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EXTENSION OF CONTINUOUS FUNCTIONALS

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§1. Introduction

This paper deals with the problem of extension of functionals under conditions which make the problems of integral and measure extension special cases of this problem.

In [2] another solution of this problem was proposed. The solution proposed in this paper is a generalization of that discussed in [1] where it was used for measure extension.

§2. Construction

Consider a lattice X. We shall call X relatively σ-complete if for every sequence of elements $\{x_n\}_{n=1}^{\infty}$, $x_n \in X$ the following implication holds:

$$x_n \leqslant x_0, \, x_0 \in X \Rightarrow \exists \bigcup_{n=1}^{\infty} x_n \in X (x_n \geqslant x_0, \, x_0 \in X \Rightarrow \bigcap_{n=1}^{\infty} x_n \in X).$$

Further we shall write $x_n \nearrow x$ to denote that $x_n \leqslant x_{n+1}$, $x = \bigcup_{n=1}^{\infty} x_n$. $x_n \searrow x$ will be interpreted analogously.

Let X be a relatively σ-complete lattice. On X define further two operations $+$ and $-$. Suppose that on X the following relations hold:

1) $x_n, y_n \in X, \, x_n \nearrow x, \, y_n \nearrow y \Rightarrow x_n \cap y_n \nearrow x \cap y$,
2) $x_n, y_n \in X, \, x_n \searrow x, \, y_n \searrow y \Rightarrow x_n \cup y_n \searrow x \cup y$,
3) $x, y \in X \Rightarrow x + y = y + x$,
4) $x, y, z \in X, \, x \geqslant y \Rightarrow x + z \leqslant y + z, \, x - z \leqslant y - z, \, z - x \geqslant z - y$,
5) $x_n, y_n \in X, x_n \nearrow x, \, y_n \nearrow y (x_n \searrow x, \, y_n \searrow y) \Rightarrow x_n + y_n \nearrow x + y \, (x_n + y_n \searrow x + y)$
6) $x_n, \, y \in X, \, x_n \nearrow x (x_n \searrow x) \Rightarrow x_n - y \nearrow x - y, \, y - x_n \searrow y - x (x_n - y \searrow x - y, \, y - x_n \nearrow y - x)$,
7) $x, y \in X, \, x \geqslant y \Rightarrow x = y + (x - y)$.
8) There exists an element $0 \in X$ such that $x - x = 0$ for every $x \in X$.

9) $x_1, y_1, x_2, y_2 \in X, x_1 \geqslant x_2, y_1 \geqslant y_2 \Rightarrow [(x_1 + y_1) - (x_2 + y_2)] \cup [(x_1 - y_2) - (x_2 - y_1)] \leqslant (x_1 - x_2) + (y_1 - y_2)$.

Definition 1. $M \in X \Rightarrow M_\sigma = \{x : x \in X, \exists x_n \in M, x_n \nearrow x\}$.

Definition 2. $M \in X \Rightarrow M_\delta = \{x : x \in X, \exists x_n \in M, x_n \searrow x\}$.

Evidently the following lemmas hold:

Lemma 1. *Let M be a lattice. Let $x_n \in M_\sigma$, $x_n \nearrow x(x_n \in M_\delta, x_n \searrow x)$. Then there exist $y_n \in M$ $y_n \leqslant x_n$ $(y_n \geqslant x_n)$ such that $y_n \nearrow x(y_n \searrow x)$.*

Lemma 2. *Let M be a lattice closed with respect to the operation $+$. Let x, $y \in M_\sigma(x, y \in M_\delta)$. Then there exist $u_n \in M$ such that $u_n \nearrow x + y(u_n \searrow x + y)$.*

Lemma 3. *Let $x \in M_\sigma$, $y \in M_\delta$. Suppose that there exist x_n, $y_n \in M$ such that $x_n \nearrow x$, $y_n \searrow y$. $x_n - y_n \in M(y_n - x_n \in M)$ for every n. Then $x - y \in M_\sigma(x - y \in M_\delta)$.*

Let $A \in X$ be a sublattice of X closed with respect to the operations $+$ and $-$ and having the following property:

(α) For every $x \in X$ there exist y, $z \in A$ such that $y \leqslant x \leqslant z$. On A define a finite real-valued functional φ_0 such that

(I) $x, y \in A, x \geqslant y \Rightarrow \varphi_0(x) \geqslant \varphi_0(y)$,

(II) $x, y \in A \Rightarrow \varphi_0(x \cup y) + \varphi_0(x \cap y) = \varphi_0(x) + \varphi_0(y)$,

(III) $x, y \in A, x \geqslant y \Rightarrow \varphi_0(x) = \varphi_0(y) + \varphi_0(x - y)$,

(IV) $x, y \in A \Rightarrow \varphi_0(x + y) \leqslant \varphi_0(x) + \varphi_0(y)$,

(V) $x_n \in A, x_n \searrow 0 \Rightarrow \lim_{n\to\infty} \varphi_0(x_n) = 0$.

Lemma 4. *If x, $y \in A_\sigma(x, y \in A_\delta)$, $x_n \nearrow x$, $y_n \nearrow y$, $(x_n \searrow x$, $y_n \searrow y)$, x_n, $y_n \in A$, $x \geqslant y$, then $\lim_{n\to\infty} \varphi_0(x_n) \geqslant \lim_{n\to\infty} \varphi_0(y_n)$. If $x \in A_\sigma \cap A_\delta$, x_n, $y_n \in A$, $x_n \nearrow x$, $y_n \searrow x$ then $\lim_{n\to\infty} \varphi_0(x_n) = \lim_{n\to\infty} \varphi_0(y_n)$.*

Proof. 1. We shall prove the first statement for $x, y \in A_\sigma$. The proof would be analogous for $x, y \in A_\delta$. For all natural n $x \cap x_n = y_n$. Evidently $x_m \cap y_n \nearrow x \cap y_n$, therefore $x_m \cap y_n \nearrow y_n$. By II and I we have $\varphi_0(y_n) = \lim_{m\to\infty} \varphi_0(x_m \cap y_n) \leqslant \lim_{m\to\infty} \varphi_0(x_m)$. *Therefore* $\lim_{n\to\infty} \varphi_0(y_n) \leqslant \lim_{m\to\infty} \varphi_0(x_m)$.

2) $x \in A_\sigma \cap A_\delta$. Evidently $y_n - x_n \searrow 0$. By V we have $\lim_{n\to\infty} \varphi_0(y_n - x_n) = 0$. By III, $\lim_{n\to\infty} \varphi_0(y_n) = \lim_{n\to\infty} \varphi_0(x_n)$.

Definition 3. $R_1 = A_\sigma \cup A_\delta$. $x \in A_\sigma(x \in A_\delta) \Rightarrow \varphi_1(x) = \lim_{n\to\infty} \varphi_0(x_n)$, $x_n \in A$, $x_n \nearrow x(x_n \searrow x)$.

Remark 1. These limits always exist and are finite, since they are limits of monotonous bounded sequences of real number.

Remark 2. The uniqueness of $\varphi_1(x)$ is guaranteed by Lemma 4.

Remark 3. $x \in A \Rightarrow \varphi_1(x) = \varphi_0(x)$.

Lemma 5. $x, y \in A_\sigma(x, y \in A_\delta)$, $x \geqslant y \Rightarrow \varphi_1(x) \geqslant \varphi_1(y)$.

Proof. A consequence of Lemma 4.

Lemma 6. $x, y \in A_\sigma(x, y \in A_\delta) \Rightarrow \varphi_1(x) + \varphi_1(y) = \varphi_1(x \cup y) + \varphi_1(x \cap y)$.

Proof. We shall consider only the case $x, y \in A_\sigma$.

Let $x_n, y_n \in A$, $x_n \nearrow x$, $y_n \nearrow y$. Then $x_n \cup y_n \nearrow x \cup y$, $x_n \cap y_n \nearrow x \cap y$. By II, for every n

$$\varphi_0(x_n) + \varphi_0(y_n) = \varphi_0(x_n \cup y_n) + \varphi_0(x_n \cap y_n),$$

and therefore for $n \to \infty$.

$$\varphi_1(x) + \varphi_1(y) = \varphi_1(x \cup y) + \varphi_1(x \cap y).$$

Lemma 7. $x, y \in A_\sigma(x, y \in A_\delta) \Rightarrow \varphi_1(x + y) \leqslant \varphi_1(x) + \varphi_1(y)$.

Proof. Analogous to that of Lemma 6.

Lemma 8. $x \in A_\sigma$, $y \in A_\delta$, $x \leqslant y \Rightarrow \varphi_1(y) = \varphi_1(x) + \varphi_1(y - x)$.

Proof. Let $x_n, y_n \in A$, $x_n \nearrow x$, $y_n \searrow y$. Then $y_n \geqslant x_n$, $y_n - x_n \searrow y - x$.

$$\varphi_0(y_n) = \varphi_0(x_n) + \varphi_0(y_n - x_n),$$

$$\varphi_1(y) = \varphi_1(x) + \varphi_1(y - x).$$

Lemma 9. $x \in A_\sigma$, $y \in A_\delta$, $x \leqslant y \Rightarrow \varphi_1(x) \leqslant \varphi_1(y)$.

Proof. Analogous to that of Lemma 8.

Lemma 10. Let $x_n \in A_\sigma(x_n \in A_\delta)$, $x_n \nearrow x(x_n \searrow x)$. Then $\varphi_1(x) = \lim\limits_{m \to \infty} \varphi_1(x_m)$.

Proof. We shall consider only the case $x_n \in A_\sigma$. By Lemma 5 $\varphi_1(x_n) \leqslant \leqslant \varphi_1(x)$, and therefore also $\lim\limits_{n \to \infty} \varphi_1(x_n) \leqslant \varphi_1(x)$. Is is now sufficient to prove the reversed inequality. According to Lemma 1 there exist $y_n \in A$ $(n = 1, 2, \ldots)$ such that $y_n \leqslant x_n$ $(n = 1, 2, \ldots)$ and $y_n \nearrow x$. By Lemma 5, $\varphi_0(y_n) \leqslant \varphi_1(x_n)$,

$$\lim_{n \to \infty} \varphi_0(y_n) \leqslant \lim_{n \to \infty} \varphi_1(x_n),$$

$$\varphi_1(x) \leqslant \lim_{n \to \infty} \varphi_1(x_n),$$

since $y_n \nearrow x$.

Lemma 11. *Let* $x \in A_\sigma$, $y \in A$, $x \geqslant y$. *Then* $\varphi_1(x) = \varphi_0(y) + \varphi_1(x - y)$.

Proof. Let $x_n \in A$ so that $x_n \geqslant y$, $x_n \nearrow x$ and use III.

Lemma 12. *Let* $x_n, x \in A_\sigma(x_n, x \in A_\delta)$, $x_n \searrow x(x_n \nearrow x)$. *Then* $\varphi_1(x) = \lim\limits_{n \to \infty} \varphi_1(x_n)$.

Proof. We shall consider only the case $x_n, x \in A_\delta$, since in the other case the proof would be analogous.

By Lemma 5, $\lim_{n\to\infty} \varphi_1(x_n) \geqslant \varphi_1(x)$. It is therefore sufficient to prove the reversed inequality. Let $x^n \in A$, $x^n \nearrow x$. By Lemma 11, $\varphi_1(x_n) = \varphi_0(x^n) + \varphi_1(x_n - x^n)$ so that $\lim_{n\to\infty} \varphi_1(x_n) = \lim_{n\to\infty} \varphi_0(x^n) + \lim_{n\to\infty} \varphi_1(x_n - x^n) = \varphi_1(x) + \lim_{n\to\infty} \varphi_1(x_n - x^n)$. Since $x_n - x^n \searrow 0$, it suffices to prove that $u_n \in A_\sigma$ $(n = 1, 2, \ldots)$, $u_n \searrow 0 \Rightarrow \lim_{n\to\infty} \varphi_1(u_n) = 0$. Let $\varepsilon > 0$. For every n there exists $v_n \in A$ such that $0 \leqslant v_n \leqslant u_n$ and $\varphi_1(u_n) < \varphi_0(v_n) + \varepsilon \,.\, 2^{-n}$. Put $w_n = \bigcap_{i=1}^{n} v_i$. Then $\varphi_1(u_n) < \varphi_0(w_n) + \varepsilon \sum_{i=1}^{n} 2^{-i}$. The proof shall be by induction.

1) For $n = 1$ — evident from the definition.

2) $\varphi_0(w_{n+1}) = \varphi_0(w_n \cap v_{n+1}) = \varphi_0(w_n) + \varphi_0(v_{n+1}) - \varphi_0(w_n \cup v_{n+1}) > \varphi_1(u_n) - \varepsilon \sum_{i=1}^{n} 2^{-i} + \varphi_1(u_{n+1}) - \varepsilon 2^{-n-1} - \varphi_1(u_n) = \varphi_1(u_{n+1}) - \varepsilon \sum_{i=1}^{n+1} 2^{-i}$.

Thus $0 \leqslant \varphi_1(u_n) < \varphi_0(w_n) + \varepsilon \sum_{i=1}^{n+1} 2^{-i}$,

$$0 \leqslant \lim_{n\to\infty} \varphi_1(u_n) \leqslant \lim_{n\to\infty} \varphi_0(w_n) + \varepsilon.$$

Since $w_n \in A$, $w_n \searrow 0$, we have $\lim_{n\to\infty} \varphi_0(w_n) = 0$, $0 \leqslant \lim_{n\to\infty} \varphi_1(u_n) \leqslant \varepsilon$. Therefore $\lim_{n\to\infty} \varphi_1(u_n) = 0$.

Lemma 13. *Let $x \in A_\sigma$, $y \in A_\delta$, $x \geqslant y$. Then $\varphi_1(x) = \varphi_1(y) + \varphi_1(x - y)$.*

Proof. Let $x_n, y_n \in A$, $x_n \nearrow x$, $y_n \searrow y$. By Lemma 11, $\varphi_1(x) = \varphi_0(x_n \cap y_m) + \varphi_1(x - (x_n \cap y_m))$. For $m \to \infty$ we have $x_n \cap y_m \searrow x_n \cap y \in A_\delta$, $x - (x_n \cap y_m) \nearrow x - (x_n \cap y) \in A_\sigma$. By Lemma 10, $\varphi_1(x) = \varphi_1(x_n \cap y) + \varphi_1(x - (x_n \cap y))$. For $n \to \infty$ we have $x_n \cap y \nearrow y$, $x - (x_n \cap y) \searrow x - y$. By Lemma 12, $\varphi_1(x) = \varphi_1(y) + \varphi_1(x - y)$.

Lemma 14. *Let $x \in A_\sigma$, $y \in A_\delta$, $x \geqslant y$. Then $\varphi_1(x) \geqslant \varphi_1(y)$.*

Proof. Analogous to that of Lemma 13.

Definition 4. $A_{\sigma\delta} = (A_\sigma)_\delta$, $A_{\delta\sigma} = (A_\delta)_\sigma$.

By Lemma 2, $A_{\sigma\delta}$ and $A_{\delta\sigma}$ are lattices closed with respect to $+$. Analogously as in Lemma 3 we could prove that if $x \in A_{\sigma\delta}$, $y \in A_{\delta\sigma}$, then $x - y \in A_{\sigma\delta}$, $y - x \in A_{\delta\sigma}$.

Lemma 15. *Let $x, y \in A_{\sigma\delta}(x, y \in A_{\delta\sigma})$, $x_n \nearrow x$, $y_n \nearrow y(x_n \searrow x, y_n \searrow y)$, $x \geqslant y$, $x_n, y_n \in A_\sigma(x_n, y_n \in A_\delta)$, then $\lim_{n\to\infty} \varphi_1(x_n) \geqslant \lim_{n\to\infty} \varphi_1(y_n)$. If $x \in A_{\sigma\delta} \cap A_{\delta\sigma}$, $x_n \in A_\delta$, $y_n \in A_\sigma$, $x_n \nearrow x$, $y_n \searrow x$, then $\lim_{n\to\infty} \varphi_1(x_n) = \lim_{n\to\infty} \varphi_1(y_n)$.*

Proof. 1) The proof of the first statement would be analogous to that of Lemma 4.

2) Let $x \in A_{\sigma\delta} \cap A_{\delta\sigma}$. Evidently $x_n \leqslant y_n$. By Lemma 13 $\varphi_1(y_n) - \varphi_1(x_n) = \varphi_1(y_n - x_n)$. Clearly $y_n - x_n \in A_\sigma$, $y_n - x_n \searrow 0$. By Lemma 12, $\lim_{n\to\infty} \varphi_1(y_n - x_n) = 0$. Therefore $\lim_{n\to\infty} \varphi_1(y_n) - \lim_{n\to\infty} \varphi_1(x_n) = \lim_{n\to\infty} \varphi_1(y_n - x_n) = 0$.

Definition 5. $R_2 = A_{\sigma\delta} \cup A_{\delta\sigma}$.

Definition 6. $x \in A_{\delta\sigma}(x \in A_{\sigma\delta}) \Rightarrow \varphi_2(x) = \lim_{n\to\infty} \varphi_1(x_n)$, $x_n \in A_\delta$, $x_n \nearrow x(x_n \in A_\sigma$, $x_n \searrow x)$.

Remark 1. These limits always exist and are finite since they are limits of bounded monotonous sequences of real numbers.

Remark 2. Lemma 15 guarantees the uniqueness of $\varphi_2(x)$.

Remark 3. $x \in R_1 \Rightarrow \varphi_2(x) = \varphi_1(x)$.

The following statements could be proved by methods analogous to those used heretofore.

Lemma 16. $x, y \in A_{\delta\sigma}(x, y \in A_{\sigma\delta}) \Rightarrow \varphi_2(x) + \varphi_2(y) = \varphi_2(x \cup y) + \varphi_2(x \cap y)$.

Lemma 17. $x, y \in A_{\delta\sigma}(x, y \in A_{\sigma\delta}) \Rightarrow \varphi_2(x + y) \leqslant \varphi_2(x) + \varphi_2(y)$.

Lemma 18. $x \in A_{\sigma\delta}$, $y \in A_{\delta\sigma}$, $x \geqslant y \Rightarrow \varphi_2(x) = \varphi_2(y) + \varphi_2(x - y)$.

Lemma 19. $x \in A_{\delta\sigma}$, $y \in A_{\sigma\delta}$, $x \leqslant y \Rightarrow \varphi_2(x) \leqslant \varphi_2(y)$.

Lemma 20. $x, y \in A_{\delta\sigma}(x, y \in A_{\sigma\delta})$, $x \geqslant y \Rightarrow \varphi_2(x) \geqslant \varphi_2(y)$.

Lemma 21. $x_n \in A_{\delta\sigma}$, $x_n \nearrow x$, $(x_n \in A_{\sigma\delta}$, $x_n \searrow x) \Rightarrow \varphi_2(x) = \lim_{n\to\infty} \varphi_2(x_n)$.

Definition 7. *We say that* $x \in S$ *iff there exist* $y \in A_{\delta\sigma}$, $z \in A_{\sigma\delta}$ *such that* $y \leqslant x \leqslant z$ *and* $\varphi_2(y) = \varphi_2(z)$.

Lemma 22. *Let* $x \in \mathrm{S}$. *Suppose further that* y_1, $y_2 \in A_{\delta\sigma}$, z_1, $z_2 \in A_{\sigma\delta}$. *Let* $y_1 \leqslant x \leqslant z_1$ and $y_2 \leqslant x \leqslant z_2$. *If, moreover,* $\varphi_2(y_1) = \varphi_2(z_1)$, $\varphi_2(y_2) = \varphi_2(z_2)$. Then $\varphi_2(z_1) = \varphi_2(z_2)$.

Proof. Evidently $z_1 \cap z_2 \in A_{\sigma\delta}$, $z_1 \cap z_2 \geqslant x$. By Lemma 19 and 20 we have $\varphi_2(y_1) \leqslant \varphi_2(z_1 \cap z_2) \leqslant \varphi_2(z_1) = \varphi_2(y_1)$, so that $\varphi_2(z_1 \cap z_2) = \varphi_2(z_1)$.
$\varphi_2(y_2) \leqslant \varphi_2(z_1 \cap z_2) \leqq \varphi_2(z_2) = \varphi_2(y_2)$, so that $\varphi_2(z_1 \cap z_2) = \varphi_2(z_2)$.
Therefore $\varphi_2(z_1) = \varphi_2(z_1 \cap z_2) = \varphi_2(z_2)$.

Definition 8. $x \in \mathrm{S} \Rightarrow \varphi(x) = \varphi_2(y) = \varphi_2(z)$, *where* y, z *are those of Definition* 7.

Remark 1. Lemma 22 ensures the uniqueness of $\varphi(x)$.

Remark 2. We see that φ is a finite functional which is an extension of φ_0.

§3. The proof

Theorem 1. $x, y \in \mathrm{S}$, $x \geqslant y \Rightarrow \varphi(x) \geqslant \varphi(y)$.

Proof. We choose $x_1, y_1 \in A_{\sigma\delta}$, $x_2, y_2 \in A_{\delta\sigma}$ such that $x_2 \leqslant x \leqslant x_1$, $y_2 \leqslant y \leqslant$

$\leqslant y_1$, $\varphi_2(x_1) = \varphi_2(x_2)$, $\varphi_2(y_1) = \varphi_2(y_2)$. Evidently $x_2 \cup y_2 \in A_{\delta\sigma}$. Then $x_2 \cup \cup y_2 \leqslant x \cup y = x \leqslant x_1$ $x_2 \leqslant x_2 \cup y_2 \leqslant x_1$. By Lemma 19 and 20 $\varphi_2(x_2) \leqslant \leqslant \varphi_2(x_2 \cup y_2) \leqslant \varphi_2(x_1) = \varphi_2(x_2)$ or $\varphi_2(x_2) = \varphi_2(x_2 \cup y_2)$. Therefore $\varphi(x) = = \varphi_2(x_2) = \varphi_2(x_2 \cup y_2) \geqslant \varphi_2(y_2) = \varphi(y)$.

Theorem 2. $x, \; y \in S \Rightarrow x \cup y, \; x \cap y \in S, \; \varphi(x) + \varphi(y) = \varphi(x \cup y) + + \varphi(x \cap y)$.

Proof. Choose x_1, y_1, x_2, y_2 as in the proof of Theorem 1. Then $x_1 \cup y_1 \geqslant \geqslant x \cup y \geqslant x_2 \cup y_2$, $x_1 \cap y_1 \geqslant x \cap y \geqslant x_2 \cap y_2$.

By Lemma 16, we have

(1) $\varphi_2(x_1) + \varphi_2(y_1) = \varphi_2(x_1 \cup y_1) + \varphi_2(x_1 \cap y_1)$,

(2) $\varphi_2(x_2) + \varphi_2(y_2) = \varphi_2(x_2 \cup y_2) + \varphi_2(x_2 \cap y_2)$.

By Lemma 19,

(3) $\varphi_2(x_1 \cap y_1) \geqslant \varphi_2(x_2 \cap y_2)$.

Substituting (3) into (1), we obtain

(4) $\varphi_2(x_1) + \varphi_2(y_1) \geqslant \varphi_2(x_1 \cup y_1) + \varphi_2(x_2 \cap y_2)$,

and subtracting (2) from (4)

$\varphi_2(x_1 \cup y_1) \leqslant \varphi_2(x_2 \cup y_2)$.

By Lemma 19,

$\varphi_2(x_1 \cup y_1) \geqslant \varphi_2(x_2 \cup y_2)$, so that

(5) $\varphi_2(x_1 \cup y_1) = \varphi_2(x_2 \cup y_2)$. Therefore $x \cup y \in S$.

By substituting (5) into (1) and subtracting (2) we obtain

$\varphi_2(x_1 \cap y_1) = \varphi_2(x_2 \cap y_2)$ so that $x \cap y \in S$. Now

$\varphi(x) + \varphi(y) = \varphi_2(x_1) + \varphi_2(y_1) = \varphi_2(x_1 \cup y_1) + \varphi_2(x_1 \cap y_1) = \varphi(x \cup y) + + \varphi(x \cap y)$.

Theorem 3. $x, y \in S \Rightarrow x - y \in S$; $x \geqslant y \Rightarrow \varphi(x) = \varphi(y) + \varphi(x - y)$.

Proof. The choice of x_1, y_1, x_2, y_2 is the same as in proving Theorem 1. Evidently $x_2 - y_1 \leqslant x - y \leqslant x_1 - y_2$, $x_2 - y_1 \in A_{\delta\sigma}$, $x_1 - y_2 \in A_{\sigma\delta}$. By Lemma 19, $\varphi_2(x_2 - y_1) \leqslant \varphi_2(x_1 - y_2)$.

By Lemma 18, axiom 9, and by Lemmas 20 and 17, we have $0 \leqslant \varphi_2(x_1 - - y_2) - \varphi_2(x_2 - y_1) = \varphi_2((x_1 - y_2) - (x_2 - y_1)) \leqslant \varphi_2((x_1 - x_2) + (y_1 - - y_2)) \leqslant \varphi_2(x_1 - x_2) + \varphi_2(y_1 - y_2) = 0$, so that $\varphi_2(x_1 - y_2) = \varphi_2(x_2 - y_1)$, which means that $x - y \in S$.

Now let $x \geqslant y$. Then $x_1 \geqslant y_2$. By Lemma 18, $\varphi(x - y) = \varphi_2(x_1 - y_2) = = \varphi_2(x_1) - \varphi_2(y_2(= \varphi(x) - \varphi(y)$.

Theorem 4. $x, \; y \in S \Rightarrow x + y \in S$, $\varphi(x) + \varphi(y) \geqslant \varphi(x + y)$.

Proof. With the same choice of x_1, y_1, x_2, y_2, we see that $x_2 + y_2 \leqslant x + + y \leqslant x_1 + y_1$, $x_2 + y_2 \in A_{\delta\sigma}$, $x_1 + y_1 \in A_{\sigma\delta}$. By Lemma 19, $\varphi_2(x_2 + y_2) \leqslant \leqslant \varphi_2(x_1 + y_1)$. By Lemmata 18, axioms 9, 20 and 17, we have $0 \leqslant \varphi_2(x_1 + + y_1) - \varphi_2(x_2 + y_2) = \varphi_2((x_1 + y_1) - (x_2 + y_2)) \leqslant \varphi_2((x_1 - x_2) + (y_1 - y_2)) =$

$= \varphi_2(x_1 - x_2) + \varphi_2(y_1 - y_2) = 0$, so that $\varphi_2(x_1 + y_1) = \varphi_2(x_2 + y_2)$, or $x + + y \in \mathrm{S}$. Now $\varphi(x + y) = \varphi_2(x_1 + y_1) \leqslant \varphi_2(x_1) + \varphi_2(y_1) = \varphi(x) + \varphi(y)$.

Theorem 5. *Let* $x_n \in \mathrm{S}$, $x_n \leqslant x_{n+1}(x_n \geqslant x_{n+1})$. *Let there exist* $a \in X$ *such that* $x_n \leqslant a(x_n \geqslant a)$. *Then*

$$x = \bigcup_{n=1}^{\infty} x_n \in \mathrm{S}\Big(x = \bigcap_{n=1}^{\infty} x_n \in \mathrm{S}\Big) \text{ and } \varphi(x) = \lim_{n\to\infty} \varphi(x_n).$$

Proof. According to (α) we may suppose that $a \in A$. The proof will be outlined for a non-decreasing sequence with the understanding that the same methods could be used for nonincreasing sequences.

Let $y_n \in \mathrm{A}_{\sigma\delta}$, $z_n \in A_{\delta\sigma}$, $z_n \leqslant x_n \leqslant y_n$ and, $\varphi_2(z_n) = \varphi_2(y_n)$, $z_n \leqslant z_{n+1}$. By Lemma 21, $z_n \nearrow z \in A_{\delta\sigma}$,

$$\varphi_2(z) = \lim_{n\to\infty} \varphi_2(z_n) = \lim_{n\to\infty} \varphi(x_n).$$

Clearly $z \leqslant x$.

Let $\varepsilon > 0$. There exists $v_n(\varepsilon) \in A_\sigma$ such that $y_n \leqslant v_n(\varepsilon) \leqslant a$, $\varphi_1(v_n(\varepsilon)) < < \varphi_2(y_n) + \varepsilon \,.\, 2^{-n}$.

Put $w_n(\varepsilon) = \bigcup_{i=1}^{n} v_i(\varepsilon)$. Then $w_n(\varepsilon) \leqslant w_{n+1}(\varepsilon)$, $w_n(\varepsilon) \in A_\sigma$, $w_n(\varepsilon) \leqslant a$. It is necessary to prove that $\varphi_1(w_n(\varepsilon)) < \varphi_2(y_n) + \varepsilon\sum_{i=1}^{n} 2^{-i}$. This we shall do by induction

1. For $n = 1$ the statement holds by definition.

2. $\varphi_1(w_{n+1}(\varepsilon)) = \varphi_1(w_n(\varepsilon) \cup v_{n+1}(\varepsilon)) = \varphi_1(w_n(\varepsilon)) + \varphi_1(v_{n+1}(\varepsilon)) - \varphi_1(w_n(\varepsilon) \cap \cap v_{n+1}(\varepsilon)) < \varphi_2(y_n) + \varepsilon\sum_{i=1}^{n} 2^{-i} + \varphi_2(y_{n+1}) + \varepsilon 2^{-n-1} - \varphi_2(y_n) = \varphi_2(y_{n+1}) + \varepsilon\sum_{i=1}^{n+1} 2^{-i}$.

Further $w_n(\varepsilon) \nearrow w(\varepsilon) \in A_\sigma$, $w(\varepsilon) \geqslant x$,
hence $\varphi_1(w(\varepsilon)) = \lim_{n\to\infty} \varphi_1(w_n(\varepsilon)) \leqslant \lim_{n\to\infty} \varphi_2(y_n) + \varepsilon$.

Put $u_n = \bigcap_{i=1}^{n} w(1/i)$. Evidently $u_n \in A_\sigma$, $z \leqslant x \leqslant u_n \leqslant w(1/n)$, $u_{n+1} \leqslant u_n$, hence $\varphi_2(z) \leqslant \varphi_1(u_n) \leqslant \varphi_1(w(1/n)) \leqslant \lim_{n\to\infty} \varphi_2(y_n) + 1/n = \lim_{n\to\infty} \varphi_2(z_n) + 1/n = = \varphi_2(z) + 1/n$, further $u_n \searrow u \in A_{\sigma\delta}$, $x \leqslant u$, $\varphi_2(z) \leqslant \varphi_2(u) \leqslant \varphi_1(u_n) \leqslant \varphi_2(z) + + 1/n$, therefore $\varphi_2(u) = \varphi_2(z)$, so that $x \in \mathrm{S}$.

Now $\varphi(x) = \varphi_2(z) = \lim_{n\to\infty} \varphi_2(z_n) = \lim_{n\to\infty} \varphi(x_n)$.

Corollary. $R_2 \subset S$.

Remark. Theorems 1—5 demonstrate that φ is the required extension of φ_0.

Theorem 6. φ is the only extension of φ_0 to S with the properties I—V.

Proof. Let there exist a functional $\psi(x)$ defined on S with the properties I—V such that $x \in \mathrm{A} \Rightarrow \psi(x) = \varphi_0(x)$.

A consequence of the properties (α) and I is the fact that $\psi(x)$ is a finite functional. Let $x \in A_\sigma$. Then there exist $x_n \in A$, $x_n \nearrow x$. By the properties III and V we have $\psi(x) = \lim_{n\to\infty} \psi(x_n)$. Therefore $\psi(x) = \lim_{n\to\infty} \psi(x_n) = \lim_{n\to\infty} \varphi(x_n) = \varphi(x)$. Analogously we could prove that if $x \in R_2 \Rightarrow \varphi(x) = \psi(x)$.

Let $x \in S$. Take $y \in A_{\sigma\delta}$, $z \in A_{\delta\sigma}$, $z \leqslant x \leqslant y$, $\varphi_2(z) = \varphi_2(y)$. By I, $\varphi(x) = \varphi_2(z) = \psi(z) \leqslant \psi(x) \leqslant \psi(y) = \varphi_2(y) = \varphi(x)$, so that $\psi(x) = \varphi(x)$.

Theorem 7. *Let $x \in X$. Suppose that there exist $y, z \in S$ such that $y \leqslant x \leqslant z$, $\varphi(y) = \varphi(z)$. Then $x \in S$.*

Proof. Take $y_1, z_1 \in A_{\sigma\delta}$, $y_2, z_2 \in A_{\delta\sigma}$ such that $y_2 \leqslant y \leqslant y_1$, $z_2 \leqslant z \leqslant z_1$, $\varphi_2(y_1) = \varphi_2(y_2)$, $\varphi_2(z_1) = \varphi_2(z_2)$. Evidently $y_2 \leqslant y \leqslant x \leqslant z \leqslant z_1$, $\varphi_2(y_2) = \varphi_2(z_1)$ and therefore $x \in S$.

§4. Measure and integral

1) Consider a set M. Let $\mathscr{X}$ be the system of all subsets of M. The operations $\cup$, $\cap$, $-$ will be interpreted in their usual set theoretic sense. Let $+$ be identical with $\cup$. Then the assumptions 1—9 hold for X.

Now let $\mathscr{A}$ be a nonempty algebra of subsets of M on which we define a finite measure μ_0. Put $\varphi_0(x) = \mu_0(x)$. Then the assumptions I—V are also satisfied. Consider the functional φ of Definition 8. By Theorems 1—5 $\mathscr{S}$ is a σ-algebra and φ a measure. By Theorem 6 it is the only extension of φ_0 to $\mathscr{S}$. By Theorem 7, φ is a complete measure.

2) Consider a set M. Let X be the system of all bounded real-valued functions defined on M. Let the operations $+$ and $-$ retain their usual sense. Let $x \cup y = \max\{x, y\}$, $x \cap y = \min\{x, y\}$. Then the assumptions 1—9 are satisfied for X.

Let A be a system of all simple integrable functions. Let μ be a finite measure on a σ-algebra. Let $\varphi_0(x) = \int x \, d\mu$. Then the assumptions I—V are also satisfied. Theorems 1—7 ensure the existence of a unique extension of φ_0 to φ where φ is an integral on $\mathscr{S}$. The integral is complete by Theorem 7. Evidently if φ_0 is linear on A, then φ is on $\mathscr{S}$.

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