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Monotone iterative technique and connectedness of the set of solutions

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IPI^It’^

Let B be an ordered Banach space with an order cone P, $U \subset B$ and let $A\colon U \to U$ be a continuous operator.

The operator A is *order preserving* if $x \leqslant y \Rightarrow A(x) \leqslant A(y)$, *strictly order preserving* if $x < y \Rightarrow A(x) < A(y)$, *strongly order preserving* if $x < y \Rightarrow A(x) \ll A(y)$. ($x \ll y$ means $y - x \in \operatorname{int} P$.)

An element $x \in U$ is called a *subequilibrium (superequilibrium)* provided $x \leqslant Ax$ ($x \geqslant Ax$). A sub- or superequilibrium is called *strict* if the strict inequality holds.

We denote $[v, w] = \{u \in B;\, v \leqslant u \leqslant w\}$.

Theorem 1. [1] *Assume that v, w, $v < w$ are a sub- and superequilibrium, respectively, of an order preserving operator A, the interval $V = [v, w] \subset U$ and $A(V)$ is a relatively compact set.*

Then $A\colon V \to V$ and there are a minimal and maximal fixed points, say ϱ, r, such that the set F of all fixed points of A in V is a subset of the interval $[\varrho, r]$.

A sequence $(x_n)_{n \in Z}$ in U such that $x_{n+1} = Ax_n$, $x_n \to x^-$ for $n \to -\infty$, $x_n \to x^+$ for $n \to \infty$, x^-, $x^+ \in U$ is called an *entire orbit* connecting x^- with x^+.

Theorem 2. [5, Proposition 2.1] *Let $u_1 < u_2$ be fixed points of a strictly order preserving continuous operator $A\colon U \to U$. Let $W = [u_1, u_2] \subset U$ and $A(W)$ be relatively compact.*

Then precisely one of the following three cases occurs:

(a) *there is another fixed point of A in W,*
(b) *there is an entire orbit consisting of strict subequilibria, connecting u_1 with u_2,*
(c) *there is an entire orbit consisting of strict superequilibria, connecting u_2 with u_1.*

Let us consider again the situation when $A\colon V \to V$, $V = [v, w]$, $v < w$ are sub- and superequilibria.

The fixed point $u \in V$ is called *stable* with respect to V if for each $\varepsilon > 0$ there is $\delta > 0$ such that $A^n(x) \in O(u, \varepsilon)$ for each $x \in O(u, \delta) \cap V$ and each $n \in N$.

Under the assumptions of stability of each fixed point $u \in V$ and the relative compactness of the set $A(V)$ it is proved that the set of fixed points of a strongly order preserving operator $A\colon V \to V$ is a continuous totally ordered curve [5, Theorem 3.3].

Under weaker assumptions we obtain the following result.

Theorem 3. *Let $A\colon V \to V$ be a strictly order preserving continuous mapping, $A(V)$ be a relatively compact set. Assume that all fixed points of A are stable with respect to V.*

Then the set $F \subset V$ of fixed points of A is connected.

The proof is based on the following lemma.

Lemma. *Let the assumptions of Theorem 3 be satisfied and let $y_1, y_2 \in V$, $y_1 < y_2$ be fixed points of A. Then there is a continuous totally ordered curve of fixed points of A connecting y_1 with y_2.*

Proof. As the set $A(V)$ is relatively compact, the set F of fixed points of A is compact and there exists a countable dense subset F_0, $\overline{F_0} = F$. We denote by $\operatorname{span}(F_0)$ the spanning set of F_0 and $E_1 = \overline{\operatorname{span}(F_0)}$. Obviously E_1 is a separable closed subspace of E. We denote by $P_1 = P \cap E_1$ the positive cone in E_1. As $y_1, y_2 \in F$, $y_1 < y_2$ we have $y_2 - y_1 \in P \cap E_1$. Moreover, the cones P, P_1 induce the same ordering on the set F.

As E_1 is a separable Banach space, there is a strictly positive linear functional $x^* \in P_1{}^*$ [2]. Obviously for each $u_1, u_2 \in F$ we have $u_1 < u_2 \Rightarrow x^*(u_1) < x^*(u_2)$. Let F_1 be the set of fixed points of A in $[y_1, y_2]$.

Denote $Z = \{U \subset F_1,\ U \text{ is a totally ordered set},\ y_1 \in U,\ y_2 \in U\}$. The set Z is inductively ordered by the set inclusion. Denote by U^+ the maximal element of Z. As U^+ is a totally ordered set, x^* is a homeomorphism of U^+ onto a closed set $x^*(U^+)$.

We claim $x^*(U^+)$ is connected. Supposing the contrary there are $u_\alpha, u_\beta \in U^+$ such that $x^*(u_\alpha) = \alpha$, $x^*(u_\beta) = \beta$, $\alpha, \beta \in x^*(U^+)$ and $(\alpha, \beta) \subset R \setminus x^*(U^+)$. That means $[u_\alpha, u_\beta]$ contains no fixed point.

The assumption of stability of fixed points implies that there is no strict superequilibrium in $O(u_\beta, \delta) \cap [u_\alpha, u_\beta]$ and no strict subequilibrium in $O(u_\alpha, \delta) \cap [u_\alpha, u_\beta]$ for δ sufficiently small.

Thus we have obtained a contradiction with Theorem 2 as neither case (a), nor cases (b), (c) occur.

$\square$

Proof of Theorem 3. Theorem 1 implies there are a minimal and a maximal fixed point ϱ, r and that $F \subset [\varrho, r]$.

If $r = \varrho$, the set F is a singleton.

If $r > \varrho$ then for each $u \in F$ there are continuous totally ordered curves of fixed points of A connecting ϱ with u and u with r. That means F is connected.

$\square$

Corollary. *Assuming $\varrho \neq r$ the set $F \subset V$ is a union of continuous totally ordered curves of fixed points of A connecting ϱ with r.*

The above Corollary completes the cascade of results of Krasnosel'skij and Lusnikov [6] concerning the relations between the type of monotony and the structure of the set of fixed points. The authors in [6] use another assumption instead of the stability of each fixed point. They assume the interval $[\varrho, r]$ is *degenerated*, i.e. there is no strict sub- or superequilibrium inside.

The following theorem presents a summary of results of Krasnosel'skij and Lusnikov (parts (a),(b) and (d)) and ours (part (c)).

Theorem 4. *Let an interval $V = [\varrho, r]$ be degenerated for a completely continuous operator $A\colon V \to V$. Then*

(a) *the set F of fixed points of A forms a continuous branch in V (i.e. F has a nonzero intersection with the boundary $\partial\Omega$ of each bounded open set Ω such that $r \in \Omega$ and $\varrho \notin \Omega$);*
(b) *if the operator A is order preserving then F contains a continuous curve;*
(c) *if the operator A is strictly order preserving then F is a union of continuous curves;*
(d) *if the operator A is strongly order preserving then F is a continuous curve.*

Remark. The assumption of degeneracy of the interval $[\varrho, r]$ can be slightly weakened by assuming that for each fixed point $x \in [\varrho, r]$ there is $0 << \delta$ such that there is no strict sub- or superequilibrium in the interval $[x, x+\delta] \cap [\varrho, r]$.

Application

We are interested in the structure of the set of solutions of the periodic boundary value problem

$$u'' + f(t,u) = 0, \tag{1}$$

$$u(0) = u(2\pi),\ u'(0) = u'(2\pi), \tag{2}$$

where $f\colon I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

We assume that there are constants a, $b \in \mathbb{R}$ such that

(i) there is a lower solution α and an upper solution β, $\alpha, \beta \in C^2(I)$, of the problem (1), (2) such that $a \leqslant \alpha(t) \leqslant \beta(t) \leqslant b$,
(ii) there is a constant $M > 0$ such that for each $u, v \in [a,b]$, $t \in I$, if $u \leqslant v$ then $f(t,v) - f(t,u) \geqslant -M^2(v-u)$,
(iii) the function $f(t,.)$ is nonincreasing in the variable x for $a \leqslant x \leqslant b$.

Using the existence result and the method of Lakshmikantham and Leela [8] we obtain that under the assumptions (i), (ii) there are maximal and minimal solutions

$r(t)$, $\varrho(t)$ of the boundary value problem (1), (2), and that for each $\eta \in [a,b] \subset C(I)$ the linear problem

$$-u'' + M^2 u = f(t,\eta) + M^2\eta, \tag{3}$$

$$u(0) = u(2\pi),\ u'(0) = u'(2\pi),$$

has the unique solution

$$u(t) = c_1 \mathrm{e}^{Mt} - c_2 \mathrm{e}^{-Mt} - \frac{\mathrm{e}^{Mt}}{2M}\int_0^t \sigma(s)\mathrm{e}^{-Ms}\,\mathrm{d}s + \frac{\mathrm{e}^{-Mt}}{2M}\int_0^t \sigma(s)\mathrm{e}^{Ms}\,\mathrm{d}s,$$

where

$$c_1 = \frac{\mathrm{e}^{2M\pi}}{2M(\mathrm{e}^{2M\pi}-1)}\int_0^{2\pi} \sigma(s)\mathrm{e}^{-Ms}\,\mathrm{d}s,$$

$$c_2 = \frac{1}{2M(\mathrm{e}^{2M\pi}-1)}\int_0^{2\pi} \sigma(s)\mathrm{e}^{Ms}\,\mathrm{d}s,$$

and

$$\sigma(t) = f(t,\eta) + M^2\eta.$$

The operator $A = V \to V$, $V = [a,b] \subset C(I)$ defined by $A(\eta) = u$, u being a solution of (3), (2) is relatively compact and strictly monotone [8].

Let $x(t)$ be a fixed point of A and let δ be a constant. We denote

$$A\left(x(t)+\delta\right) = x(t) + \varepsilon(t).$$

From the definition of A we obtain that $\varepsilon(t)$ is a solution of the boundary value problem

$$-\varepsilon(t)'' + M^2\varepsilon(t) = F(t),$$

$$\varepsilon(0) = \varepsilon(2\pi),\ \varepsilon'(0) = \varepsilon'(2\pi),$$

where $F(t) = f(t, x+\delta) - f(t,x) + M^2\delta$.

The assumption (iii) implies that $|\varepsilon(t)| \leqslant |\delta|$ and that each fixed point of the operator A is stable. Theorem 3 implies that the set of solutions of the boundary value problem (1), (2) is connected.

Remark. Our example is only an illustrative one. The direct computation yields that $r(t) - \varrho(t) = c_0$, where c_0 is a nonnegative constant and the solution set has the form $S = \{\varrho(t) + c;\ c \in [0, c_0]\}$. See [9, Theorem].

As our second application we give the result concerning the structure of the set of solutions $x(t) \in C(I)$, $I = [0,1]$) of the integral equation

(4) $$x(t) = \int_I K(t,s) f(s, x(s))\,\mathrm{d}s.$$

We assume that

(i) the function $K(t,s)\colon I \times I \to \mathbb{R}$ is continuous and $0 \leqslant K(t,s)$,

(ii) the function $f(t,.)$ is increasing in the variable x,

(iii) there is a lower solution α and an upper solution β, $\alpha, \beta \in C(I)$, $\alpha(t) \leqslant \beta(t)$,

(iv) the function f is continuous and there is $\delta > 0$ such that

$$f(t, u+\delta) - f(t,u) < \tfrac{\delta}{m}$$

for each $t \in I$ and $u \in [\alpha(t), \beta(t)]$, where $m \in \mathbb{R}$ and

$$m = \int_0^1 K(t_1, s)\,\mathrm{d}s = \max_{t \in I} \int_0^1 K(t,s)\,\mathrm{d}s.$$

(v) $K(t,s)$ is not identically zero in any subset $I \times [s_1, s_2]$, $s_1, s_2 \in I$.

The operator $A\colon V \to V$, $V = [\alpha, \beta] \subset C(I)$ defined by $A(\eta) = u$, where

$$u(t) = \int_I K(t,s) f(s, \eta(s))\,\mathrm{d}s,$$

is relatively compact and strictly monotone.

Let $x(t)$ be a fixed point of A, and let $\delta > 0$ be a constant.

Then

$$A\left(x(t) + \delta\right) = x(t) + \varepsilon(t).$$

where $\varepsilon(t)$ is given by

$$\varepsilon(t) = \int_I K(t,s) F(s)\,\mathrm{d}s,$$

$F(t) = f(t, x(t) + \delta) - f(t, x(t))$.

The assumption (iv) implies $\varepsilon(t) \leqslant \delta$.

Thus the solution $x(t)$ is stable. Theorem 3 implies that the set of solutions of the integral equation (4) bounded between $\alpha(t)$ and $\beta(t)$ is connected.

In the paper [3] it is proved that under assumptions somewhat weaker then (i)–(iii) the set of solutions is a complete lattice. Adding the assumptions (iv), (v) we obtain that this lattice is connected (in topology of $C(I)$) and is either a singleton or a union of totally ordered continuous curves.

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