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ON THE EXTENSION OF A MEASURE ON LATTICES

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Every measure γ defined on a subalgebra R of a σ-complete Boolean algebra H can be extended to a measure γ on the smallest σ-algebra S over R. In the present paper we prove this theorem for a certain type of not necessarily distributive lattices (σ-continuous, orthocomplemented, modular).

We have only two new definitions. All other definitions will be used according to [2]. If b is an element of a complemented lattice H, then we denote by $C(b)$ the set of all complements of the element b. A non-empty subset R of an orthocomplemented lattice S is called a lattice ring if $a \cup b$, $a \cap b$, $a \cap b^{\perp} \in R$ for any $a, b \in R$. A lattice σ-ring is a σ-complete lattice ring.

A real-valued function γ defined on a lattice ring R is called a measure if it fulfills the following three conditions(1):

(1) If $x_n \nearrow x$, $x_n \in R$ $(n = 1,2, \ldots)$, $x \in R$, then $\lim\limits_{n\to\infty} \gamma(x_n) = \gamma(x)$.

(2) $\gamma(x) + \gamma(y) = \gamma(x \cup y) + \gamma(x \cap y)$ for every $x, y \in R$.

(3) $\gamma(0) = 0$ and $\gamma(x) \geqq 0$ for every $x \in R$.

Theorem 1. *Let H be a σ-continuous, modular, complemented (orthocomplemented) lattice. Let R be a sublattice of H and let for any $a, b \in R$ and any $b' \in C(b)$ the following holds: $a \cap b' \in R$ $(a \cap b^{\perp} \in R)$. Let γ be a finite measure on R.*

Then there exists a set $N \subset R$ and a finite real-valued function $\bar{\gamma}$ on N with the following properties:

(4) *N is a conditionally σ-complete sublattice of H.*

(5) *$\bar{\gamma}$ is an extension of γ, i. e. $\bar{\gamma}(a) = \gamma(a)$ for $a \in R$.*

(6) *$\bar{\gamma}(x) + \bar{\gamma}(y) = \bar{\gamma}(x \cup y) + \bar{\gamma}(x \cap y)$ for any $x, y \in N$.*

(7) *$\bar{\gamma}$ is a non-negative and non descendent function.*

(8) *If $x_n \in N$, $x_n \nearrow x$ $(x_n \searrow x)$ and $\{\gamma(x_n)\}$ is bounded, then $x \in N$ and $\bar{\gamma}(x) = \lim\limits_{n\to\infty} \bar{\gamma}(x_n)$.*

(9) *Let $\gamma^*(x) = \inf \gamma_0(b)$ for $x \in H$, where the infimum is taken over all*

(1) Cf. Theorem 4.

elements $b \geqq x$ *such that there exists a sequence* $\{a_n\}$ *of elements of* R, $a_n \nearrow b$. *Here* $\gamma_0(b) = \lim_{n\to\infty} \gamma(a_n)$. *Then* $\bar{\gamma}(x) = \gamma^*(x)$ *for all* $x \in N$.

Proof. Denote by B the set of all $b \in H$ for which there exists a sequence $\{a_n\}$ of elements of R such that $a_n \nearrow b$. Let $c \leqq d$, $c, d \in B$, $c_n \nearrow c$, $d_n \nearrow d$, $c_n, d_n \in R$. From (1), (2) and σ-continuity of H it follows that

$$\gamma(c_m) = \lim_{n\to\infty} \gamma(c_m \cap d_n) \leqq \lim_{n\to\infty} \gamma(d_n)$$

hence

$$\lim_{n\to\infty} \gamma(c_n) \leqq \lim_{n\to\infty} \gamma(d_n). \tag{10}$$

Hence we can define the function γ_0 on the set B by the equality

$$\gamma_0(b) = \lim_{n\to\infty} \gamma(a_n),$$

where $a_n \nearrow b$, $a_n \in R$ $(n = 1,2,\dots)$. The function γ_0 is non-negative, non descendant, subadditive and coincident on R with γ.

For an arbitrary element $d \in H$ we define

$$\gamma^*(d) = \inf\{\gamma_0(b) : d \leqq b \in B\}.$$

The function γ^* is a extension of γ_0, non descendant (and hence also non-negative) and subadditive.

Now we shall prove the following property of γ:

(11) If $y_n, z_n \in R$, $y_n \nearrow y \in H$, $z_n \searrow z \in H$, $z \leqq y$, then $\inf \gamma(z_n) \leqq \sup \gamma(y_n)$.

By the definition $y \in B$, hence

$$\gamma^*(y) = \gamma_0(y) = \lim \gamma(y_n) = \sup \gamma(y_n). \tag{12}$$

Now we shall distinguish two cases: complemented resp. orthocomplemented lattices.(2)

Let H be complemented and let $x \cap y' \in R$ for any $x, y \in R$ and any $y' \in C(y)$. Then we have:

(13) If $z_n \searrow z$, $z_n \in R$, $z \in H$, then there exist a sequence $\{z'_n\}$ and an element z' such that $z' \in C(z)$, $z'_n \in C(z_n)$ $(n = 1,2,\dots)$ and $z'_n \nearrow z'$.(3)

It follows from the σ-continuity of H that $z_1 \cap z'_n \nearrow z_1 \cap z'$. Since $z_1 \cap z'_n \in R$, we have by (12)

$$\gamma^*(z_1 \cap z') = \lim \gamma^*(z_1 \cap z'_n), \quad z_1 \cap z'_n \nearrow z_1 \cap z'. \tag{14}$$

(2) If H is orthocomplemented we suppose less about R.

(3) [2], I., Lemma 1.9 and 1.13.

If H is orthocomplemented, then clearly (13) holds and hence (14) too. Now let $z_n \searrow z$ and $z'_n \in C(z_n)$, $z' \in C(z)$ be those complements for which (14) holds (we examine simultaneously both cases, complemented and orthocomplemented). From (14) it follows that

$$(15) \qquad \gamma^*(z_1 \cap z') = \lim \gamma^*(z_1 \cap z'_n) = \lim (\gamma(z_1) - \gamma(z_n)) = \\ = \gamma(z_1) - \lim \gamma(z_n).$$

Evidently

$$(16) \qquad \gamma^*(z_1) = \gamma^*(z \cup (z_1 \cap z')) \leqq \gamma^*(z) + \gamma^*(z_1 \cap z').$$

From (15) and (16) it follows that $\gamma^*(z) \geqq \lim \gamma^*(z_n)$.
Because the opposite inequality is evident, we have

$$(17) \qquad \gamma^*(z) = \lim \gamma(z_n) = \inf \gamma(z_n).$$

From (17), (12) and from the fact that γ^* is non descendant it follows (11).

In [1] the following theorem is proved (Theorem 5): If γ is a non descendant function on a sublattice of a σ-continuous lattice S, fulfilling the conditions (2) and (11), then there exists a conditionally σ-continuous sublattice $N \subset R$ of the lattice S such that the function $\bar{\gamma}$, defined on N by the equality $\bar{\gamma}(d) = \gamma^*(d)$ fulfills on N the conditions (4), (6), (8) (and evidently also (9)). We have found out that $\bar{\gamma}$ fulfills also (5) and (7).

Lemma 1. *Let H be a σ-continuous, orthocomplemented lattice, $R \subset H$ be a lattice ring, S the smallest lattice σ-ring over R and M the smallest monotonous set over R.*(4) *Then $S = M$.*

Proof. Since S is a monotonous set, we have $M \subset S$. For the proof of the reverse inclusion it suffices to prove that M is a ring. Let $\circ$ be an arbitrary operation of $\cup$, $\cap$. Let $x \in R$ be an arbitrary but fixed element, $G = \{y \in M : x \circ y \in M\}$. Evidently $G \supset R$, G is a monotonous set, hence $G \supset M$. Hence for each $x \in R$ and each $y \in M$ we have $x \circ y \in M$. Now let us take a fixed $y \in M$ and put $K = \{x \in M : x \circ y \in M\}$. Since according to the previous K is a monotonous set and $K \supset R$, we have $K \supset M$, hence M is closed under the lattice operations. Similarly we prove that $a \cap b^\perp \in M$ for all $a, b \in M$.

Lemma 2. *Let H be a σ-continuous, orthocomplemented, modular lattice. Let $R \subset H$ be a lattice ring, γ a finite measure on R, S the smallest lattice σ-ring over R, γ^* the function defined in Theorem 1 by (9). Let F be the smallest set over R fulfilling the following condition:*

(4) A set $K \subset H$ is monotonous if it contains the supremum and the infimum of every monotonous (i. e. non descendant, or non ascendent) sequence of K.

(α) *If either* $x_n \nearrow x$ *or* $x_n \searrow x$ *and* $\{\gamma^*(x_n)\}$ *is bounded,* $x_n \in F$ $(n = 1, 2, \ldots)$, *then* $x \in F$.[5]

Then

$$F = \{d \in S : \gamma^*(d) < \infty\}. \tag{18}$$

Proof. According to Lemma 1 we have $S = M$, where M is the smallest monotonous set over R. Clearly $R \subset F \subset N \cap M$. Because γ^* is finite on N, we have $F \subset \{d \in M : \gamma^*(d) < \infty\}$. Let $d \in M$, $\gamma^*(d) < \infty$. According to the definition of γ^*, there exists an element $e \in B$ for which $d \leqq e$, $\gamma^*(e) < \infty$, $e = \bigcup_{n=1}^{\infty} a_n$, where $a_n \in R$. Put $P = \{f \in M : f \cap e \in F\}$. Evidently $P \supset R$. P is a monotonous set, because $0 \leqq f \cap e = e$ and $\gamma^*(e) < \infty$ for all $f \in M$. Hence $P \supset M$ and $d = d \cap e \in F$. Therefore we have proved (18).

Theorem 2. *Let H be a σ-continuous, orthocomplemented, modular lattice. Let $R \subset H$ be a lattice ring, γ a finite measure on R, S the smallest lattice σ-ring over R. Then there exists a measure $\bar{\gamma}$ on S that is a extension of γ.*

Proof. Put $\bar{\gamma}(d) = \gamma^*(d)$ for all $d \in S$. $\bar{\gamma}$ is a extension of γ, it is non-negative and $\bar{\gamma}(0) = 0$. It remains to be proved that $\bar{\gamma}$ satisfies the conditions (1) and (2).

Let $\{x_n\}$ be a sequence of elements of S, $x_n \nearrow x$. Clearly $\lim \bar{\gamma}(x_n) \leqq \bar{\gamma}(x)$, hence the equality holds, if $\lim \bar{\gamma}(x_n) = \infty$. Let $\lim \bar{\gamma}(x_n) < \infty$. Then by (18) it is $x_n \in F \subset N$ for all n and $\{\bar{\gamma}(x_n)\}$ is bounded. Hence according to Theorem 1 we have $\bar{\gamma}(x) = \lim \bar{\gamma}(x_n)$ and the property (1) is proved.

The property (2) is fulfilled if at least one of the expressions $\bar{\gamma}(x)$, $\bar{\gamma}(y)$ is equal to ∞. In the reverse case we have $x, y \in F \subset N$ and (2) follows from Theorem 1.

Theorem 3. *Let H be a σ-continuous, modular, orthocomplemented lattice. Let $R \subset H$ be a lattice ring, γ a σ-finite measure on R (i. e. each element of R is majorized by the supremum of a sequence of elements of R of a finite measure). Then there exists a σ-finite measure $\bar{\gamma}$ on the smallest lattice σ-ring over R that is a extension of γ. The measure $\bar{\gamma}$ is determined uniquely.*

Proof. Put $A = \{e \in R : \gamma(e) < \infty\}$. A is a lattice ring. According to Theorem 2 there exists a measure $\bar{\gamma}$ on the smallest lattice σ-ring $S(A)$ over A that is a extension of γ. We shall prove that $S(A) = S$. We have $S(A) \subset S$ because $A \subset R$. On the other side, each element of R is a supremum of a countable number of elements of A. Actually, let $a \in R$, $a \leqq \bigcup_{n=1}^{\infty} a_n$, $a_n \in A$. Put

[5] Since the set N from Theorem 1 has the property (α), there exists such a set F.

$b_n = \bigcup_{i=1}^{n} a_i$ Clearly $b_n \in A$, $b_n \nearrow \bigcup_{n=1}^{\infty} a_n$. Further $b_n \cap a \nearrow \bigcup a_n \cap a = a$ because H is σ-continuous. Hence $\{b_n \cap a\}$ is a sequence of elements of A such that a is its supremum. Therefore $R \subset S(A)$, $S \subset S(A)$, hence $\bar{\gamma}$ is a measure on S.

The measure $\bar{\gamma}$ is σ-finite, because the set $P = \{d \in S : d \leqq \bigcup_{n=1}^{\infty} a_n, a_n \in R\}$ is monotonous and it contains R.

Let γ_1 be any measure on S being a extension of γ. Let F be the smallest set over A fu'fi'ling tho property (α) (see Lemma 2). According to Theorem 1 the set $Q = \{x \in S : \gamma_1(x) = \bar{\gamma}(x)\}$ fulfills the property (α), it contains A, hence $Q \supset F$ and γ_1 coincides with $\bar{\gamma}$ on F. Let $e \in S$. From (18) and the σ-finiteness of $\bar{\gamma}$ it follows that there exists a sequence $\{e_n\}$, $e_n \in F$, $e_n \nearrow e$. Therefore $\gamma_1(e) = \lim \gamma_1(e_n) = \lim \bar{\gamma}(e_n) = \bar{\gamma}(e)$.

Theorem 4. *Let H be a σ-complete, modular, complemented (resp. orthocomplemented) lattice. Let R be a sublattice of the lattice H and let $a \cap b' \in R$ for all $a, b \in R$ and $b' \in C(b)$ (resp. $a \cap b^{\perp} \in R$ for all $a, b \in R$). Let γ be a non-negative real-valued function on R, $\gamma(0) = 0$. Then γ is a measure if and only if γ is σ-additive, i. e. if $\gamma(\bigcup_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} \gamma(a_n)$ for any disjoint sequence of elements of R such that $\bigcup_{n=1}^{\infty} a_n \in R$.*

A sequence $\{x_n\}$ is called disjoint if for any two disjoint set of indices α, β we have $\bigcup_{i \in \alpha} x_i \cap \bigcup_{j \in \beta} x_j = 0$.

Proof. 1. Let γ be a measure. From the property (2) it follows that $\gamma(x \cup y) = \gamma(x) + \gamma(y)$ for any two disjoint elements x, y. This leads by induction $\gamma(\bigcup_{i=1}^{n} x_i) = \sum_{i=1}^{n} \gamma(x_i)$ for any finite disjoint set $\{x_1, \ldots, x_n\}$ of elements. Finaly, let $\{x_n\}$ be any disjoint sequence of elements of R. Put $y_n = \bigcup_{i=1}^{n} x_i$. Then $y_n \nearrow \bigcup_{n=1}^{\infty} x_n$, hence

$$\gamma(\bigcup_{n=1}^{\infty} x_n) = \lim \gamma(y_n) = \lim \gamma(\bigcup_{i=1}^{n} x_i) = \lim \sum_{i=1}^{n} \gamma(x_i) = \sum_{n=1}^{\infty} \gamma(x_n).$$

2. Let γ be σ-additive. First we shall prove (1). Let $x_n \nearrow x$, $x_n \in R$, $x \in R$. Put $y_1 = x_1$, $y_n = x_n \cap x_{n-1}^{\perp}$ $(n = 2,3, \ldots)$ in the case of the orthocomplementarity of H. In the case of the complementarity let us construct $x_n' \in C(x_n)$ arbitrarily and put $y_n = x_n \cap x_{n-1}'$ $(n = 2,3, \ldots)$. In both cases $x = \bigcup_{n=1}^{\infty} x = \bigcup_{n=1}^{\infty} y_n$. Therefore

$$\gamma(x) = \gamma(\bigcup_{n-1}^{\infty} y_n) = \sum_{n-1}^{\infty} \gamma(y_n) = \lim \sum_{i=1}^{n} \gamma(y_i) =$$
$$= \lim (y(x_1) + \sum_{i=1}^{n} (\gamma(x_i) - \gamma(x_{i-1}))) = \lim \gamma(x_n),$$

if $\gamma(x_n) < \infty$ for all n. If $\gamma(x_n) = \infty$ for at least one n, then the equality $\gamma(x) = \lim \gamma(x_n)$ follows from the fact that γ is not decreasing.

The proof will be completed if we prove (2). Let $x, y \in R$, $(x \cap y)' \in C(x \cap y)$. Since $x \cap y \leqq x$, $x \cap y \leqq y$ we have

$$x = (x \cap y) \cup [x \cap (x \cap y)'], \quad y = (x \cap y) \cup [y \cap (x \cap y)'].$$

From it and from the additivity of γ we get (under the assumptions for complemented lattices)

$$\begin{aligned} (19) \quad & \gamma(x) + \gamma(y) = \gamma(x \cap y) + \gamma(x \cap (x \cap y)') + \gamma(x \cap y) + \\ & + \gamma(y \cap (x \cap y)') = 2\gamma(x \cap y) + \gamma([x \cap (x \cap y)'] \cup \\ & \cup [y \cap (x \cap y)']) = \gamma(x \cap y) + \gamma(z), \end{aligned}$$

where $z = (x \cap y) \cup [x \cap (x \cap y)'] \cup [y \cap (x \cap y)']$. If H is orthocomplemented then we take $(x \cap y)' = (x \cap y)^{\perp}$ and (19) holds for this complement.

As $x \cap (x \cap y)' \leqq x$, $y \cap (x \cap y)' \leqq y$, we have $z \leqq x \cup y$. Further, according to the modular law ($x \cap y \leqq x$) we have

$$z = \{x \cap [(x \cap y) \cup (x \cap y)']\} \cup [y \cap (x \cap y)'] = x \cup [y \cap (x \cap y)'] \geqq x.$$

Symmetrically, we have $z \geqq y$, hence $z \geqq x \cup y$. Hence we proved that $z = x \cup y$. From it and (19) 2 follows.

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