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ON A PAIR OF MANIFOLDS WITH CONNECTION

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In paper [5] the manifold with connection is considered as a quintuple $\mathscr{S}(B, E, \Phi, \sigma, C)$, where $E(B, F, G, P)$ is a fibre bundle ($F = G/H$, $\dim F > \dim B$, H is a closed subgroup of G) associated to the principle bundle $P(B, G)$, $\Phi = PP^{-1}$ is a groupoid associated to P, C is a connection of order 1 on Φ, σ is a global section of E with the following property: the development $C^{1}(x)(\sigma)$ of σ by means of C is a regular jet for any $x \in B$. In the present paper we consider the manifold with connection as it is considered in [5].

On the torsion form of a pair of manifolds with connection

Let the $\mathscr{S}(B, E, \Phi, \sigma, C)$ be a manifold with connection. Kolář using Švec's definition in [6] defines the torsion form of the manifold $\mathscr{S}$ as fosllows: Let $\mu_0 : G \to G/H$ be the canonical projection. Let Ω be the curvature form of the connection C. Let R be the reduction of the principal bundle P determined by the section σ. Then

$$u_*\mu_{0*}(\Omega), \quad u \in \pi^{-1}(x) \cap R$$

is the torsion form of $\mathscr{S}$ at the point $x \in B$. We consider the torsion form like the above one.

1. In this paper the index i will have the values *1*, *2*. Let V, T_1, T_2 be vector spaces ($\dim T_i = v_i$, $\dim V = m$, $m < \min(v_1, v_2)$), $\gamma : T \to T_1 \oplus T_2$ be a isomorphism, $pr_i : T_1 \oplus T_2 \to T_i$ is the natural homomorphism. Let $\xi : V \to T$ be a monomorphism with the following property: $pr_i\gamma\xi : V \to T_i$ are monomorphisms.

Denote $Z = \operatorname{im} \xi$, $Z_i = \operatorname{im}(pr_i\gamma\xi)$; $\dim z = \dim Z_i = m$. The restriction of the homomorphism $pr_i\gamma$ to Z determines the isomorphism $\eta_i : Z \to Z_i$ and thus $\eta_i\xi : V \to Z_i$ is an isomorphism. Let ω be a vector *2*-form on V with values in T determined by a tensor $t \in T \otimes^2 \wedge V^*$. Then $\omega_i = pr_i\gamma\omega$ is a vector *2*-form on V with values in T_i. Let $\varepsilon : T \to T/Z$, $\varepsilon_i : T_i \to T_i/Z_i$ be natural homomorphisms. The form $\varepsilon\omega_i$, resp. $\varepsilon_i\omega_i$, will be called *ξ-reduction of* ω, or of ω_i, respectively. Denote $S = \gamma^{-1}(Z_1 \oplus Z_2) \subset T$. Obviously $Z \subset S$. Let

$\mu : T \to T/S$ be the natural homomorphism. The vector *2*-form $\mu\omega$ on V with values in T/S will be called *ξ-semireduction of ω*. Let $S_2 = \gamma^{-1}(O \oplus Z_2)$ and let μ_2 be the natural homomorphism $T \to T/S_2$. The form $\mu_2\omega$ will be called *the second ξ-semireduction of ω*. Similarly $\mu_2\omega$ is the first ξ-semireduction of the form ω.

Let the ξ-reduction of ω, resp. of ω_i, vanish. Then the form ω, resp. ω_i, is a *2*-form with values in Z, or in Z_i, respectively, and thus the forms $\xi^{-1}\omega$, $(\eta_i\xi)^{-1}\omega_i$ are *2*-forms on V with values in V.

Definition 1. *We shall speak that the forms ω_1, ω_2 form a ξ-reduction pair (shortly an r-pair) if their ξ-reductions vanish and if*

(1) $$(\eta_1\xi)^{-1}\omega_1 = (\eta_2\xi)^{-1}\omega_2.$$

The following lemmas are obvious.

Lemma 1. *The form ω vanishes if and only if the forms ω_1, ω_2 vanish.*

Lemma 2. *The ξ-semireduction of ω vanishes if and only if the ξ-reductions of the forms ω_1 and ω_2 vanish.*

Lemma 3. *The second ξ-reduction of the form ω vanishes if and only if the forms ω_1 and the ξ-reduction of $\omega_2$vanish.*

A similar lemma can be expressed about the first ξ-reduction of ω.

Lemma 4. *The ξ-reduction of the form ω vanishes if and only if the forms ω_1 and ω_2 form a ξ-reduction pair.*

Proof of Lemma 4. Let the ξ-reduction of ω vanish. Then it is obvious that ξ-reductions of the forms ω_1 and ω_2 vanish and $\xi^{-1}\omega = (\eta_i\xi)^{-1}\omega_i$. Conversely let the forms ω_1 and ω_2 form a ξ-reduction pair. Let $u_1 \in V$, $u_2 \in V$. As the forms $\varepsilon_1\omega_1$, $\varepsilon_2\omega_2$ vanish, $\omega_i(u_1, u_2) \in Z_i$ and thus there are $s_1 \in S_1$, $s_2 \in S_2$ unambiguously, so that $\omega(u_1, u_2) = s_1 + s_2$. Denote $\omega_i(u_1, u_2) = pr_i\gamma(s_1 + s_2) = z_i \in Z_i$, $\eta_i^{-1}(z_i) = y_i \in Z$. When we use (1), we get

$$(\eta_1\xi)^{-1}\omega_1(u_1, u_2) = (\eta_2\xi)^{-1}\omega_2(u_1, u_2)$$

and thus $y_1 = y_2 = y$. As $pr_i\gamma(y) = z_i$ thus $y = s_1 + s_2$ and thus $\varepsilon\omega(u_1, u_2) = 0$, i. e. the ξ-reduction of the form ω vanishes.

Note 1. Let the ξ-semireduction of ω vanish. The form ω is a *2*-form with values in $S = \gamma^{-1}(Z_1 + Z_2)$ and its reduction can be called *the jumbled reduction of the forms ω_1 and ω_2*. The jumbled reduction of ω_1, ω_2 is a *2*-form with values in S/Z and it vanishes if and only if ω_1 and ω_2 form a ξ-reduction pair.

2. In this paper we shall use the standard notation of the theory of jets (see [2]). Our considerations are in the category C^∞. Let M, V_1, V_2 be differentiable manifolds; $\dim M = m$, $\dim V_i = v_i$. Denote $p_i : V_1 \times V_2 \to V_i$ the natural projection. The following assertions are obvious:

a) $X \in \bar{J}^r_x(M, V_1 \times V_2) \Rightarrow p_i X \in \bar{J}^r_x(M, V_i)$.

b) $X_1 \in \bar{J}^r_x(M, V_1)$, $X_2 \in \bar{J}^r_x(M, V_2) \Rightarrow$ there is a unique jet $X \in \bar{J}^r_x(M, V_1 \times V_2)$ so that $p_i X = X_i$. X is regular if some jet of the jets X_1, X_2 is regular.

c) $X \in \bar{J}^r_x(M, V_1 \times V_2)$ is semiholonomic, resp. holonomic if and only if $p_i X$ are semiholonomic, resp. holonomic.

Definition 2. *Let $X_1 \in \bar{J}^r_x(M, V_1)$, $X_2 \in \bar{J}^r_x(M, V_2)$. We shall speak that jets X_1, X_2 are holonomicly connected if there is a semiholonomic r-frame $\bar{h}$ at $x \in M$ so that jets $X_1\bar{h}$ and $X_2\bar{h}$ are holonomic.*

Let N, M be differentiable manifolds. Let $X \in \bar{J}^r(M, N)$. The contact element kX at the point $\beta^0_r X \in N$ determined by X is a set of jets $X\bar{h}\bar{L}^r_m$ where h is a semiholonomic frame at $\alpha X \in M$ and $\bar{L}^r_m$ is the group of invertible r-jets on R^m from O into O. We shall speak that kX is holonomic if there is in $X\bar{h}\bar{L}^r_m$ a holonomic jet.

Lemma 5. *Let $X \in \bar{J}^r_x(M, V_1 \times V_2)$. Then kX is holonomic if and only if $p_1 X$ and $p_2 X$ are holonomicly connected.*

Proof. Let kX be holonomic. Then there is a frame $\bar{h}$ at $x \in M$ so that Xh is a holonomic jet. Hence $p_i(X\bar{h})$ is holonomic. But $p_i(X\bar{h}) = (p_i X)\bar{h}$ and thus $p_1 X$, $p_2 X$ are holonomicly connected. Conversely let $p_1 X$ and $p_2 X$ be holonomicly connected. Then there is a semiholonomic r-frame $\bar{h}$ so that $(p_i X)\bar{h}$ are holonomic. It results from the assertion c, that $X\bar{h}$ is a holonomic jet and thus kX is holonomic.

Lemma 6. *Let N, M, V be differentiable manifolds*; $\dim N = n$, $\dim M = m < \dim V = v$. *Let $X \in \bar{J}^2_a(N, M)$, $Y \in \bar{J}^2_{\beta^0{}_2 x}(M, V)$ and let Y be regular and holonomic. Then YX is holonomic if and only if X is a holonomic jet.*

Proof. Let h_1 be a holonomic 2-frame at $a \in N$, h_2 be a holonomic 2-frame at $\beta^0_2 X$ and h_3 be a holonomic 2-frame at $\beta^0_2 Y$. Let Y have in the frames h_2, h_3 the co-ordinates:

$$Y \equiv h_3^{-1} Y h_2 = (y^\beta_p, y^\beta_{p,j}), \ \beta = 1, 2, \ldots, v; \ p, j = 1, 2, \ldots, m.$$

Let X have in the frames h_1, h_2 the co-ordinates:

$$X \equiv h_2^{-1} X h_1 = (a^k_u, a^k_{u,t}), \quad k = 1, 2, \ldots, m; \quad u, t = 1, 2, \ldots, n.$$

Then YX has the co-ordinates

$$YX \equiv (h_3^{-1} Y h_2)(h_2^{-1} X h_1) = (v^u_\beta, v^\beta_{u,t}),$$

where

$$v^\beta_u = y^\beta_k a^k_u, \quad v^\beta_{u,t} = y^\beta_{p,j} a^p_u a^j_t + y^\beta_k a^k_{u,t}.$$

It is obvious that if X is holonomic then YX is holonomic. Let YX be holonomic. Then $O = v^\beta_{u,t} - v^\beta_{t,u} = v^\beta_{[u,t]}$.

As $y^{\beta}_{[p,j]} = O$ we have

(2) $$O = v^{\beta}_{[u,t]} = y^{\beta}_{k} a^{k}_{[u,t]}.$$

As Y is regular we can suppose without loss of generality that det $(y^{\beta}_{k}) \neq 0$, where $\beta, k = 1, 2, \ldots, m$. Thus we get from (2) for any stable index $[u, t]$ and for $\beta = 1, 2, \ldots, m$ a homogeneous system of equations with the unknowns $a^{k}_{[u,t]}$, $k = 1, 2, \ldots, m$, the determinant of which does not vanish. Thus $a^{k}_{[u,t]} = O$. Q. E. D.

Lemma 7. *Let N, M, V be differentiable manifolds,* dim N = dim M. *Let $X \in \bar{J}^{r}_{a}(N, M)$ be a regular holonomic r-jet. Let $Y \in \bar{J}^{r}_{\beta^{0}_{r}}(M, V)$. Then YX is holonomic if and only if Y is holonomic.*

Proof. It is obvious that if Y is holonomic, YX is holonomic, too. Let YX be holonomic. As X is regular and dim N = dim M X is invertible and thus X^{-1} is holonomic. Hence $(YX)X^{-1} = Y$ is holonomic.

Lemma 8. *Let $X \in \bar{J}^{2}(M, V_1 \times V_2)$,* dim $M <$ dim V_2. *Let p_2X be holonomic and regular. Then kX is holonomic if and only if p_1X is holonomic.*

Proof. If p_1X is also holonomic, then X is holonomic and thus kX is holonomic. Let kX be holonomic. Then there is a semiholonomic 2-frame $\bar{h}$ at αX so that $X\bar{h}$ is holonomic and thus $p_i(X\bar{h}) = (p_iX)\bar{h}$ is holonomic. As p_2X is holonomic, then from Lemma 6 we get: $\bar{h}$ is holonomic. Then from Lemma 7 we get: p_1X is holonomic.

Let us suppose dim $M = m <$ min (dim $V_1 = v_1$, dim $V_2 = v_2$). Let $X \in \bar{J}^{2}_{x}(M, V_1 \times V_2)$ be a regular semiholonomic jet with this characteristic: p_1X, p_2X are regular, too. Denote

$$T \equiv T_{\beta^{0}_{2}}(V_1 \times V_2), \quad T_i \equiv T_{\beta^{0}_{2}p_1X}(V_i) = p_{i*}T, \quad V \equiv T_X(M).$$

We can identify $T \equiv T_1 \oplus T_2$. Let h_1 be a holonomic 2-frame at $x \in M$ and h_2 be a holonomic 2-frame at $\beta^{0}_{2}X \in V_1 \times V_2$. Let $(x^{\gamma}_{p}, x^{\gamma}_{p,j})$, $\gamma = 1, 2, \ldots, v_1 + 1, \ldots, v_1 + v_2$; $p, j = 1, 2, \ldots, m$ be co-ordinates of the jet X in the frames h_1 and h_2. Then $(x^{\alpha}_{p}, x^{\alpha}_{p,j})$ $\alpha = 1, 2, \ldots, v_1$ are co-ordinates of the jet p_1X in the frames h_1, h_2 and $(x^{v_1+\beta}_{p}, x^{v_1+\beta}_{p,j})$, $\beta = 1, 2, \ldots, v_2$, are co-ordinates of p_2X in the frames h_1 and p_2h_2. Difference tensors (the notion of the difference tensor of a semiholonomic 2-jet was introduced by Kolář in [5]) determined by the jets X, p_iX have the components $\Delta(X) \in T \otimes^2 \wedge V^* : x^{\gamma}_{[p,j]}, \gamma = 1, 2, \ldots, v_1, v_1 + 1, \ldots, v_1 + v_2$, $\Delta(p_1X) \in T_1 \otimes^2 \wedge V^* : x^{\alpha}_{[p,j]}$, $\alpha = 1, 2, \ldots, v_1$ $\Delta(p_2X) \in T_2 \otimes^2 \wedge V^* : x^{v_1+\beta}_{p,j}$, $\beta = 1, 2, \ldots, v_2$; $p, j = 1, 2, \ldots, m$. From this we obviously get

(3) $$\Delta(p_iX) = p_{i*}\Delta(X).$$

Vector 2-forms determined by $\Delta(X)$, $\Delta(p_iX)$ will be called *difference forms* *of jets* X, p_iX and denoted ω, ω_i. From (3) we get: $\omega_i = dp_i\omega$. When we denote $\xi \equiv ds \equiv s_*$, where $s(y)$ is a local mapping such that $\beta_2^1 X = j_x^1 s(y)$, we can with regard to the regularity of jets X, p_1X, p_2X do all considerations of paragraph 1. Now the subspaces Z, Z_i are contact subspaces determined by the jets $\beta_2^1 X$, $\beta_2^1 p_i X$. Instead of the ξ-reduction and the ξ-semireduction we shall speak of the reduction and the semireduction of the difference form.

Kolář proved in [5]: The reduction of the difference form of the semiholonomic *2*-jet X vanishes if and only if the contact element kX is holonomic. Hence we get from Lemma 2.

Lemma 9. *The semireduction of the difference form* ω *of the jet* $X \in \bar{J}_x^2(M, V_1 \times V_2)$ *vanishes if and only if contact elements* kp_1X, kp_2X *are holonomic.*

From Lemma 3 we get.

Lemma 10. *The second semireduction of the difference form* ω *vanishes if and only if* p_1x *is holonomic and if the contact element* kp_2X *is holonomic.*

From Lemma 5 we get.

Lemma 11. *The reduction of the difference form* ω *vanishes if and only if the jets* p_1X, p_2X *are holonomicly connected.*

Corollary of Lemmas 4 and 11. The difference forms ω_1, ω_2 of the jets p_1X, p_2X form an r-pair if and only if the jets p_1X, p_2X are holonomicly connected.

Now Lemma 8 can be expressed as follows:

Lemma 8'. *Let the difference form* ω_2 *of the jet* p_2X *vanish. Then the reduction of the difference form* ω *of the jet* $X \in \bar{J}_x^2(M, V_1 \times V_2)$ *vanishes if and only if the difference form* ω_1 *vanishes.*

3. Application for the torsion form. We first recall some notions of the theory of spaces with connection; see [2] and [5]. Let $P(B)$ $G, \pi)$ be a principal fibre bundle. The Lie-groupoid associated to the principle fibre bundle P is a set of equivalence classes $\Phi = P \times P/G$ with the projections a and b, which are defined as follows: $\Theta = \{(u_1, u_2)\}$, a $\Theta = \pi(u_2)$, $b\Theta = \pi u_1$. Further $\Theta_1 . \Theta_2 \quad \{(u_1, u_2)\} . \{(u_2, u_3)\} = \{(u_1, u_3)\}$ and $1_x = \{(u, u)\}$, (where $\pi u - x$) is the unit of Φ over $x \in B$. Kolář in [4] uses the modified form of Ehresmann's definition of the connection on Φ. An element of connection of the order r on Φ at $x \in B$ is a jet $X \in \tilde{J}_x^r(a^{-1}(x), b, B)$ such that $\beta_r^0 X = 1_x$. Denote $\tilde{Q}_x^r(\Phi)$ the set of elements of connection of the order r on Φ at $x \in B$. The connection of the order r on Φ is a section $C_r : B \to \tilde{Q}^r(\Phi) = \bigcup_{x \in B} Q_x^r(\Phi)$. C_r' is the first prolongation of the connection C_r. If $C_1(x) = j_x^1 \varrho(t)$, then

$$C_1'(x) = j_x^1 C_1(t) . \varrho(t), \tag{4}$$

where $C_1(t) \,.\, \varrho(t)$ is the image of the jet $C_1(t)$ in the mapping $\varrho(t) \in a^{-1}(x) \subset \Phi$.

Let $E(B, F, G, P)$ be a fibre bundle associated to the principal fibre bundle $P(B, G, \pi) \,.\, \Phi$ is a groupoid of operators on E:

$$\Theta = (u_1, u_2) \in \Phi, \quad f = (u_2, v) \in E_{\pi(u_2)} \Rightarrow \Theta(f) = (u_1, v) \in E_{\pi(u_1)}$$

where $v \in F$. Let σ be a section on E. $C_r^{-1}(x)(\sigma)$ is the development of the section σ by means of the element $C_r(x)$. Further we shall use:

(5) $$C_1^{-1}(x)(\sigma) = j_x^1(\varrho^{-1}(t)\,[\sigma(t)])$$

where

$$C_1(x) \quad j_x^1\varrho(t)\,.$$

(6) $$C_1'^{-1}(x)(\sigma) = C_1^{-1}(x)(j_x^1[C_1^{-1}(t)(\sigma)]); \quad \text{see [4] or [5]}.$$

Let ${}^1P(B, G_1, \pi_1)$, ${}^2P(B, G_2, \pi_2)$ be principle fibre bundles. Denote $P_x \equiv {}^1P_x \times {}^2P_x$, where ${}^iP_x = \pi_i^{-1}(x) \,.\, P = \bigcup_{x \in B} P_x$ is the fibre product of ${}^1P$ and ${}^2P$. The projection π on P is defined by $\pi(P_x) = x \,.\, P$ has the structure of the principle fibre bundle $P(B, G_1 \times G_2, \pi)$, where the group $G_1 \times G_2$ acts on P on the right according to the rule

$$(P_x)G_1 \times G_2 - ({}^1P_x)G_1 \times ({}^2P_x)G_2\,.$$

Let ${}^i\Phi$ be a Lie groupoid associated to ${}^iP$, Φ be a Lie groupoid associated to P. As ${}^i\Phi = {}^iP \times {}^iP/G_i$ and $\Phi = P \times P/G_1 \times G_2$, then any couple $({}^1\Theta, {}^2\Theta)$ where ${}^i\Theta \in {}^i\Phi$ and $a_1({}^1\Theta) = a_2({}^2\Theta)$, $b_1({}^1\Theta) = b_2({}^2\Theta)$ (a_i, b_i are projections on ${}^i\Phi$) determines a unique element $\Theta \in \Phi$ and conversely. Then Φ is such a set of couples $({}^1\Theta, {}^2\Theta)$, ${}^i\Theta \in {}^i\Phi$, that $a_1({}^1\Theta) = a_2({}^2\Theta)$, $b_1({}^1\Theta) - b_2({}^2\Theta)$. Denote $\tilde{p}_i : \Phi \to {}^i\Phi$ the map defined by $\tilde{p}_i({}^1\Theta, {}^2\Theta) = {}^i\Theta$. Let ${}^iC$ be the connection of order 1 on ${}^i\Phi$. It is easy to see that there is a unique connection C_1 of order 1 on Φ such that $\tilde{p}_iC_1 = {}^iC_1$. Let ${}^iE(B, F_i, G_i, {}^iP)$ be a fibre bundle associated with ${}^iP$. Denote $E_x = {}^1E_x \times {}^2E_x$. The fibre product $E = \bigcup_{x \in B} E_x$ can be identified with the fibre bundle $E(B, F_1 \times F_2, G_1 \times G_2, P)$ associated to P on which the group $G_1 \times G_2$ acts on the left according to the rule

$$G_1 \times G_2(F_1 \times F_2) = G_1(F_1) \times G_2(F_2)\,.$$

Denote $\overline{p}_i : E \to {}^iE$ maps determined by natural projections $E_x \to {}^iE_x$ for any $x \in B$. Let ${}^i\sigma$ be a global section on ${}^iE$. Then there is a unique section on E determined by

$$\sigma(x) = [{}^1\sigma(x), {}^2\sigma(x)] \in E_x\,.$$

Definition 3. *A pair of manifolds with connection is a couple of manifolds*

$$\mathcal{S}_1(B, {}^1E, {}^1\Phi, {}^1\sigma, {}^1C_1), \quad \mathcal{S}_2(B, {}^2E, {}^2\Phi, {}^2\sigma, {}^2C_1)\,.$$

It is clear from the foregoing consideration that there is a unique manifold with the connection $\mathscr{S}(B, E, \Phi, \sigma, C_1)$, which is determined by the couple of manifolds with connection. This manifold we shall call the representative of the pair.

The following relations result from (4), (5), (6)

$$\tilde{p}_i C'_1 = {}^i C'_1$$

$$p_i C'_1{}^1(x)(\sigma) = \tilde{p}_i C'_1{}^1(x)(\overline{p}_i(\sigma)).$$

Then

$$\overline{p}_i[C'_1{}^1(x)(\sigma)] = {}^i C'_1{}^1(x)({}^i\sigma). \tag{7}$$

Let ψ_i be the torsion form of $\mathscr{S}_i$ and ψ be the torsion form of $\mathscr{S}$, which we shall call *the torsion form of a pair* of manifolds with connection. Kolář showed in [5] that the torsion form of a manifold with connection was able to be identified at $x \in B$ with $-\Delta C'_1{}^1(x)(\sigma)$. The following relation

$$\overline{p}_{i*}\psi = \psi_i$$

results from (3) and (7).

Nov, Lemmas 9, 10, 11 imply

Theorem 1. *The semireduction of the torsion form of a pair of manifolds with connection vanishes if and only if ψ_1 and ψ_2 vanish; i. e. if and only if the contact elements $k^1 C'_1{}^1(x)^1(\sigma)$, ${}^2C'_1{}^1(x)({}^2\sigma)$ are holonomic.*

Theorem 2. *The second semireduction of the torsion form of a pair of manifolds with connection vanishes if and only if ψ_1 vanishes and the reduction of ψ_2 vanishes i. e. if and only if the jet ${}^1C'_1{}^1(x)({}^1\sigma)$ is holonomic and the contact element $k^2 C'^{-1}_1(x)$ $({}^2\sigma)$ is holonomic.*

Theorem 3. *The reduction of ψ vanishes if and only if ψ_1 and ψ_2 determine r-pair; i. e. if and only if jets ${}^1C'_1{}^1(x)(\sigma_1)$, ${}^2C'^{-1}_1(x)(\sigma_2)$ are holonomicly connected.*

We are going to determine the co-ordinate condition for the vanishing of the reduction of the torsion form of the pair of manifolds with connection. Let us recall some notations:

$$F_i = G_i/H_i, \quad \underline{H}_i = T_e(H_i), \quad {}^ie_1, {}^ie_2, \ldots, {}^ie_{r_1}$$

is a basis in G_i, i_R or R, resp. is the reduction of the principle fibre bundle ${}^iP$, or P, respectively, which is determined by the section ${}^i\sigma$, or σ, resp. Let ${}^i\varphi$, or ${}^i\Omega$ resp. be the restrinction of the fundamental form of the connection ${}^i\Gamma$, which represents the connection ${}^iC$ on ${}^iP$ (see [4]), or of the curvature form of this connection resp., with regard to a local section ${}^i\nu : U \to {}^iR$, $U \subset B$.

$$^i\varphi = {}^i\omega^s \otimes {}^ie_s + {}^i\omega^\lambda \otimes {}^ie_\lambda, \quad s = 1, 2, \ldots, n_i = \dim F_i$$

$$\lambda = n_i + 1, \ldots, r_i = \dim G_i,$$

where ${}_ie_\lambda \in \underline{H}_i$. We can suppose that ${}^i\omega^1; {}^i\omega^2; \ldots, {}^i\omega^m$, $m = \dim B$, are independent on the section ${}^i\nu$. Then

$$^i\omega^\alpha = {}^ia_k^{\alpha i}\omega^k, \quad \alpha = m+1, \ldots, n_i; \quad k = 1, 2, \ldots, m$$

and

$$^2\omega^k = b_j^{k\,1}\omega^j, \det |b_j^k| \neq 0,\ j,\ k = 1, \ldots, m.$$

The form ${}^i\Omega$ can be written

$$^i\Omega = {}^i\Omega^s \otimes {}^ie_s + {}^i\Omega^\lambda \otimes {}^ie_\lambda,$$

$$s = 1, 2, \ldots, n_i, \quad \lambda = n_i + 1, \ldots, r_i.$$

Let $p_i : P \to {}^iP$ be the natural projection. Let ε be a scalar form and f be a function on ${}^iP$. We will denote

$$p_i^*\varepsilon \equiv \bar{\varepsilon}, \quad fp_i \equiv \bar{f}.$$

$\varphi = {}^1\varphi dp_1 + {}^2\varphi dp_2$ is a fundamental form of the connection Γ on P restricted to the section $\nu : U \to R$ $(\nu(x) = [{}^1\nu(x), {}^2\nu(x)])$ and thus

$$^i\bar{\omega}^\alpha = {}^i\bar{a}_k^{\alpha i}\bar{\omega}^k$$

$$^2\bar{\omega}^k = \bar{b}_j^{k\,1}\bar{\omega}^j, \quad \det |\bar{b}_k^j| \neq 0.$$

Likewise $\Omega = {}^1\Omega\, dp_1 + {}^2\Omega\, dp_2$ is a restriction of the curvature form of the connection Γ on P with regard to the section ν. The reduction of the torsion form of the manifold $\mathcal{S}$ vanishes if and only if

$$^i\bar{\Omega}^\alpha = {}^ia_k^{\alpha i}\bar{\Omega}^k,$$

$$^2\bar{\Omega}^k = \bar{b}_j^k\, {}^1\bar{\Omega}^j, \quad \text{see [5];}$$

and thus the reduction of the torsion form of the pair of manifolds with connection vanishes if and only if

$$^i\Omega^\alpha = {}^ia_k^{\alpha i}\Omega^k,$$

$$^2\Omega^k = b_j^{k\,1}\Omega^j.$$

Point similarity and point equivalence of manifolds of the pair of manifolds with connection

4. Let $F = G/H$ be a homogeneous space in which the Lie group G acts on the left; c is the class in F determined by H. Let B be a differentiable

manifold. Let U be an open set in $B, x \in U, f \in F$. Let $U \to f$ be a constant mapping from B in F. The r-jet of this mapping will be denoted $f^{(r)}_x$. $X \in \in \tilde{J}^r_x(B, G)$, we shall denote $X_f = X(f^{(r)}_x)$, where the symbol on the right-hand side denotes the r-th anholonomic prolongation of the operation of the group G on F.

Definition 4. *Let ${}^1F = G/H_1$, ${}^2F = G/H_2$ be homogeneous spaces. We shall speak that the jets $X \in \tilde{J}^r_x(B, {}^1F)$, $Y \in \tilde{J}^r_x(B, {}^2F)$ are G-adjoint if there are a jet $Z \in \tilde{J}^r_x(B, G)$ and the points $f_1 \in {}^1F$, $f_2 \in {}^2F$ so that $X = Z_{f_1}$, $Y = Z_{f_2}$.*

Let $\mathscr{S}_1(B, {}^1E, \Phi, {}^1\sigma, C)$, $\mathscr{S}_2(B, {}^2E, \Phi, {}^2\sigma, C)$ be a pair of manifolds with connection. Now ${}^1E$, ${}^2E$ are fibre bundles associated to $P(B, G)$. Let ${}^iF = = G/H_i$ be their type fibres. We shall denote $p \,.\, g$ the operation of the group G on P; ${}^iR$ is the reduction of the principal fibre bundle P determined by the section ${}^i\sigma$; ${}^iR_x$ is the fibre of ${}^iR$ over $x \in B$. Let $r_1 \in {}^1R_x$, $r_2 \in {}^2R_x$. It is obvious that ${}^iR_x = r_i \,.\, H_i$. The equality $r_1 \,.\, g = r_2$ determines a map $\varkappa : {}^1R_x \times \times {}^2R_x \to G$. Let $r \in {}^1R_x$, $\tilde{r} \in {}^2R_x$, then $r \,.\, h_1 = r_1$, $r_2 \,.\, h_2 = \tilde{r}(h_i \in H_i)$ and thus $r \,.\, h_1 g h_2 = \tilde{r}$. Hence $H_1 g H_2 = \operatorname{im} \varkappa$. $H_1 g H_2$ is a class of the decomposition of the group G by the double module (H_1, H_2), i. e. $H_1 g H_2 \in G/(H_1, H_2)$; see [1]. We shall denote $D \equiv G(H_1, H_2)$. Thus we get the map $q : B \to D$; $q(x) = H_1 g H_2$.

Definition 5. *We shall say that manifolds $\mathscr{S}_1$, $\mathscr{S}_2$ of a pair of manifolds with connection which have a common principal fibre bundle, are D-similar at $x \in B$ when there is a neighbourhood U of $x \in B$ and $d \in D$ so that $q(U) = d$.*

Let $\Gamma(p)$ be the representative of the connection C at $p \in P$, $\Gamma^{-1}(p)({}^i\sigma)$ be the development of the section ${}^i\sigma$ by means of $\Gamma(p)$; see [4].

Theorem 4. *The manifolds $\mathscr{S}_1$, $\mathscr{S}_2$ of a pair of manifolds with connection, which have a common principal fibre bundle P, are D-similar at $x \in B$ if and only if the jets $\Gamma^{-1}(p)({}^1\sigma)$, $\Gamma^{-1}(p)({}^2\sigma)$ are G-adjoint ($\pi(p) = x$).*

Proof. Let $p \in P_x$. Let $\Gamma(p) = j^1_x\varrho(t)$, where $\varrho(t)$ is a local section on (B, π, P) defined on a neighbourhood U of $x \in B$. Let $\mathscr{S}_1$, $\mathscr{S}_2$ be D-similar. Let $q(U) = d \in D$. Let $g_0 \in G$ be a representative of d. Then there is a local section $\mu(t) = r_t$ of $({}^1R, \pi, B)$ defined on U, so that $r_t \,.\, g_0$ is a local section on $({}^2R, \pi, B)$. Now ${}^1\sigma(t) = (r_t, c_1)$, ${}^2\sigma(t) = (r_t \,.\, g_0, c_2)$, where $c_i \in {}^iF$ is the element determined by the class H_i in G/H_i. Let us denote $g_t \in G$ the elements determined by $\varrho(t) \,.\, g_t = r_t$. We get the mapping $\delta : U \to G$, $\delta(t) = g_t$. Now

$$\Gamma^{-1}(p)({}^1\sigma) = j^1_x\varrho^{-1}(t)({}^1\sigma(t)) = j^1_x[\varrho^{-1}(t)(r_t, c_1)] =$$

$$= j^1_x[\varrho^{-1}(t)(\varrho(t) \,.\, g_t, c_1)] = j^1_x[\varrho^{-1}(t)(\varrho(t), g_t(c_1))] = j^1_x[g_t(c_1)].$$

$$\Gamma^{-1}(p)({}^2\sigma) = j^1_x[\varrho^{-1}(t)({}^2\sigma(t))] = j^1_x[\varrho^{-1}(t)(r_t \,.\, g_0, c_2)] =$$

$$= j^1_x[\varrho^{-1}(t)(r_t, g_0(c_2))] = j^1_x[g_t g_0(c_2)]$$

and thus $\Gamma^{-1}(p)(^1\sigma)$ and $\Gamma^{-1}(p)(^2\sigma)$ are G-adjoint. Conversely let $\Gamma^{-1}(p)(^1\sigma)$ and $\Gamma^{-1}(p)(^2\sigma)$ are G-adjoint; $\pi(p) = x$. Then

$$\Gamma^{-1}(p)(^i\sigma) = j^1_x[g_t(f_i)],$$

where g_t is a mapping $\delta : U \to G$, $\delta(t) = g_t$ and $f_i \in {}^iF$. Let $f_i = s_i(c_i)$ $s_i \in G$. Let $s_2 = s_1 \cdot g_0$. From the definition of the development of the section by means of $\Gamma(p)$ we get

$$(8) \qquad \varrho^{-1}(t)(^1\sigma(t)) = g_t s_1(c_1) = \varrho^{-1}(t)(\varrho(t), g_t s_1(c_1)) =$$

$$= \varrho^{-1}(t)(\varrho(t) \cdot g_t s_1, c_1),$$

$$(9) \qquad \varrho^{-1}(t)(^2\sigma(t)) = g_t s_2(c_2) = \varrho^{-1}(t)(\varrho(t), g_t s_2(c_2)) =$$

$$= \varrho^{-1}(t)(\varrho(t) \cdot g_t s_1 g_0, c_2).$$

From (8) and (9) we get: $\varrho(t) \cdot g_t s_1 \in {}^1R_t$, $\varrho(t) \cdot g_t s_1 g_0 \in {}^2R_t$ and thus $g_0 \in q(t) \in D$ for any $t \in U$, i. e. the map $q(t)$ is constant on U. The manifolds $\mathscr{S}_1$, $\mathscr{S}_2$ are D-similar at $x \in B$. Q. E. D.

5. Let ${}^iX \in \bar{J}^x_r(B, F)$. We shall say that ${}^1X$, ${}^2X$ are G-congruent if there is $g_0 \in G$, so that ${}^2X = g_0{}^1X$.

Let us consider a special pair of manifolds with connection $\mathscr{S}_1(B, E, P, {}^1\sigma, C)$, $\mathscr{S}_2(B, E, P, {}^2\sigma, C)$. $C^{(r)}(x)$ denotes the r-th prolongation of C at $x \in B$, $\Gamma^{(r)}(p)$ (where $\pi(p) = x$) denotes the representative of $C^{(r)}(x)$ at $p \in P_x$, $\Gamma^{(r)-1}(p)(\sigma)$ denotes the $(r + 1)$-th development of the section σ into F and thus $\Gamma^{(r)-1}(p)(\sigma) \in \bar{J}^{r+1}_x(B, F)$. It is obvious that if $\Gamma^{(r)-1}(p)(^1\sigma)$, $\Gamma^{(r)-1}(p)(^2\sigma)$ are G-congruent, $\Gamma^{(r)-1}(p \cdot g)(^1\sigma)$, $\Gamma^{(r)-1}(p \cdot g)(^2\sigma)$ are G-congruent, too.

Definition 6. *We shall say that $\mathscr{S}_1$, $\mathscr{S}_2$ are G-equivalent of the order $(r + 1)$ at $x \in B$ if the jets $\Gamma^{(r)-1}(p)(^1\sigma)$, $\Gamma^{(r)-1}(p)(^2\sigma)$ are G-congruent* $(\pi(p) = x)$.

Note. Let $\mathscr{S}_1$, $\mathscr{S}_2$ be G-equivalent of the order 2 at $x \in B$. Then: $\Gamma'^{-1}(p)(^1\sigma)$ is holonomic $\Leftrightarrow \Gamma'^{-1}(p)(^2\sigma)$ is holonomic. Then: $\psi_1 = 0 \Leftrightarrow \psi_2 = 0$. We obtain: If $\mathscr{S}_1$, $\mathscr{S}_2$ are G-equivalent of the order 2 at $x \in B$, the first semireduction, the 2-nd semireduction, respectively, of the torsion form of the pair $\mathscr{S}_1$, $\mathscr{S}_2$ vanishes if and only if the torsion form vanishes.

It is easy to prove the following characteristic of the G-equivalence of the order 1 of the manifolds $\mathscr{S}_1$, $\mathscr{S}_2$: $\mathscr{S}_1$, $\mathscr{S}_2$ are G-equivalent of the order 1 at $x \in B$ if and only if there are a jet $Y \in J^1_x(B, G)$, $g_0 \in G$ and $p \in P_x$, so that $\Gamma(p) \cdot Y \in J^1_x({}^1R, \pi, B)$ and $\Gamma(p) \cdot g_0 Y \in J^1_x({}^2R, \pi, B)$, where the symbols $\Gamma(p) \cdot Y$ and $\Gamma(p) \cdot g_0 Y$ indicate the first prolongation of the operation of the group G on P.

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