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ROUTE SYSTEMS OF A CONNECTED GRAPH

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Summary. The concept of a route system was introduced by the present author in [3]. Route systems of a connected graph G generalize the set of all shortest paths in G. In this paper some properties of route systems are studied.

Keywords: route systems, shortest paths, geodetic graphs

AMS classification: 05C12, 05C38

0. Before giving the definition of a route system we need to introduce some auxiliary notions.

Let G be a graph (in the sense of [1], for example, i.e. a finite undirected graph with no loops or multiple edges) with a vertex set $V(G)$. We denote by $\mathcal{W}_N(G)$ the set of all sequences

$$u_0, \ldots, u_i, \tag{0}$$

where $i \geqslant 0$ and $u_0, \ldots, u_i \in V(G)$. Similarly as in [4], instead of (0) we write $u_0 \ldots u_i$. If $v_0, \ldots, v_j \in V(G)$ and $\alpha = v_0 \ldots v_j$, where $j \geqslant 0$, then we put $A\alpha = v_0$, $Z\alpha = v_j$, $\|\alpha\| = j$ and $\bar{\alpha} = v_j \ldots v_0$. If $u_0, \ldots, u_k$, $w_0, \ldots, w_m \in V(G)$, $\beta = u_0 \ldots u_k$ and $\gamma = w_0 \ldots w_m$, where $k, m \geqslant 0$, then we write $\beta\gamma = u_0 \ldots u_k w_0 \ldots w_m$. We denote by $*$ the empty sequence in the sense that $\alpha * = \alpha = * \alpha$ for every $\alpha \in \mathcal{W}_N(G)$, $** = *$ and $\bar{*} = *$. Put $\mathcal{W}(G) = \mathcal{W}_N(G) \cup \{*\}$. If $\mathcal{M} \subseteq \mathcal{W}_N(G)$ and $u, v \in V(G)$, then we denote

$$\mathcal{M}_{(u,v)} = \{\alpha \in \mathcal{M};\ A\alpha = u \text{ and } Z\alpha = v\}$$

and

$$\mathcal{M}^{(u,v)} = \{\alpha \in \mathcal{M};\ \text{there exist } \beta, \gamma, \delta \in \mathcal{W}(G) \text{ such that } \alpha = \beta\gamma\delta \text{ and } \gamma \in \mathcal{M}_{(u,v)}\}.$$

Let $v_0, \ldots, v_i \in V(G)$, where $i \geqslant 0$; we say that $v_0 \ldots v_i$ is a path in G if the vertices $v_0, \ldots, v_i$ are mutually distinct and the vertices v_j and v_{j+1} are adjacent in G for each integer j, $0 \leqslant j < i$. We denote by $\mathcal{P}(G)$ the set of all paths in G. Let $\alpha \in \mathcal{W}_N(G)$; we say that α is a shortest path in G if $\alpha \in \mathcal{P}(G)$ and $\|\alpha\| \leqslant \|\beta\|$ for every $\beta \in \mathcal{P}(G)$ such that $A\alpha = A\beta$ and $Z\alpha = Z\beta$. We denote by $\mathcal{S}(G)$ the set of all shortest paths in G.

Let G be a connected graph, and let $\mathcal{R} \subseteq \mathcal{P}(G)$. We will say that $\mathcal{R}$ is a *semi-route system* on G in the following Axioms I–IV are fulfilled for arbitrary $u, v \in V(G)$ and α, β, γ, $\delta \in \mathcal{W}(G)$:

I if u and v are adjacent, then $uv \in \mathcal{R}$;

II if $\alpha \in \mathcal{R}$, then $\bar{\alpha} \in \mathcal{R}$;

III if $u\alpha v \in \mathcal{R}$, then $u\alpha \in \mathcal{R}$;

IV if $\alpha u\beta v\gamma$, $u\delta v \in \mathcal{R}$, then $\alpha u\delta v\gamma \in \mathcal{R}$.

Moreover, we say that $\mathcal{R}$ is a *route system* on G if it is a semi-route system on G and the following Axiom V is fulfilled for arbitrary $u, v \in V(G)$:

V there exist $\alpha \in \mathcal{R}$ such that $A\alpha = u$ and $Z\alpha = v$.

Let G be a connected graph. Consider a route system $\mathcal{R}$ on G; if $u, v \in V(G)$, then we denote

$$d_{\mathcal{R}}(u, v) = \min(\|\alpha\|;\ \alpha \in \mathcal{R},\ A\alpha = u \text{ and } Z\alpha = v).$$

It is easy to see that $\mathcal{S}(G)$ is a route system on G. Note that $\mathcal{S}(G)$ is the only route system on G if and only if G is a tree, cf. [3]. Instead of $d_{\mathcal{S}(G)}$ we will write d only. Obviously, if $u, v \in V(G)$, then $d(u, v)$ is the distance between u and v in G.

The following theorem was proved in [4]:

Theorem 0. *Let G be a connected graph, and let $\mathcal{R}$ be a route system on G. Then $\mathcal{R} = \mathcal{S}(G)$ if and only if the following conditions (1)–(3) hold for arbitrary $u, v, x, y \in V(G)$ and $\alpha, \beta \in \mathcal{W}(G)$:*

(1) *if $u\alpha xv \in \mathcal{R}$, then $uv \notin \mathcal{R}$;*

(2) *if $u\alpha xy$, $uv\beta y$, $vu\alpha x \in \mathcal{R}$, then $v\beta yx \in \mathcal{R}$;*

(3) *if xy, $uv\alpha x \in \mathcal{R}$, $u\varphi yx \in \mathcal{R}$ for no $\varphi \in \mathcal{W}(G)$ and $uv\psi y \in \mathcal{R}$ for no $\psi \in \mathcal{W}(G)$, then $v\alpha xy \in \mathcal{R}$.*

1. Let G be a connected graph, and let $\mathcal{R}$ be a semi-route system on G. We say that $\mathcal{R}$ is *geodetic* if the following Axiom VI is fulfilled for arbitrary $u, v \in V(G)$:

$$\text{VI} \quad |\mathcal{R}_{(u,v)}| \leqslant 1.$$

Thus, if $\mathcal{R}$ is a route system on G, then it is geodetic if and only if $|\mathcal{R}_{(u,v)}| = 1$ for every pair of vertices u and v of G.

Example. Let G be a connected graph of diameter two. Put $\mathcal{S} = \mathcal{S}(G)$. For every pair of vertices u and v of distance two in G we choose exactly one path in $\mathcal{S}_{(u,v)}$, say a path α_{uv}, such that $\alpha_{vu} = \bar{\alpha}_{uv}$. Denote

$$\begin{aligned}\mathcal{R} = \{u;\ u \in V(G)\} \cup \\ \cup \{vw;\ v \text{ and } w \text{ are adjacent vertices of } G\} \cup \\ \cup \{\alpha_{xy};\ x, y \in V(G) \text{ and } d(x,y) = 2\}.\end{aligned}$$

It is not difficult to see that $\mathcal{R}$ is a geodetic route system on G.

Let G be a connected graph. Consider a route system $\mathcal{R}$ on G. If $u, v \in V(G)$, then we denote by $N_{\mathcal{R}}(u,v)$ the set of all $w \in V(G)$ such that there exists $\alpha \in \mathcal{W}(G)$ with the property that $uw\alpha \in \mathcal{R}_{(u,v)}$. Similarly as in [3] we denote

$$\#_{\mathcal{R}}(x,y) = \{x\} \cup \{z \in V(G);\ N_{\mathcal{R}}(z,x) - N_{\mathcal{R}}(z,y) \neq \emptyset\}$$

for any $x, y \in V(G)$. The mapping $\#_{\mathcal{R}}$ has its origin in the author's study of mathematical models in semiotics.

It is not difficult to see that if G is a connected graph and $\mathcal{R}$ is a geodetic route system on G, then $\#_{\mathcal{R}}(u,v) = \#_{\mathcal{R}}(v,u)$.

Lemma 1. *Let G be a connected graph, and let $\mathcal{R}$ be a route system on G. Assume that $\mathcal{R}$ is not geodetic. Then there exists a pair of adjacent vertices u and v of G such that $\#_{\mathcal{R}}(u,v) \neq \#_{\mathcal{R}}(v,u)$.*

Proof. Since $\mathcal{R}$ is not geodetic, there exist $v, w \in V(G)$ such that $|\mathcal{R}_{(w,v)}| \geqslant 2$ and $|\mathcal{R}_{(x,y)}| = 1$ for any $x, y \in V(G)$ with the property that $d_{\mathcal{R}}(x,y) < d_{\mathcal{R}}(w,v)$. Since $|\mathcal{R}_{(w,v)}| \geqslant 2$, there exist distinct $\alpha, \beta \in \mathcal{R}_{(w,v)}$ such that $\|\alpha\| = d_{\mathcal{R}}(w,v)$. Then α and β have no common vertex different from v and w (otherwise, combining Axioms II and III, we easily get $\alpha = \beta$, which is a contradiction). We distinguish two cases:

1. Let $d_{\mathcal{R}}(w,v) = 1$. Then $\alpha = wv$. Since $\beta \neq \alpha$, there exist $u \in V(G)$ and $\gamma \in \mathcal{W}(G)$ such that $\beta = w\gamma uv$. Axiom IV implies that if $\delta \in \mathcal{R}_{(w,u)}$, then

$\delta v \in \mathcal{R}_{(w,v)}$. Hence $w \notin \#_{\mathcal{R}}(u,v)$. Recall that $\alpha = wv$. We have $wvu \notin \mathcal{R}_{(w,u)}$ (otherwise, Axiom IV would imply that $w\gamma uvu \in \mathcal{R}$, which is a contradiction). Thus $w \in \#_{\mathcal{R}}(v,u)$.

2. Let $d_{\mathcal{R}}(w,v) \geqslant 2$. Then there exist $u \in V(G)$ and $\gamma \in \mathcal{W}(G)$ such that $\alpha = w\gamma uv$. According to Axiom III, $w\gamma u \in \mathcal{R}_{(w,u)}$. It follows from the definition of $d_{\mathcal{R}}$ that $d_{\mathcal{R}}(w,u) < d_{\mathcal{R}}(w,v)$. This implies that $\mathcal{R}_{(w,u)} = \{w\gamma u\}$. Hence, $w \notin \#_{\mathcal{R}}(u,v)$. Clearly, u does not lie on β. Moreover, we see that $\beta u \notin \mathcal{R}_{(w,u)}$. Thus $w \in \#_{\mathcal{R}}(v,u)$, which completes the proof of lemma. □

Combining Lemma 1 with the above observation we get:

Theorem 1. *Let G be a connected graph, and let $\mathcal{R}$ be a route system on G. Then $\mathcal{R}$ is geodetic if and only if $\#_{\mathcal{R}}(u,v) = \#_{\mathcal{R}}(v,u)$ for every pair of vertices u and v of G.*

If G is a connected graph and $u, v \in V(G)$, then instead of $\#_{\mathcal{S}(G)}(u,v)$ we will write $\#(u,v)$. Note that the mapping $\#$ was introduced in [2].

A connected graph G is called geodetic if $\mathcal{S}(G)$ is a geodetic route system on G.

Corollary 1. *A connected graph G is geodetic if and only if $\#(u,v) = \#(v,u)$ for every pair of vertices u and v of G.*

2. In this section we will prove that if G is a connected graph and $\mathcal{R}$ is a route system on G, then there exists a subset of $\mathcal{R}$ which is a geodetic route system on G. In fact, we will prove a more general result for semi-route systems.

If G is a connected graph, then we define $b(G) = |E(G)| - |V(G)| + 1$, where $E(G)$ is the edge set of G.

Theorem 2. *Let G be a connected graph, and let $\mathcal{R}$ be a semi-route system on G. Then there exists a geodetic semi-route system $\mathcal{R}^*$ on G with the properties that $\mathcal{R}^* \subseteq \mathcal{R}$ and*

(4) $$\mathcal{R}^*_{(u,v)} \neq \emptyset \text{ if and only if } \mathcal{R}_{(u,v)} \neq \emptyset$$
for every pair of vertices u and v of G.

Proof. We proceed by induction on $b(G)$. Obviously, $b(G) \geqslant 0$. First, let $b(G) = 0$. Then G is a tree, and therefore, $\mathcal{R} = \mathcal{S}(G)$. We put $\mathcal{R}^* = \mathcal{R}$.

Let now $b(G) \geqslant 1$. Then there exists $a \in E(G)$ such that $G - a$ is connected. Let r and s be the vertices incident with a. Axiom I implies that $\mathcal{R}_{(r,s)} \neq \emptyset$. There exists $\alpha \in \mathcal{R}_{(r,s)}$ such that

(5) $$\|\alpha\| \geqslant \|\alpha'\| \quad \text{for every } \alpha' \in \mathcal{R}_{(r,s)}.$$

There exist adjacent $v, w \in V(G)$ and $\xi, \zeta \in \mathcal{W}(G)$ such that $\alpha = \xi vw\zeta$. Then $vw \in \mathcal{R}$. Combining Axiom IV with (5) we get

$$\mathcal{R}_{(v,w)} = \{vw\}. \tag{6}$$

Let e be the edge incident with v and w. We see that $G - e$ is connected. Since $\mathcal{R} \subseteq \mathcal{P}(G)$, it is clear that $\mathcal{R}^{(v,w)} \cap \mathcal{R}^{(w,v)} = \emptyset$. Denote

$$\hat{\mathcal{R}} = \mathcal{R} - (\mathcal{R}^{(v,w)} \cup \mathcal{R}^{(w,v)}). \tag{7}$$

It is easy to see that $\hat{\mathcal{R}}$ is a semi-route system on $G - e$ such that $\hat{\mathcal{R}}_{(t,u)} \subseteq \mathcal{R}_{(t,u)}$ for every pair of vertices t and u of G. Since $b(G - e) = b(G) - 1$, the induction hypothesis implies that there exists a geodetic semi-route system $\mathcal{T}$ on $G - e$ with the properties that $\mathcal{T} \subseteq \hat{\mathcal{R}}$ and

(8) $\mathcal{T}_{(t,u)} \neq \emptyset$ if and only if $\hat{\mathcal{R}}_{(t,u)} \neq \emptyset$ for every pair of vertices t and u of G.

Consider arbitrary vertices z and z' of G such that $\mathcal{T}_{(z,z')} \neq \emptyset$. Recall that $\mathcal{T}$ is geodetic. We denote by $\tau_{zz'}$ the only element of $\mathcal{T}_{(z,z')}$; note that if $z = z'$, then $\tau_{zz'} = z$.

Consider arbitrary vertices x and y of G such that $\mathcal{R}_{(x,y)} \neq \emptyset$ and $\mathcal{T}_{(x,y)} = \emptyset$. As follows from (7) and (8),

$$\mathcal{R}_{(x,y)} \subseteq \mathcal{R}^{(v,w)} \cup \mathcal{R}^{(w,v)}.$$

Recall that $\mathcal{R}^{(v,w)} \cap \mathcal{R}^{(w,v)} = \emptyset$. If $\mathcal{R}_{(x,y)} \subseteq \mathcal{R}^{(v,w)}$, then we put $\tilde{x} = v$ and $\tilde{y} = w$; if $\mathcal{R}_{(x,y)} \subseteq \mathcal{R}^{(w,v)}$, then we put $\tilde{x} = w$ and $\tilde{y} = v$. Since $\mathcal{R}_{(x,y)} \neq \emptyset$, it follows from Axioms II and III that $\mathcal{R}_{(x,\tilde{x})} \neq \emptyset \neq \mathcal{R}_{(y,\tilde{y})}$. We wish to show that

$$\mathcal{R}_{(x,\tilde{x})} \cap (R^{(v,w)} \cup \mathcal{R}^{(w,v)}) = \emptyset = \mathcal{R}_{(\tilde{y},y)} \cap (\mathcal{R}^{(v,w)} \cup \mathcal{R}^{(w,v)}). \tag{9}$$

We assume, to the contrary, that (9) does not hold. Without loss of generality, let $\mathcal{R}_{(x,\tilde{x})} \cap \mathcal{R}^{(v,w)} \neq \emptyset$. As follows from (6), there exist $\beta, \gamma \in \mathcal{W}(G)$ such that $\beta vw\gamma \in \mathcal{R}_{(x,\tilde{x})}$. Since $\tilde{x} \in \{v, w\}$ and $\mathcal{R} \subseteq \mathcal{P}(G)$, we get $\gamma = *$. Thus $\tilde{x} = w$. This implies that $\mathcal{R}_{(x,y)} \subseteq \mathcal{R}^{(w,v)}$. Recall that $\mathcal{R}_{(x,y)} \neq \emptyset$. According to (6), there exist $\varphi, \psi \in \mathcal{W}(G)$ such that $\varphi wv\psi \in \mathcal{R}_{(x,y)}$. Since $\beta vw \in \mathcal{R}$, Axiom IV implies that $\beta vwv\psi \in \mathcal{R}$, which is a contradiction. Thus (9) holds. We get $\mathcal{T}_{(x,\tilde{x})} \neq \emptyset \neq \mathcal{T}_{(\tilde{y},y)}$. This implies that $\tau_{x\tilde{x}}\tau_{\tilde{y}y} \in \mathcal{R}$.

For arbitrary vertices t and u of G such that $\mathcal{R}_{(t,u)} \neq \emptyset$ we define

$$\sigma_{tu} = \tau_{tu} \text{ if } \mathcal{T}_{(t,u)} \neq \emptyset \quad \text{and} \quad \sigma_{tu} = \tau_{t\tilde{t}}\tau_{\tilde{u}u} \text{ if } \mathcal{T}_{(t,u)} = \emptyset.$$

We put

$$\mathcal{R}^* = \{\sigma_{tu};\ t, u \in V(G) \text{ such that } \mathcal{R}_{(t,u)} \neq \emptyset\}.$$

Certainly, $\mathcal{R}^* \subseteq \mathcal{P}(G)$. It is easy to see that $\mathcal{R}^*$ is a geodetic semi-route system on G. Moreover, it is clear that (4) holds. Thus the theorem is proved. □

Corollary 2. *Let G be a connected graph. A route system $\mathcal{R}$ on G is geodetic if and only if no proper subset of $\mathcal{R}$ is a route system on G.*

Corollary 3. *For every connected graph G there exists a geodetic route system on G.*

3. Let G be a connected graph. We say that a route system $\mathcal{R}$ on G is *maximal* (or *minimal*) if $\mathcal{R}$ is a proper subset of no route system on G (or no proper subset of $\mathcal{R}$ is a route system on G, respectively). Corollary 2 asserts that a route system on G is minimal if and only if it is geodetic. Recall that $\mathcal{S}(G)$ is a route system on G. We will ask when $\mathcal{S}(G)$ is (or is not) a maximal route system on G.

Theorem 3. *Let G be a connected bipartite graph. Then $\mathcal{S}(G)$ is a maximal route system on G.*

Proof. We assume, on the contrary, that there exists a route system $\mathcal{R}$ on G such that $\mathcal{S}(G) \subsetneq \mathcal{R}$. As follows from Axioms I and II, there exist distinct $u, v, w \in V(G)$ and $\alpha \in \mathcal{W}(G)$ with the properties that

$$u\alpha vw \in \mathcal{R} - \mathcal{S}(G) \quad \text{and} \quad u\alpha v \in \mathcal{S}(G).$$

Hence, $d(u, w) \neq \|u\alpha vw\| = d(u, v)+1$. Since G has no odd cycle, it is routine to show that $d(u, w) = d(u, v) - 1$. This means that there exists $\beta \in \mathcal{W}(G)$ such that $u\beta wv \in \mathcal{S}(G)$. Since $\mathcal{S}(G) \subseteq \mathcal{R}$, we have $u\beta wv \in \mathcal{R}$. Recall that $u\alpha vw \in \mathcal{R}$. Axiom IV implies that $u\beta vwv \in \mathcal{R}$, and thus $u\beta vwv \in \mathcal{P}(G)$, which is a contradiction. Thus the theorem is proved. □

Clearly, a geodetic graph has no odd cycle if and only if it is a tree.

Theorem 4. *Let G be a geodetic graph different from a tree. Then $\mathcal{S}(G)$ is not a maximal route system on G.*

Proof. Clearly, there exists an odd cycle in G. It is routine to prove that there exist $x, y \in V(G)$ and $\varrho, \sigma \in \mathcal{W}(G)$ such that $x\varrho y \in \mathcal{S}(G)$, $x\sigma y \in \mathcal{P}(G)$, $\|x\sigma y\| = \|x\varrho y\| + 1$, and σ has no common vertex with ϱ.

Consider arbitrary $\varphi, \psi \in \mathcal{W}(G)$ such that $\varphi x\varrho y\psi \in \mathcal{S}(G)$. Suppose $\varphi x\sigma y\psi \notin \mathcal{P}(G)$. Then σ has a common vertex with $\varphi\psi$. Without loss of generality, we assume that σ has a common vertex with φ. Then there exist $\xi_1, \xi_2, \zeta_1, \zeta_2 \in \mathcal{W}(G)$ and $t \in V(G)$ such that $\varphi = \xi_1 t \xi_2$ and $\sigma = \zeta_1 t \zeta_2$. Since $\xi_1 t \xi_2 x \varrho y \psi \in \mathcal{S}(G)$, we have $t\xi_2 x \varrho y \in \mathcal{S}(G)$. Hence $\|x\sigma y\| > \|t\zeta_2 y\| \geqslant \|t\xi_2 x \varrho y\| \geqslant 1 + \|x\varrho y\| = \|x\sigma y\|$, which is a contradiction. Thus $\varphi x \sigma y \psi \in \mathcal{P}(G)$. Suppose $\varphi x \sigma \notin \mathcal{S}(G)$. Put $\lambda = \varphi x \varrho$ and $\mu = \varphi x \delta$. There exists $\omega \in \mathcal{S}(G)$ such that $A\omega = A\mu$ and $Z\omega = Z\mu$. We have $\|\omega\| < \|\mu\|$, and thus $\|\omega y\| < \|\mu y\| = \|\lambda y\| + 1$. This implies that $\omega y \in \mathcal{S}(G)$. Since G is geodetic, $\omega y = \lambda y$. Thus $\omega = \lambda$. Recall that $\sigma \neq *$. We have $Z\lambda = Z\sigma$, which is a contradiction. Thus $\varphi x \sigma y \in \mathcal{S}(G)$. Analogously, $\sigma y \psi \in \mathcal{S}(G)$. We have proved the following statement:

$$\text{(10)} \qquad \text{if } \varphi x \varrho y \psi \in \mathcal{S}(G), \text{ then } \varphi x \sigma y \psi \in \mathcal{P}(G) \text{ and } \varphi x \sigma,\ \sigma y \psi \in \mathcal{S}(G) \text{for any } \varphi, \psi \in W(G).$$

Denote

$$\mathcal{T} = \{\varphi x \sigma y \psi;\ \varphi, \psi \in \mathcal{W}(G) \text{ such that } \varphi x \varrho y \psi \in \mathcal{S}(G)\},$$

$\bar{\mathcal{T}} = \{\bar{\alpha};\ \alpha \in \mathcal{T}\}$ and $\mathcal{R} = \mathcal{S}(G) \cup \mathcal{T} \cup \bar{\mathcal{T}}$. Obviously, $\mathcal{S}(G) \subsetneq \mathcal{R}$. As follows from (10), $\mathcal{R} \subseteq \mathcal{P}(G)$. W want to prove that $\mathcal{R}$ is a route system on G. Certainly, $\mathcal{R}$ fulfills Axioms I, II and V.

Consider arbitrary $u, v \in V(G)$ and $\alpha \in \mathcal{W}(G)$. Suppose $u\alpha v \in \mathcal{R}$. If $u\alpha v \in \mathcal{S}(G)$, then $u\alpha \in \mathcal{S}(G)$. Assume that $u\alpha v \notin \mathcal{S}(G)$. Without loss of generality, let $u\alpha v \in \mathcal{T}$. Then there exist $\varphi, \psi \in \mathcal{W}(G)$ such that $\varphi x \varrho y \psi \in \mathcal{S}(G)$ and $u\alpha v = \varphi x \sigma y \psi$. If $\psi \neq *$, then $u\alpha \in \mathcal{T}$. If $\psi = *$, then $u\alpha = \varphi x \sigma$, and according to (10), $\varphi x \sigma \in \mathcal{S}(G)$. Hence $\mathcal{R}$ fulfills Axiom III.

Consider arbitrary $u, v, w \in V(G)$, $\alpha, \beta, \gamma, \delta \in \mathcal{W}(G)$. Suppose $\alpha u \beta v \gamma, u\delta v \in \mathcal{R}$. We distinguish two cases:

1. Let $\alpha u \beta v \gamma \in \mathcal{S}(G)$. If $u\delta v \in \mathcal{S}(G)$, then $\alpha u \delta v \gamma \in \mathcal{S}(G)$. Suppose $u\delta v \notin \mathcal{S}(G)$. Without loss of generality, we assume that $u\delta v \in \mathcal{T}$. Then there exist $\varphi, \psi \in \mathcal{W}(G)$ such that $\varphi x \varrho y \psi \in \mathcal{S}(G)$ and $u\delta v = \varphi x \sigma y \psi$. We have $\alpha\varphi x \varrho y \psi\gamma \in \mathcal{S}(G)$, and thus $\alpha u \delta v \gamma = \alpha \varphi x \sigma y \psi \gamma \in \mathcal{T}$.

2. Let $\alpha u \beta v \gamma \notin \mathcal{S}(G)$. Without loss of generality, we assume that $\alpha u \beta v \gamma \in \mathcal{T}$. Then there exist $\varphi, \psi \in \mathcal{W}(G)$ such that $\varphi x \varrho y \psi \in \mathcal{S}(G)$ and $\alpha u \beta v \gamma = \varphi x \sigma y \psi$. According to (10), $\varphi x \sigma, \sigma y \psi \in \mathcal{S}(G)$. Recall that G is geodetic. If both u and v belong to $\varphi x \sigma$, then $u\delta v \in \mathcal{S}(G)$, and thus $\alpha u \delta v \gamma = \alpha u \beta v \gamma$. If both u and v belong to $\sigma y \psi$, then we obtain the same result. Let now u belong to φx and v belong to $y\psi$. There exist $\lambda, \mu \in \mathcal{W}(G)$ such that $\beta = \lambda \sigma \mu$. Obviously, $\alpha u \lambda \varrho \mu v \gamma \in \mathcal{S}(G)$. If $u\delta v \in \mathcal{S}(G)$, then $\alpha u \delta v \gamma = \varphi x \varrho y \psi \in \mathcal{S}(G)$.

Suppose $u\delta v \notin \mathcal{S}(G)$. Then there exist $\xi, \zeta \in \mathcal{W}(G)$ such that either (a) $\xi x \varrho y \zeta \in \mathcal{S}(G)$ and $u\delta v = \xi x \sigma y \zeta$ or (b) $\xi y \bar{\varrho} x \zeta \in \mathcal{S}(G)$ and $u\delta v = \xi y \bar{\sigma} x \zeta$. First, let $u\delta v =$

$\xi x\sigma y\zeta$. Since G is geodetic, we have $\xi x\varrho y\zeta = u\lambda\varrho\mu v$. Hence $\alpha u\delta v\gamma = \alpha u\beta v\gamma$. Let now $u\delta v = \xi y\bar{\sigma}x\zeta$. Then $u \neq x$. We have $\xi y\bar{\varrho}x\zeta \in \mathcal{S}(G)$. Since $u \neq x$, $\lambda \neq *$. There exists $\tau \in \mathcal{W}(G)$ such that $\lambda = \tau x$. Since G is geodetic and $u\tau x \in \mathcal{S}(G)$, we have $u\tau x = \xi y\bar{\varrho}x$. Recall that $\alpha u\lambda\varrho\mu v\gamma \in \mathcal{S}(G)$. Hence $\alpha\xi y\bar{\varrho}x\varrho\mu v\gamma \in \mathcal{S}(G)$. Since $\mu v\gamma = y\psi$, we conclude that $y\bar{\varrho}x\varrho y \in \mathcal{S}(G)$, which is a contradiction.

Thus $\mathcal{R}$ fulfills Axiom IV. The proof is complete. □

Conjecture. *Let G be a connected graph. Then $\mathcal{S}(G)$ is a maximal route system on G if and only if G is bipartite.*

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