(t_0)}(\Phi),\ldots,R_{x_n(t_0)}(\Phi)\right)^T,\\
\int_{t_0}^{t} R_A(\Phi,s)\,\mathrm{d}s &= \left(\int_{t_0}^{t} R_{A_{kj}}(\Phi,s)\,\mathrm{d}s\right),\\
\int_{t_0}^{t} R_f(\Phi,s)\,\mathrm{d}s &= \left(\int_{t_0}^{t} R_{f_1}(\Phi,s)\,\mathrm{d}s,\ldots,\int_{t_0}^{t}(\Phi,s)\,\mathrm{d}s\right)^T,\\
\|R_A(\Phi,t)\| &= \sqrt{\sum_{k,j=1}^{n} |R_{A_{kj}}(\Phi,t)|^2},\\
\|R_f(\Phi,t)\| &= \sqrt{\sum_{j=1}^{n} |R_f(\Phi,t)|^2},\\
\|R_A(\Phi,t)\|_K &= \sup_{t\in K}\|R_A(\Phi,t)\|,\\
\|R_f(\Phi,t)\|_K &= \sup_{t\in K}\|R_f(\Phi,t)\|.
\end{aligned}
$$

We say that a generalized function G is real valued if it admits a real valued representative.

Starting with those elements of $\mathscr{E}_0$ which are real valued we obtain in this way an algebra $\bar{\mathbf{R}}^1$ containing $\mathbf{R}^1$ as subalgebra. Thus $\bar{\mathbf{C}} = \bar{\mathbf{R}}^1 + \mathrm{i}\bar{\mathbf{R}}^1$, where $\mathrm{i}^2 = -1$ (see[4]).

Let a_{kj}, $b_j \in \mathscr{N}[\mathbf{R}^1]$, m_{kj}, $p_j \in \mathscr{J}$; $q_j \in \mathbf{C}$, $r_j \in \bar{\mathbf{C}}$; A_{kj}, f_j, $x_j \in \mathscr{G}(\mathbf{R}^1)$ for k, $j = 1, \ldots, n$. Then we write $a = (a_{kj}) \in \mathscr{N}^{n\times n}[\mathbf{R}^1]$, $b = (b_1,\ldots,b_n)^T \in \mathscr{N}^n\mathbf{R}^1]$, $m = (m_{kj}) \in \mathscr{J}^{n\times n}$, $p = (p_1,\ldots,p_n)^T \in \mathscr{J}^n$, $q = (q_1,\ldots,q_n)^T \in \mathbf{C}^n$, $r = (r_1,\ldots,r_n)^T \in \bar{\mathbf{C}}^n$, $A = (A_{kj}) \in \mathscr{G}^{n\times n}(\mathbf{R}^1)$, $x = (x_1,\ldots,x_n)^T \in \mathscr{G}^n(\mathbf{R}^1)$, $R_A(\Phi,t) \in \mathscr{E}_M^{n\times n}[\mathbf{R}^1]$ and $R_x(\Phi,t) \in \mathscr{E}_M^n[\mathbf{R}^1]$.

We say that $x = (x_1,\ldots,x_n) \in \mathscr{G}^n(\mathbf{R}^1)$ is a solution of the system (1.0) if x satisfies the system (1.0) identically in $\mathscr{G}^n(\mathbf{R}^1)$.

Let i denote the usual square root of -1. We shall consider the problems

$$\begin{cases} Z'(t) = \mathrm{i}A(t)Z(t) & (3.0) \\ Z(t_0) = I, \qquad t_0 \in \mathbf{R}^1 & (3.1) \end{cases}$$

and

$$\begin{cases} Z'(t) = A(t)Z(t) & (3.2) \\ Z(t_0) = I, \qquad t_0 \in \mathbf{R}^1, & (3.3) \end{cases}$$

where I denotes the identity matrix.

First we introduce a hypothesis H:

Hypothesis H

(3.4) $$A \in \mathscr{G}^{n\times n}(\mathbf{R}^1);$$

(3.5) the matrix $A \in \mathscr{G}^{n\times n}(\mathbf{R}^1)$ admits a representative $R_A(\Phi, t)$ such that

(a) $R_{A_{kj}}(\Phi, t) \in \mathbf{R}^1$ for every $\Phi \in \mathscr{A}_1$ and $k, j = 1, \dots, n$,

(b) $R_A(\Phi, t) = \big(R_A(\Phi, t)\big)^T$ for every $\Phi \in \mathscr{A}_1$;

(3.6) the matrix A admits a representative $R_A(\Phi, t)$ such that $R_A(\Phi, t)$ has the property (3.5) (a) and $R_A(\Phi, t) = -\big(R_A(\Phi, t)\big)^T$ for every $\Phi \in \mathscr{A}_1$;

(3.7) the matrix A admits a representative $R_A(\Phi, t)$ with the following property: for every K there is $N \in \mathbf{N}$ such that for every $\Phi \in \mathscr{A}_N$ there are constants $c > 0$ and $\eta_0 > 0$ such that

$$\left\| \int_{t_0}^{t} \|R_A(\Phi_\varepsilon, s)\| \, \mathrm{d}s \right\|_K \leqslant c \quad \text{for} \quad 0 < \varepsilon < \eta_0.$$

(3.8) the matrix A admits a representative $R_A(\Phi, t)$such that

(c) $R_{A_{kj}}(\Phi, t) = 0$ for every $\Phi \in \mathscr{A}_1$, $k < j$ and $n > 1$;

(d) for every K there is $N \in \mathbf{N}$ such that for every $\Phi \in \mathscr{A}_N$ there are $c > 0$ and $\eta_0 > 0$ such that

$$\left\| \int_{t_0}^{t} R_{A_{jj}}(\Phi_\varepsilon, s) \, \mathrm{d}s \right\|_K \leqslant c \quad \text{for } 0 < \varepsilon < \eta_0 \text{ and } j = 1, \dots, n.$$

Now we will give theorems on the existence and uniqueness of solutions of homogeneous systems.

Theorem 3.1. *Let the assumptions* (3.4)–(3.5) *be satisfied. Then the problem* (3.0)–(3.1) *has exactly one solution in* $\mathcal{G}^{n\times n}(\mathbf{R}^1)$.

Remark 3.1. Theorem 3.1 is similar to Theorem 3.2.4 and Theorem 3.5.4 in [4]. The author assumes in Theorem 3.2.4 that $A \in \mathcal{G}(\mathbf{R}^1)$ and A has a compact support. Theorem 3.5.4 is proved for generalized functions with values in Banach spaces. The proof of Theorem 3.1 is slightly different from that of Theorem 3.5.4.

Proof of Theorem 3.1. We consider the system of differential equations

$$\begin{cases} Z'(t) = \mathrm{i}R_A(\Phi,t)Z(t) & (3.9) \\ Z(t_0) = I, & (3.10) \end{cases}$$

where $R_A(\Phi,t)$ is a representative of A satisfying (3.5).

For a fixed $\Phi \in \mathcal{A}_1$ this problem is the classical linear Cauchy problem. It has exactly one solution $R_Z(\Phi,t)$ on $\mathbf{R}^1$. We are going to prove $R_Z(\Phi,t) \in \mathcal{E}_M^{n\times n}[\mathbf{R}^1]$. Let $R_V(\Phi,t)$ be a solution of the adjoint systems to the system (3.9). Then, by (3.5), we get

$$D_1 R_V(\Phi,t) = -\mathrm{i}R_A(\Phi,t)R_V(\Phi,t) \tag{3.11}$$

and

$$\big(R_V(\Phi,t)\big)^* R_Z(\Phi,t) = c(\Phi), \tag{3.12}$$

where $\big(R_V(\Phi,t)\big)^* = \big(\overline{R_V(\Phi,t)}\big)^T$ and $c(\Phi)$ is a constant dependent on Φ.

Putting

$$R_V(\Phi,t) = R_Z(\Phi,t)$$

we have

$$\big(R_Z(\Phi_\varepsilon,t)\big)^* R_Z(\Phi_\varepsilon,t) = \big(R_Z(\Phi_\varepsilon,t_0)\big)^* R_Z(\Phi_\varepsilon,t_0) = I. \tag{3.13}$$

Hence

$$\|R_Z(\Phi_\varepsilon,t)\|_K \leqslant \sqrt{n}. \tag{3.14}$$

Relations (3.9) and (3.14) yield

$$\|D_r R_Z(\Phi_\varepsilon,t)\|_K \leqslant c\varepsilon^{-N} \quad \text{for} \quad 0<\varepsilon<\eta_0. \tag{3.15}$$

Thus $R_Z(\Phi,t) \in \mathcal{E}_M^{n\times n}[\mathbf{R}^1]$. If we define Z as the class of $R_Z(\Phi,t)$ in $\mathcal{G}^{n\times n}(\mathbf{R}^1)$, then Z is a solution of the problem (3.0)–(3.1). To prove uniqueness of the solution of the problem (3.0)–(3.1) we observe that by (3.9) and (3.13)

$$D_1\big((R_Z(\Phi,t))^*\big) = -\mathrm{i}\big(R_Z(\Phi,t)\big)^* R_A(\Phi,t) \tag{3.15}$$

and

$$\big(R_Z(\Phi,t)\big)^* = \big(R_Z(\Phi,t)\big)^{-1} \in \mathcal{E}_M^{n\times n}[\mathbf{R}^1]. \tag{3.16}$$

We define Z^* as the class of $\big(R_Z(\Phi,t)\big)^*$. Then

$$(Z^*)' = -\mathrm{i}Z^*A. \tag{3.17}$$

Let $Y \in \mathcal{G}_M^{n\times n}(\mathbf{R}^1)$ be another solution of the problem (3.0)–(3.1). We denote

$$U = Z^*(Y-Z). \tag{3.18}$$

By (3.17)–(3.18) we obtain

$$\begin{aligned} U' &= (Z^*)'(Y-Z) + Z^*(Y'-Z') \\ &= -\mathrm{i}Z^*A(Y-Z) + \mathrm{i}Z^*(AY-AZ) = 0 \end{aligned} \tag{3.19}$$

and

$$R_U(\Phi,t_0) = \big(R_Z(\Phi,t_0)\big)^*\big(R_Y(\Phi,t_0) - R_Z(\Phi,t_0)\big) \in \mathcal{J}^{n\times n}. \tag{3.20}$$

The last relations yield

$$U(t_0) = 0. \tag{3.21}$$

Applying Theorem 2.3.1 from [4] and relations (3.19)–(3.21) we can see that

$$U = 0 \quad \text{in} \quad \mathcal{G}^{n\times n}(\mathbf{R}^1). \tag{3.22}$$

or equivalently

$$\big(R_Z(\Phi,t)\big)^*\big(R_Y(\Phi,t) - R_Z(\Phi,t)\big) \in \mathcal{N}^{n\times n}[\mathbf{R}^1]. \tag{3.23}$$

On the other hand, (3.16) and (3.23) yield

$$R_Z(\Phi,t)\big(R_Z(\Phi,t)\big)^*\big(R_Y(\Phi,t) - R_Z(\Phi,t)\big) \in \mathcal{N}^{n\times n}[\mathbf{R}^1] \tag{3.24}$$

and consequently

$$R_Y(\Phi,t) - R_Z(\Phi,t) \in \mathcal{N}^{n\times n}[\mathbf{R}^1]. \tag{3.25}$$

This proves the theorem. □

Theorem 3.2. *Let the assumptions* (3.4) *and* (3.6) *be fulfilled. Then the problem* (3.2)–(3.3) *has exactly one solution in* $\mathcal{G}^{n\times n}(\mathbf{R}^1)$.

Proof. The proof of Theorem 3.2 is similar to that of Theorem 3.1. We start from the problem

$$\begin{cases} Z'(t) = R_A(\Phi,t)Z(t) & (3.26)\\ Z(t_0) = I,\ t_0 \in \mathbf{R}^1, & (3.27)\end{cases}$$

where $R_A(\Phi,t)$ has the property (3.6).

First we prove relations (3.13)–(3.14) and (3.16), where $R_Z(\Phi,t)$ denotes a solution of the problem (3.26)–(3.27). Uniqueness follows from (3.18) and (3.22). Indeed,

$$\begin{aligned} U' &= (Z^*)'(Y-Z) + Z^*(Y'-Z') \\ &= -Z^*A(Y-Z) + Z^*(AY-AZ) = 0 \end{aligned} \tag{3.28}$$

where

$$(Z^*)' = Z^*A^* = -Z^*A. \tag{3.29}$$

Using relations (3.22)–(3.25) we have

$$Y = Z, \tag{3.30}$$

which completes the proof of Theorem 3.2. □

Theorem 3.3. *Let the assumptions* (3.4) *and* (3.7) *be fulfilled. Then the problem* (3.2)–(3.3) *has exactly one solution if* $\mathcal{G}^{n\times n}(\mathbf{R}^1)$.

Proof. We consider the problem (3.26)–(3.27), where $\Phi \in \mathcal{A}_1$ and $R_A(\Phi,t)$ has the property (3.7). Using the Gronwall inequality we have

$$\|R_Z(\Phi_\varepsilon,t)\|_K \leqslant \sqrt{n}\exp\left(\left\|\int_{t_0}^t \|R_A(\Phi_\varepsilon,s)\|\mathrm{d}s\right\|_K\right) \leqslant \sqrt{n}\exp c. \tag{3.31}$$

By (3.31) and (3.26) there is $N \in \mathbf{N}$ such that for $\Phi \in \mathcal{A}_N$ we have

$$\|D_r R_Z(\Phi_\varepsilon,t)\|_K \leqslant c_r\varepsilon^{-N} \quad \text{for} \quad 0<\varepsilon<\eta_0. \tag{3.32}$$

Hence $R_Z(\Phi,t) \in \mathcal{E}_M^{n\times n}[\mathbf{R}^1]$. Denoting by Z the class of $R_Z(\Phi,t)$ in $\mathcal{G}^{n\times n}(\mathbf{R}^1)$, we get that Z is a solution of the problem (3.2)–(3.3). Let $Y \in \mathcal{G}^{n\times n}(\mathbf{R}^1)$ be another solution of the problem (3.2)–(3.3).

Then

$$D_1 R_Y(\Phi, t) = R_A(\Phi, t) R_Y(\Phi, t) + R_m(\Phi, t), \tag{3.33}$$

where

$$R_m(\Phi, t) \in \mathscr{N}^{n\times n}[\mathbf{R}^1] \tag{3.34}$$

and

$$R_Y(\Phi, t) - I \in \mathscr{J}^{n\times n}. \tag{3.35}$$

In view of (3.26) and (3.33)–(3.35) we deduce that (for $q \geqslant N_1$ and $\Phi \in \mathscr{A}_q$)

$$\begin{aligned}(3.36)\ &\|R_Z(\Phi_\varepsilon, t) - R_Y(\Phi_\varepsilon, t)\|_K \\ &\leqslant \left(\|R_Y(\Phi_\varepsilon, t_0) - I\|_K + \|R_m(\Phi_\varepsilon, t)\|_K\right) \exp\left(\left\| \int_{t_0}^{t} \|R_A(\Phi_\varepsilon, s)\| \mathrm{d}s \right\|_K\right) \\ &\leqslant c_0 \varepsilon^{\alpha(q) - N_1} \qquad \text{for } 0 < \varepsilon < \eta_0.\end{aligned}$$

On the other hand, by (3.36), (3.26) and (3.33) we have

$$\|D_1\left(R_Z(\Phi_\varepsilon, t) - R_Y(\Phi_\varepsilon, t)\right)\|_K \leqslant c_1 \varepsilon^{\alpha(q) - N_2} \quad \text{for} \quad 0 < \varepsilon < \tilde{\eta}_0. \tag{3.37}$$

This yields

$$R_Z(\Phi, t) - R_Y(\Phi, t) \in \mathscr{N}^{n\times n}[\mathbf{R}^1]. \tag{3.38}$$

and Theorem 3.3 is proved. □

R e m a r k 3.2. Let δ denote the generalized function which admits as a representative the function $R(\Phi, t) = \Phi(-t)$, where $\Phi \in \mathscr{A}_1$. Then the problem

$$\begin{cases} x'(t) = \varepsilon^{-1} \Phi\left(\dfrac{-t}{\varepsilon}\right) x(t) \\ x(-1) = 1 \end{cases} \tag{3.39}$$

has exactly one solution $\tilde{R}_x(\Phi_\varepsilon, t)$. Since

$$\tilde{R}_x(\Phi_\varepsilon, t) = \exp\left(\int_{-1}^{t} \varepsilon^{-1} \Phi\left(\frac{-s}{\varepsilon}\right) \mathrm{d}s\right) \quad \text{for small } \varepsilon > 0 \tag{3.40}$$

and

$$\left| \int_{-1}^{t} \left| \varepsilon^{-1} \Phi\left(\frac{-s}{\varepsilon}\right) \right| \mathrm{d}s \right| \leqslant \int_{-\infty}^{\infty} |\Phi(t)| \,\mathrm{d}t < \infty. \tag{3.41}$$

we conclude that $R_x(\Phi_\varepsilon, t)$ satisfies the condition (3.7). Thus the problem

$$\begin{cases} x' = \delta x \\ x(-1) = 1 \end{cases} \tag{3.42}$$

has exactly one solution $x \in \mathcal{G}(\mathbf{R}^1)$, which is a class of $\tilde{R}_x(\Phi, t)$.

Theorem 3.4. *Let the assumptions* (3.4) *and* (3.8) *be satisfied. Then the problem* (3.2)–(3.3) *has exactly one solution in* $\mathcal{G}^{n\times n}(\mathbf{R}^1)$.

P r o o f. We examine the problem

$$\begin{cases} D_1 R_{z_{1k}}(\Phi, t) = R_{A_{11}}(\Phi, t) R_{z_{1k}}(\Phi, t) & (3.43)_{1k} \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ D_1 R_{z_{nk}}(\Phi, t) = R_{A_{n1}}(\Phi, t) R_{z_{1k}}(\Phi, t) + \ldots + R_{A_{nk}}(\Phi, t) R_{z_{nk}}(\Phi, t) & (3.43)_{nk} \\ R_{z_{kj}}(\Phi, t_0) = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k, \end{cases} & (3.44) \end{cases}$$

where j, $k = 1, \ldots, n$ and $R_A(\Phi, t)$ satisfies (3.8).

By $(3.43)_{1k}$ and (3.8) we infer that (for $\Phi \in \mathscr{A}_{N_1}$)

$$\|R_{z_{1k}}(\Phi_\varepsilon, t)\|_K \leqslant \exp\left(\left\| \int_{t_0}^{t} R_{A_{11}}(\Phi_\varepsilon, t)\, \mathrm{d}s \right\|_K\right) \tag{3.45}$$

and

$$\|D_r R_{Z_{1K}}(\Phi_\varepsilon, t)\|_K \leqslant c_1 \varepsilon^{-N_1} \quad \text{for} \quad 0 < \varepsilon < \eta_{01} \quad \text{and} \quad k = 1, \ldots, n. \tag{3.46}$$

If $n > 1$, then $(3.43)_{nk}$ and (3.8) imply (for $\Phi \in \mathscr{A}_{N_2}$)

$$\begin{aligned} \|R_{z_{2k}}(\Phi_\varepsilon, t)\|_K &\leqslant \exp\left(\left\| \int_{t_0}^{t} R_{A_{22}}(\Phi_\varepsilon, s)\, \mathrm{d}s \right\|_K\right) \\ &\times \left(1 + \left\| \int_{t_0}^{t} \left(\exp\left(-\int_{t_0}^{s} R_{A_{22}}(\Phi_\varepsilon, \tau)\, \mathrm{d}\tau\right)\right) R_{A_{21}}(\Phi_\varepsilon, s) R_{z_1}(\Phi_\varepsilon, s)\, \mathrm{d}s \right\|_K\right) \\ &\leqslant c_2 \varepsilon^{-N_2} \end{aligned} \tag{3.47}$$

for $0 < \varepsilon < \eta_{02}$ and $N_2 \in \mathbf{N}$.

Hence, by $(3.43)_{2k}$ we have

$$\|D_r R_{z_{2k}}(\Phi_\varepsilon, t)\|_K \leqslant \tilde{c}_2 \varepsilon^{-N_2'}, \quad 0 < \varepsilon < \tilde{\eta}_{02}, \quad N_2' \in \mathbf{N}. \tag{3.48}$$

We show by induction that

(3.49)
$$\|R_{z_{\overline{m}k}}(\Phi_\varepsilon,t)\|_K \leqslant c_{\overline{m}}\varepsilon^{-N_{\overline{m}}}$$

and

(3.50)
$$\|D_r R_{z_{\overline{m}k}}(\Phi_\varepsilon,t)\|_K \leqslant \tilde{c}_{\overline{m}}\varepsilon^{-N'_{\overline{m}}}$$

for $0 < \eta < \tilde{\eta}_0$, $N_{\overline{m}}, N'_{\overline{m}} \in \mathbf{N}$, $\overline{m} = 1, \ldots, n$ and $k = 1, \ldots, n$.
Therefore

(3.51)
$$R_Z(\Phi,t) \in \mathscr{E}_M^{n\times n}[\mathbf{R}^1].$$

If Z denotes the class of $R_Z(\Phi,t)$ in $\mathscr{G}^{n\times n}(\mathbf{R}^1)$, then Z is a solution of the problem (3.2)–(3.3). Now we shall prove the uniqueness of solution as the problem (3.2)–(3.3). Let $Y \in \mathscr{G}^{n\times n}(\mathbf{R}^1)$ be another solution of the problem (3.2)–(3.3). Then the relations (3.33)–(3.5) are valid. We set

(3.52)
$$R_U(\Phi_\varepsilon,t) = R_Z(\Phi_\varepsilon,t) - R_Y(\Phi_\varepsilon,t).$$

Using $(3.43)_{1k}$ and (3.33)–(3.35) we conclude that there is $N_{1k} \in \mathbf{N}$ such that for $q \geqslant N_{1k}$ and $\Phi \in \mathscr{A}_q$

(3.53)
$$\|R_{u_{1k}}(\Phi_\varepsilon,t)\|_K \leqslant q_{1k}\varepsilon^{\alpha(q)-N_{1k}} \exp\left(\left\|\int_{t_0}^t R_{A_{11}}(\Phi_\varepsilon,s)\,\mathrm{d}s\right\|_K\right),$$

where

(3.54)
$$R_U(\Phi_\varepsilon,t) = \big(R_{u_{kj}}(\Phi_\varepsilon,t)\big), \quad 0<\eta<\eta_0 \quad \text{and} \quad q_{1k} \in \mathbf{R}^1.$$

In view of (3.33) and $(3.43)_{1k}$ we have (for $q \geqslant N_{1k}$ and $\Phi \in \mathscr{A}_q$)

(3.55)
$$\|D_r R_{u_{1k}}(\Phi_\varepsilon,t)\|_K \leqslant \tilde{g}_{1k}\varepsilon^{\alpha(q)-N'_{1k}}$$

for small ε.
Thus

(3.56)
$$R_{u_{1k}} \in \mathscr{N}[\mathbf{R}^1].$$

Similarly (if $n > 1$)

(3.57)
$$\|R_{u_{2k}}(\Phi_\varepsilon,t)\|_K \leqslant g_{2k}\varepsilon^{\alpha(q)-N_{2k}},$$

where $N_{2k} \in \mathbf{N}$, $g_{2k} \in \mathbf{R}^1$ and $K = 1, \ldots, n$.

Taking into account (3.57) and $(3.43)_{2k}$ we infer that

$$\|D_r R_{u_{2k}}(\Phi_\varepsilon, t)\|_K \leqslant \tilde{g}_{2k} \varepsilon^{\alpha(q)-N'_{2k}} \tag{3.58}$$

for small ε, $g_{2k} \in \mathbf{R}^1$ and $N'_{2k} \in \mathbf{N}$. So

$$R_{2k}(\Phi, t) \in \mathcal{N}[\mathbf{R}^1]. \tag{3.59}$$

Finally, by induction we get

$$\|R_{u_{jk}}(\Phi_\varepsilon, t)\|_K \leqslant g_{jk} \varepsilon^{\alpha(q)-N_{jk}}$$

and

$$\|D_r R_{u_{jk}}(\Phi_\varepsilon, t)\|_K \leqslant \tilde{g}_{jk} \varepsilon^{\alpha(q)-N'_{jk}} \tag{3.60}$$

for small ε, N_{jk}, $N'_{jk} \in \mathbf{N}$ and $j, k = 1, \ldots, n$, which yields $R_U \in \mathcal{N}^{n \times n}[\mathbf{R}^1]$, and this completes the proof of Theorem 3.3. □

Theorem 3.5. *Let the assumptions* (3.4)–(3.5) *be fulfilled. Then the problem*

$$\begin{cases} x'(t) = \mathrm{i}A(t)x(t) & (3.0)' \\ x(t_0) = 0, \qquad t_0 \in \mathbf{R}^1 & (3.1)' \end{cases}$$

has only the trivial solution in $\mathcal{G}^n(\mathbf{R}^1)$.

P r o o f. If x is a solution of the problem (3.0)′–(3.1)′ in $\mathcal{G}^n(\mathbf{R}^1)$, then

$$D_1 R_x(\Phi, t) = \mathrm{i} R_A(\Phi, t) R_x(\Phi, t) + R_m(\Phi, t), \tag{3.61}$$

where

$$R_m(\Phi, t) \in \mathcal{N}^n[\mathbf{R}^1], \quad \Phi \in \mathcal{A}_1 \quad \text{and} \quad R_A(\Phi, t) \text{ satisfies (3.5)}. \tag{3.62}$$

Let

$$\tilde{R}_m(\Phi, t_0) \in \mathcal{I}^n. \tag{3.63}$$

Equalities (3.61)–(3.63) yield

$$R_x(\Phi, t) = R_Z(\Phi, t) R_c(\Phi, t), \tag{3.63'}$$

where $R_Z(\Phi, t)$ is a solution of the problem (3.9)–(3.10) and

$$(3.64) \qquad \begin{cases} D_1 R_c(\Phi, t) = \left(R_Z(\Phi, t)\right)^{-1} R_m(\Phi, t) \\ R_c(\Phi, t_0) = \tilde{R}_m(\Phi, t_0). \end{cases}$$

By (3.16), (3.62) and (3.64) we have

$$(3.65) \qquad D_1 R_c(\Phi, t) \in \mathscr{E}_M^n[\mathbf{R}^1]$$

and

$$(3.66) \qquad D_1 R_c(\Phi, t) \in \mathscr{N}^n[\mathbf{R}^1].$$

On the other hand,

$$(3.67) \qquad R_c(\Phi, t) = \int_{t_0}^{t} \left(R_Z(\Phi, x)\right)^{-1} R_m(\Phi, s)\, \mathrm{d}s + \tilde{R}_m(\Phi, t_0),$$

therefore

$$(3.68) \qquad R_c(\Phi, t) \in \mathscr{E}_M^n[\mathbf{R}^1]$$

and

$$(3.69) \qquad R_c(\Phi, t) \in \mathscr{N}^n[\mathbf{R}^1].$$

Using (3.63) and (3.68)–(3.69) we deduce that

$$(3.70) \qquad R_x(\Phi, t) \in \mathscr{N}^n[\mathbf{R}^1],$$

which completes the proof of Theorem 3.5. □

Theorem 3.6. *We assume that* $A \in \mathscr{G}^{n\times n}(\mathbf{R}^1)$ *and at least one of the conditions* (3.6)–(3.8) *is satisfied. Then the problem*

$$\begin{matrix} (3.2)' \\ (3.3)' \end{matrix} \qquad \begin{cases} x'(t) = A(t)x(t) \\ x(t_0) = 0 \end{cases}$$

has only the trivial solution in $\mathscr{G}^n(\mathbf{R}^1)$.

In the case (3.6) the proof of Theorem 3.6 is similar the to proof of Theorem 3.5. To this purpose we examine the equality

$$(3.71) \qquad D_1 R_x(\Phi, t) = R_A(\Phi, t) R_x(\Phi, t) + R_m(\Phi, t)$$

where $R_m(\Phi,t) \in \mathcal{N}^n[\mathbf{R}^1]$, $\Phi \in \mathcal{A}_1$ and $R_A(\Phi,t)$ has the property (3.6). Next, we show that $R_x(\Phi,t) \in \mathcal{N}^n[\mathbf{R}^1]$. Now we shall prove Theorem 3.6 in the cases (3.7)–(3.8). We observe that

$$(3.72) \qquad \big(R_Z(\Phi,t)\big)^{-1} \in \mathcal{E}_M^{n\times n}[\mathbf{R}^1],$$

where $R_Z(\Phi,t)$ is a solution of the problem (3.26)–(3.27).

Indeed, by the classical results of the theory of differential equations we have

$$(3.73) \qquad D_1\big(\big(R_Z(\Phi,t)\big)^{-1}\big) = -\big(R_Z(\Phi,t)\big)^{-1}\big(R_A(\Phi,t)\big)^{*}.$$

Hence we can obtain similar estimates for elements of the matrix $\big(R_Z(\Phi_\varepsilon,t)\big)^{-1}$ as for elements of the matrix $R_Z(\Phi_\varepsilon,t)$ in relations (3.31)–(3.32) and (3.46)–(3.50). Applying relations (3.63)–(3.69) and (3.71)–(3.72) we can prove Theorem 3.6 analogously as in the cases (3.5)–(3.6), which implies our assertion.

Theorem 3.7. *We assume that conditions* (3.4)–(3.5) *are satisfied. Then every solution x of the equation* (3.0)′ *has a representation*

$$(3.74) \qquad x = Zc,$$

where Z is a solution of the problem (3.0)–(3.1), $c = (c_1, \ldots, c_n)^T$ *and c_j are constant generalized functions on $\mathbf{R}^1$ for $j = 1, \ldots, n$.*

Theorem 3.8. *Let $A \in \mathcal{G}^{n\times n}(\mathbf{R}^1)$ and let at least one of the assumptions* (3.6)–(3.8) *be satisfied. Then every solution x of the equation* (3.2)′ *has a representation* (3.74), *where Z is a solution of the problem* (3.2)–(3.3).

Proofs of Theorems 3.7–3.8. Let x be a solution of the equation (3.0)′ of (3.2)′. Then, by arguments similar to those given in the proof of Theorem 3.5, we have (for a fixed $t_0 \in \mathbf{R}^1$)

$$R_x(\Phi,t) = R_Z(\Phi,t)R_c(\Phi,t), \quad D_1R_c(\Phi,t) = \big(R_Z(\Phi,t)\big)^{-1}R_m(\Phi,t),$$

where

$$R_c(\Phi,t_0) = R_{x(t_0)}(\Phi) \quad \text{and} \quad R_m(\Phi,t) \in \mathcal{N}^n[\mathbf{R}^1].$$

Hence, taking into account (3.66) and Theorem 2.2.1 from [4], we obtain (3.74). □

Remark 3.3. If the matrix $A \in \mathcal{G}^{n\times n}(\mathbf{R}^1)$ has not properties (3.6)–(3.8), then the solution $R_Z(\Phi,t)$ of the problem (3.26)–(3.27) need not be moderate. In fact, let δ denote the generalized function on $\mathbf{R}^1$ (the delta Dirac distribution) which admits as a representative the function $R_\delta(\Phi, t) = \Phi(-t)$, where $\Phi \in \mathscr{A}_1$. Then, taking into account the problem

$$x'(t) = \big(\Phi(-t)\big)' x(t), \quad x(-1) = 1,$$

we have

$$\hat{R}_x(\Phi, t) = x(t) = \exp\big(\Phi(-t) - \Phi(-1)\big). \tag{3.75}$$

Since there is $\Phi \in \mathscr{A}_q$ such that $\Phi(0) = 1$ for $q = 1, \dots$ (see [4], pp. 7–11), therefore (for small $\varepsilon > 0$)

$$\hat{R}_x(\Phi_\varepsilon, 0) = \exp(\varepsilon^{-1}) \quad \text{and} \quad \exp\big(\Phi(-t)\big) \notin \mathscr{E}_M[\mathbf{R}^1].$$

It is not difficult to show that the problem

$$\begin{cases} x' = (\delta^2)' x \\ x(-1) = 1 \end{cases} \tag{3.76}$$

has not solution in $\mathscr{G}(\mathbf{R}^1)$. Indeed, if the problem (3.76) has a solution $x \in \mathscr{G}(\mathbf{R}^1)$, then

$$D_1 R_x(\Phi, t) = \big((\Phi(-t))^2\big)' R_x(\Phi, t) + R_m(\Phi, t)$$

where $R_x(\Phi, -1) - 1 \in \mathscr{I}$ and $R_m(\Phi, t) \in \mathscr{N}[\mathbf{R}^1]$.

Hence we get

$$\begin{aligned} (3.77)\ & R_x(\Phi_\varepsilon, 0) \\ &= \exp\big(\varepsilon^{-2}\Phi^2(0)\big)\Big(\int_{-1}^{0}\Big(\exp\Big(-\varepsilon^{-2}\Phi^2\Big(\frac{-s}{\varepsilon}\Big)\Big)\Big) R_m(\Phi_\varepsilon, s)\,\mathrm{d}s + 1 + \tilde{R}_m(\Phi_\varepsilon)\Big) \end{aligned}$$

for $\tilde{R}_m(\Phi) \in \mathscr{I}$ and for a suitably small $\varepsilon > 0$.

Let $\Phi(0) = 1$ and $\Phi \in \mathscr{A}_q$ for $q = 1, \dots$. Then there is $N \in \mathbf{N}$ such that for $q \geqslant N$ and $\Phi \in \mathscr{A}_q$

$$\Big(\exp\Big(\frac{-1}{\varepsilon^2}\Phi^2\Big(\frac{-s}{\varepsilon}\Big)\Big)\Big)|R_m(\Phi_\varepsilon, s)| \leqslant \frac{1}{4} \quad \text{and} \quad |\tilde{R}_m(\Phi_\varepsilon)| \leqslant \frac{1}{4}$$

hold for $0 < \varepsilon < \eta_0$ and $s \in K$.

So, by virtue of (3.77) we obtain

$$\frac{1}{2}\exp(\varepsilon^{-2}) \leqslant R_x(\Phi_\varepsilon, 0) \leqslant \frac{3}{2}\exp(\varepsilon^{-2}).$$

Consequently, $R_x(\Phi_\varepsilon, t) \notin \mathscr{E}_M[\mathbf{R}^1]$, which is impossible. Thus the problem (3.76) has no solution in $\mathscr{G}(\mathbf{R}^1)$.

Remark 3.4. We define the matrices A_1, A_2, A_3 and A_4 by

$$A_1(t) = \begin{pmatrix} \delta^{(k)}(t) & \delta^{(q)}(t) \\ \delta^{(q)}(t) & \delta^{(q)}(t+1) \end{pmatrix}, \quad A_2(t) = \begin{pmatrix} 0 & \delta^{(q)}(t) \\ -\delta^{(q)}(t) & 0 \end{pmatrix},$$

$$A_3(t) = \begin{pmatrix} \delta(t) & 2\delta(t-1) \\ 3\delta(t+1) & 4\delta(t+3) \end{pmatrix}, \quad A_4(t) = \begin{pmatrix} \delta(t) & 0 \\ p(t) & \delta(t+1) \end{pmatrix},$$

where $p \in \mathscr{G}(\mathbf{R}^1)$ and δ denotes the delta Dirac distribution. It is not difficult to verify that the matrix A_1 has property (3.5), the matrix A_2 has property (3.6), the matrix A_3 has property (3.7) and the matrix A_4 has property (3.8).

Now we sill give two; theorems on the independence of solutions of the problems (3.0)–(3.1) and (3.2)–(3.3) on representatives of the matrix A.

Theorem 3.9. *We assume that*

(3.78) $$A \in \mathscr{G}^{n\times n}(\mathbf{R}^1),$$

(3.79) *the matrix A fulfils at least one of the conditions* (3.5)–(3.6),

(3.80) *$R_Z(\Phi, t)$ is a solution of the problem* (3.9)–(3.10),

(3.81) $$R_m(\Phi, t) \in \mathscr{N}^{n\times n}[\mathbf{R}^1],$$

$\tilde{R}_Y(\Phi, t)$ is a solution of the problem

(3.82) $$\begin{cases} Y'(t) &= i\big(R_A(\Phi,t) + R_m(\Phi,t)\big)Y(t) \\ Y(t_0) &= I, \qquad t_0 \in \mathbf{R}^1. \end{cases}$$

Then $\tilde{R}_Y(\Phi, t) \in \mathscr{E}_M^{n\times n}[\mathbf{R}^1]$.

Theorem 3.10. *We assume that*

(3.83) *contitions* (3.78) *and* (3.81) *are satisfied,*

(3.84) *a matrix A fulfils at least one of the properties* (3.7)–(3.8),

(3.85) *$R_Z(\Phi, t)$ is a solution of the problem* (3.26)–(3.27),

$\tilde{R}_Y(\Phi, t)$ is a solution of the problem

(3.86) $$\begin{cases} Y'(t) &= \big(R_Y(\Phi,t) + R_m(\Phi,t)\big)Y(t) \\ Y(t_0) &= I, \qquad t_0 \in \mathbf{R}^1. \end{cases}$$

Then $\tilde{R}_Y(\Phi,t) \in \mathscr{E}_M^{n\times n}[\mathbf{R}^1]$.

P r o o f o f T h e o r e m s 3.9 a n d 3.10. If A fulfills at least one of the conditions (3.5)–(3.8), then $R_Z(\Phi,t) \in \mathscr{E}_M^{n\times n}[\mathbf{R}^1]$ and $\big(R_Z(\Phi,t)\big)^{-1} \in \mathscr{E}_M^{n\times n}[\mathbf{R}^1]$. Let $\tilde{R}_Y(\Phi,t)$ be a solution of the problem (3.82) or (3.86). Then

$$(3.87)\quad \tilde{R}_Y(\Phi_\varepsilon,t) = R_Z(\Phi_\varepsilon,t) + \int_{t_0}^{t} R_Z(\Phi_\varepsilon,t)\big(R_Z(\Phi_\varepsilon,s)\big)^{-1} R_m(\Phi_\varepsilon,s)\tilde{R}_Y(\Phi_\varepsilon,s)\,ds.$$

Hence, be the Gronwall inequality, we get $\tilde{R}_Y(\Phi,t) \in \mathscr{E}_M^{n\times n}[\mathbf{R}^1]$. □

4. System of nonhomogeneous differential equations

Theorem 4.1. *We assume that conditions* (3.4)–(3.5) *are satisfied and* $f \in \mathscr{G}^n(\mathbf{R}^1)$. *Then the problem*

$$\begin{cases} x'(t) = \mathrm{i}A(t)x(t) + f(t) & (4.0)\\ x(t_0) = x_0, \qquad t_0 \in \mathbf{R}^1,\ x_0 \in \bar{\mathbf{C}}^n & (4.1)\end{cases}$$

has exactly one solution in $\mathscr{G}^n(\mathbf{R}^1)$.

Theorem 4.2. *We assume that* $A \in \mathscr{G}^{n\times n}(\mathbf{R}^1)$, $f \in \mathscr{G}^n(\mathbf{R}^1)$, *and at least one of conditions* (3.6)–(3.8) *is satisfied. Then the problem*

$$\begin{cases} x'(t) = A(t)x(t) + f(t) & (4.2)\\ x(t_0) = x_0, \qquad t_0 \in \mathbf{R}^1,\ x_0 \in \bar{\mathbf{C}}^n & (4.3)\end{cases}$$

has exactly one solution in $\mathscr{G}^n(\mathbf{R}^1)$.

P r o o f s o f T h e o r e m s 4.1–4.2. The uniqueness of solutions of the problems (4.0)–(4.1) and (4.2)–(4.3) follows from Theorem 3.5 and 3.6. Now, we shall prove existence of a solution of the problem (4.0)–(4.1). To this purpose we consider the problem

$$\begin{cases} x'(t) = \mathrm{i}R_A(\Phi,t)x(t) + R_f(\Phi,t) & (4.0)'\\ x(t_0) = R_{x_0}(\Phi), & (4.1)'\end{cases}$$

where $R_A(\Phi,t)$ satisfies (3.5). Let $R_x(\Phi,t)$ be a solution of the problem (4.0)′–(4.1)′. Then

$$(4.4)\qquad R_x(\Phi,t) = R_Z(\Phi,t)R_v(\Phi,t),$$

where

$$
\begin{cases} D_1 R_v(\Phi,t) = \left(R_Z(\Phi,t)\right)^{-1} R_f(\Phi,t) & (4.5)\\ R_v(\Phi,t_0) = R_{x_0}(\Phi) & (4.6)\end{cases}
$$

and $R_Z(\Phi,t)$ is a solution of the problem (3.9)–(3.10).

By (3.16) and (4.5) we get

$$
(4.7)\qquad \left(R_Z(\Phi,t)\right)^{-1} \in \mathscr{E}_M^{n\times n}[\mathbf{R}^1], \quad D_1 R_v(\Phi,t) \in \mathscr{E}_M^{n\times n}[\mathbf{R}^1]
$$

and

$$
(4.8)\qquad R_v(\Phi,t) = \int_{t_0}^{t} \left(R_Z(\Phi,s)\right)^{-1} R_f(\Phi,s)\,\mathrm{d}s + R_{x_0}(\Phi).
$$

Thus

$$
(4.9)\qquad R_v(\Phi,t) \in \mathscr{E}_M^{n}[\mathbf{R}^1],
$$

If we define x as the class of $R_Z(\Phi,t)R_v(\Phi,t)$, then x is a solution of the problem (4.0)–(4.1), which completes the proof of Theorem 4.1. □

The proof of existence of a solution of the problem (4.2)–(4.3) is similar to the proof of Theorem 4.1. Indeed let, $R_x(\Phi,t)$ be a solution of the problem

$$
\begin{cases} x'(t) = R_A(\Phi,t)x(t) + R_f(\Phi,t) & (4.2)'\\ x(t_0) = R_{x_0}(\Phi). & (4.3)'\end{cases}
$$

Then $R_x(\Phi,t)$ has the properties (4.4)–(4.9), which completes the proof of the theorem.

Theorem 4.3. *We assume that conditions* (3.4)–(3.5) *are satisfied and* $f \in \mathscr{G}^n(\mathbf{R}^1)$. *Moreover, we assume that Q is a solution of equation* (4.0). *Then every solution x of equation* (4.0) *has a representation*

$$
(4.10)\qquad x = Zc + Q,
$$

where Z is a solution of the problem (3.0)–(3.1), $c = (c_1, \ldots, c_n)^T$ *and c_j are generalized constant functions on $\mathbf{R}^1$ for $j = 1, \ldots, n$.*

Theorem 4.4. *Let* $A \in \mathscr{G}^{n\times n}(\mathbf{R}^1)$, $f \in \mathscr{G}^n(\mathbf{R}^1)$. *Let Q be a solution of equation* (4.2), *and let at least one of conditions* (3.6)–(3.8) *be satisfied. Then every solution*

x of equation (4.2) *has a representation* (4.10), *where Z is a solution of the problem* (3.2)–(3.3).

Proofs of Theorems 4.3 and 4.4. We see that x defined by (4.10) is a solution of the equation (4.0) or (4.2). Next, we consider equalities

$$Q' = \mathrm{i}AQ + f \tag{4.11}$$

and

$$Q' = AQ + f. \tag{4.12}$$

In view of the relations (4.0), (4.11), (4.2) and (4.12) we have

$$(x - Q)' = \mathrm{i}A(x - Q)$$

or

$$(x - Q)' = A(x - Q).$$

Applying Theorem 4.1 and 4.2 to the last equalities we get

$$(x - Q) = Zc,$$

which completes the proofs. □

Remark 4.1. If f is a piecewise continuous function on $\mathbf{R}^1$, we define $R_f(\Phi, t)$ as follows:

$$R_f(\Phi, t) = \int_{-\infty}^{\infty} f(t + u)\Phi(u)\, du \quad \text{(see [4])}, \tag{4.13}$$

where $\Phi \in \mathscr{A}_1$. Obviously, $R_f(\Phi, t) \in \mathscr{E}_M[\mathbf{R}^1]$. Let f_1, f_2 be continuous functions defined by

$$f_1(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ t, & \text{if } t > 0 \end{cases} \tag{4.14}$$

and

$$f_2(t) = \begin{cases} t, & \text{if } t \leqslant 0, \\ 0, & \text{if } t > 0. \end{cases} \tag{4.15}$$

Then the classical product is 0. Their product in the Colombeau algebra $\mathscr{G}(\mathbf{R}^1)$ is non zero (see [4], p. 16). On the other hand, if g_1, $g_2 \in C^\infty$, then the classical product and the product in $\mathscr{G}(\mathbf{R}^1)$ give rise to the same element of $\mathscr{G}(\mathbf{R}^1)$. Hence we deduce:

Corollary 4.1. *Let all elements of a matrix A and of a vector f be C^∞ functions on $\mathbf{R}^1$. Moreover, let $x_0 \in \mathbf{C}^n$. Then the classical and the generalized solutions (i.e. solutions in the Colombeau algebra) of the problem (4.2)–(4.3) give rise to the same elements of $\mathcal{G}^n(\mathbf{R}^1)$.*

Indeed, let x and y be respectively a classical solution and generalized solution of the problem (4.2)–(4.3) (because the matrix A fulfills condition (3.7)). Then

$$u'(t) = A(t)u(t), \quad u(t_0) = 0$$

where $u(t) = x(t) - y(t)$. Hence, by Theorem 3.6 we infer that $x = y$.

Corollary 4.2. *If all elements of the matrix A are piecewise continuous functions, then the matrix A has the property (3.7).*

Example 4.1. If necessary, we denote the product in $\mathcal{G}(\mathbf{R}^1)$ by $\odot$ to avoid confusion with the classical product. We consider the equations

$$x'(t) = f_1(t)x(t) + f_2'(t), \tag{4.16}$$

$$x'(t) = f_1(t) \odot x(t) + f_2'(t), \tag{4.16$'$}$$

where f_1 and f_2 are defined by (4.14)–(4.15).

If is easy to show that $x = f_2$ is a classical solution of the equation (4.16) (in the Carathéodory sense). On the other hand, $x = f_2$ is not a solution of the equation (4.16)′ in the Colombeau algebra $\mathcal{G}(\mathbf{R}^1)$ (because $f_1 \odot f_2$ is not zero in $\mathcal{G}(\mathbf{R}^1)$).

Remark 4.2. It is known that every distribution is moderate (see [5]). On the other hand, L. Schwartz proves in [19] that there does not exist an algebra $\tilde{A}$ such that the algebra $\mathcal{C}(\mathbf{R}^1)$ of continuous functions on $\mathbf{R}^1$ is a subalgebra of $\tilde{A}$, the function 1 is the unit element in $\tilde{A}$, elements of $\tilde{A}$ are "C^∞" with respect to a derivation which coincides with the usual one in $C^1(\mathbf{R}^1)$, and such that the usual formula for the derivation of a product holds. As a consequence multiplication in $\mathcal{G}(\mathbf{R}^1)$ does not coincide with usual multiplication of continuous functions. To repair the consistency problem for multiplication we give the definition introduced by J. F. Colombeau in [5].

An element u of $\mathcal{G}(\mathbf{R}^1)$ is said to admit a member $w \in \mathcal{D}'(\mathbf{R}^1)$ as the associated distribution, if it has a representative $R_u(\Phi_\varepsilon, t)$ with the following property: for every $\psi \in \mathcal{D}(\mathbf{R}^1)$ there is $N \in \mathbf{N}$ such that for every $\Phi \in \mathcal{A}_N(\mathbf{R}^1)$ we have

$$\int_{-\infty}^{\infty} R_u(\Phi_\varepsilon, t)\psi(t)\,dt \to w(\psi) \quad \text{as} \quad \varepsilon \to 0. \tag{4.17}$$

If u admits an associated distribution w, then this associated distribution is unique (see [4], p. 64).

R e m a r k 4.4. The authors of [3] have considered a delta sequence $\delta_n \in \mathscr{D}(\mathbf{R}^1)$ with the following properties:

(i) there is a sequence of positive numbers α_n converging to 0 such that $\delta_n(t) = 0$ for $|t| \geqslant \alpha_n$ and $\int_{-\infty}^{\infty} \delta_n(t)\,dt = 1$,

(ii) for every positive integer k there is a positive integer M_k such that

$$\alpha_n^k \int_{-\infty}^{\infty} |\delta_n^{(k)}(t)|\,dt \leqslant M_k \quad \text{for} \quad n \in \mathbf{N}.$$

Definition 4.1. Let S and U be two given distributions on $\mathbf{R}^1$. If for every delta sequence the product $(S * \delta_n)(U * \delta_n)$ admits a limit in $\mathscr{D}'(\mathbf{R}^1)$ if $n \to \infty$, we define $SU \in \mathscr{D}'(\mathbf{R}^1)$ as this limit (the asterisk denotes the convolution, see [3], p. 242).

The following theorem has been proved in [5] (p. 107):

Theorem C. *If the product SU exists, then the product $S \odot U \in \mathscr{G}(\mathbf{R}^1)$ admits an associated distribution which is SU.*

Definition 4.2. We say that $x \in \mathscr{G}^n(\mathbf{R}^1)$ is a weak solution of the system (1.0) if $\left(x_k' - \sum_{j=1}^{n} A_{kj} \odot x_j - f_k\right) \in \mathscr{G}(\mathbf{R}^1)$ is associated to the zero distribution for $k = 1, \ldots, n$ (see [4]).

Definition 4.3. Let $A_{kj}, x_j, f_k \in \mathscr{D}'(\mathbf{R}^1)$ and let $A_{kj}x_j$ denote products in the sense of Definition 4.1 ($k, j = 1, \ldots, n$). Then we say that $x = (x_1, \ldots, x_n)$ is a distributional solution of the system (1.0).

Theorem 4.5. *Let x be a distributional solution of the system* (1.0). *Then $x \in \mathscr{G}^n(\mathbf{R}^1)$ is a weak solution of the system* (1.0).

P r o o f. Let

$$\begin{aligned} R_{A_{kj}}(\Phi_\varepsilon, t) &= (A_{kj} * \beta_\varepsilon)(t), \\ R_{x_j}(\Phi_\varepsilon, t) &= (x_j * \beta_\varepsilon)(t), \\ R_{f_k}(\Phi_\varepsilon, t) &= (f_k * \beta_\varepsilon)(t), \\ y_k(\Phi_\varepsilon, t) &= R_{x_k'}(\Phi_\varepsilon, t) - \sum_{j=1}^{n} R_{A_{kj}}(\Phi_\varepsilon, t) R_{x_j}(\Phi_\varepsilon, t) - R_{f_k}(\Phi_\varepsilon, t), \end{aligned}$$

where

$$\beta_\varepsilon(t) = \varepsilon^{-1}\Phi\left(\frac{-t}{\varepsilon}\right), \quad \Phi \in \mathscr{A}_1(\mathbf{R}^1) \quad \text{and} \quad k = 1, \ldots, n. \tag{4.18}$$

Then $y_k(\Phi_\varepsilon, t) \in \mathscr{E}_M[\mathbf{R}^1]$ and $y_k(\Phi_\varepsilon, t)$ is a representative of $(x'_k - \sum_{j=1}^{n} A_{kj} \odot x_j - f_k)$ for $k = 1, \ldots, n$. On the other hand, Theorem C and the convergence in $\mathscr{D}'(\mathbf{R}^1)$ imply

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} y_k(\Phi_\varepsilon, t)\psi_k(t)\, dt = 0,$$

where $\psi_k \mathscr{D}(\mathbf{R}^1)$ and $k = 1, \ldots, n$. □

The last inequality completes the proof of Theorem 4.5.

Remark 4.5. By Theorem 4.2 (property (3.7)) we observe that if all elements of a matrix A and of a vector f are locally integrable functions and $x_0 \in \mathbf{C}^n$, then the problem (4.2)–(4.3) has exactly one solution $x \in \mathscr{G}^n(\mathbf{R}^1)$. By virtue of Theorem 4.5 we deduce that x admits an associated vector distribution $w = (w_1, \ldots, w_n)$ use k-th component is

$$w_k(\psi_k) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} R_{x_k}(\Phi_\varepsilon, t)\psi_k(t)\, dt,$$

where $\psi_k \in \mathscr{D}(\mathbf{R}^1)$, $k = 1, \ldots, n$, $R_x(\Phi_\varepsilon, t)$ is a solution of the problem

$$\begin{cases} x'(t) = R_A(\Phi_\varepsilon, t)x(t) + R_f(\Phi_\varepsilon, t) \\ x(t_0) = x_0, \qquad t_0 \in \mathbf{R}^1,\ x_0 \in \mathbf{C}^n \end{cases}$$

and $R_A(\Phi_\varepsilon, t) = \int_{-\infty}^{\infty} A(t + \varepsilon u)\Phi(u)\, du$, $R_f(\Phi_\varepsilon, t) = \int_{-\infty}^{\infty} f(t + \varepsilon u)\Phi(u)\, du$.

Remark 4.6. It was shown in [13] that the problem

$$\begin{cases} x'(t) = 2\delta(t)x(t) \\ x(-1) = 0 \end{cases}$$

has a solution $x = cH$, where δ denotes the Dirac delta distribution, H denotes the Heaviside function and c denotes a constant. On the other hand, the problem

$$\begin{cases} x'(t) = 2\delta(t) \odot x(t) \\ x(-1) = 0 \end{cases} \tag{4.19}$$

has only the trivial solution in $\mathscr{G}(\mathbf{R}^1)$ (Theorem 3.3). Hence $x = H$ is not a solution of the problem (4.19). It is not difficult to observe that $x = 0$ and $x = H$ are weak solutions of the problem (4.19).

Remark 4.7. In many papers conditions are given which guarantee existence of distributional solutions of ordinary differential equations (for example in [7], [11], [12]–[16]). Non-continuous solutions of ordinary differential equations can be considered in an other way (for example, see [8], [9], [10], [17], [18]).

Remark 4.8. Let $A_{kj} = B'_{kj}$, where B_{kj} are functions of finite variation for k, $j = 1, \ldots, n$ and the derivative is meant in the distributional sense. Moreover, let

$$R_{A_{kj}}(\Phi_\varepsilon, t) = (B_{kj} * \delta_\varepsilon)'(t), \tag{4.20}$$

where δ_ε is defined by (4.18). Then [1] yields

$$\int_K |R_{A_{kj}}(\Phi_\varepsilon, t)| \, \mathrm{d}t \leqslant c < \infty \quad \text{for} \quad \Phi \in \mathscr{A}_1 \quad \text{and} \quad k, j = 1, \ldots, n.$$

Hence the matrix A has the property (3.7).

It is worth noting that if A is a matrix such that $A \in \mathscr{G}^{n\times n}(\mathbf{R}^1)$, $A_{jj} = B'_{jj}$, B_{jj} are continuous functions, $j = 1, \ldots, n$, the derivative is meant in the distributional sense, $A_{kj} = 0$ for $k < j$ and $n > 1$, then the matrix A has the property (3.8). (The last fact follows by (4.20).)

Remark 4.9. The definition of generalized functions on an open interval $(a, b) \subset \mathbf{R}^1$ is almost the same as the definition in the whole $\mathbf{R}^1$ (see [4]). In this paper we have proved theorems on generalized solutions of linear differential equations in the case $(a, b) = (-\infty, \infty)$. It is not difficult to observe that the above proved theorems are also true in the case when generalized functions A_{kj}, f_k and x_k are considered on an interval (a, b) for $k, j = 1, \ldots, n$. To this purpose it is necessary to formulate the properties (3.4)–(3.8) on the interval (a, b).

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