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COMPACTIFICATION OF PRODUCTS

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INTRODUCTION

Given a set E, an algebra B of bounded real valued functions on E will be called a function algebra if:

(a) B is closed in the uniform norm.
(b) B separates the points of E.
(c) B contains the constant functions.

Let $E^{\wedge}$ denote the set of all non-zero homomorphisms on B to the real numbers. We may regard E as a subset of $E^{\wedge}$ by means of the evaluation homomorphism given by each point of E. Then a given $f \in B$ may be extended to $E^{\wedge}$ by defining $f^{\wedge}(\psi) = \psi(f)$, for $\psi \in E^{\wedge}$. Then $B^{\wedge} = \{f^{\wedge} : f \in B\}$ is a function algebra on $E^{\wedge}$. Give $E^{\wedge}$ the weak topology induced by the functions of $B^{\wedge}$. $E^{\wedge}$ is completely regular in this topology. Also

(1) E is dense in $E^{\wedge}$,
(2) $E^{\wedge}$ is compact,
(3) $B^{\wedge}$ consists of all continuous real valued functions on $E^{\wedge}$.

$E^{\wedge}$ will be called the B-compactification of E. It is unique, except possibly for a homeomorphism which leaves E pointwise fixed. We note that the relevant properties of $E^{\wedge}$ may be established without involving the Tychonoff theorem. If E is a completely regular space, and $B = C(E)$ is the function algebra of all bounded continuous functions on E, then $E^{\wedge}$ is the Stone—Čech compactification βE of E.

Consider now a family $(E_\alpha)_{\alpha \in I}$ of completely regular spaces such that $\mathop{\mathsf{X}}\limits_{\substack{\alpha\in I\\ \alpha\neq\alpha_0}} E_\alpha$ is infinite for each $\alpha_0 \in I$. In this situation, Glicksberg [3] has proved

Theorem A. $\mathop{\mathsf{X}}\limits_{\alpha\in I} E_\alpha$ *is pseudocompact if and only if* $\beta(\mathop{\mathsf{X}}\limits_{\alpha\in I} E_\alpha) = \mathop{\mathsf{X}}\limits_{\alpha\in I} (\beta E_\alpha)$.

Motivated by this theorem, our discussion firstly centres on the following question: If $(E_\alpha)_{\alpha\in I}$ is a given family of sets and B_α is a function algebra on E_α, let $E_\alpha^\wedge$ be the B_α-compactification of E_α. Let $E = \mathop{\mathsf{X}}\limits_{\alpha\in I} E_\alpha$. Is there a function algebra B on E such that if $E^\wedge$ is the B-compactification of E, then $E^\wedge = \mathop{\mathsf{X}}\limits_{\alpha\in I} E_\alpha^\wedge$? This question is answered in the affirmative by taking for B the closure of the tensor product algebra $\bigotimes\limits_{\alpha\in I} B_\alpha$ on E. This enables us to obtain a criterion for the pseudo-compactness of a topological product. A further corollary is the Tychonoff theorem.

Secondly we consider a set E on which a binary operation S is defined. We characterise those function algebras B on E such that the binary operation S on E may be extended to one $S^\wedge$ on $E^\wedge$, is that $S^\wedge : E^\wedge \times E^\wedge \to E^\wedge$ is continuous. Our discussion here shall depend heavily on

Theorem B. *For $i = 1,2$ let B_i be a function algebra on E_i. Let $t : E_1 \to E_2$ be a map. Then $B_2 \circ t \subseteq B_1$ if and only if t has a continuous extension $t^\wedge$, where $t^\wedge : E_1^\wedge \to E_2^\wedge$. When this is the case,* $(f_2 \circ t)^\wedge = f_2^\wedge \circ t^\wedge$ for each $f_2 \in B_2$.

Applications to the cases where (E, S) denotes a semigroup and group and to a result of Comfort and Ross, are then considered.

COMPACTIFICATION OF PRODUCTS

Let $(E_\alpha)_{\alpha\in I}$ be a family of sets and let B_α be a function algebra on E_α. $E_\alpha^\wedge$ shall denote the B_α-compactification of E_α. $B_\alpha^\wedge$ is the extended algebra. Let $E = \mathop{\mathsf{X}}\limits_{\alpha\in I} E_\alpha$. Given $f_\alpha \in B_\alpha$, we may regard f_α as a function of E by defining $f_\alpha(x) = f_\alpha(x_\alpha)$, for $x \in E$. Then finite sums of functions of the form $f = f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_n}$, clearly form an algebra A on E. We write $A = \bigotimes\limits_{\alpha\in I} B_\alpha$ and it is the (tensor) product algebra on E. We let B be the uniform closure of A. Write $B = \overline{\bigotimes\limits_{\alpha\in I}} B_\alpha$ is the closed (tensor) product algebra on E. B is obviously a function algebra on E, and the B-compactification of E is denoted by $E^\wedge$.

Lemma 1. *There is a bijection from $E^\wedge$ onto $\mathop{\mathsf{X}}\limits_{\alpha\in I} E_\alpha^\wedge$.*

Proof. Let $\psi \in \mathop{\mathsf{X}}\limits_{\alpha\in I} E_\alpha^\wedge$. ψ_α is a non-zero homomorphism on B_α. If $f \in A$ write $f = \Sigma_{i=1}^n f_{\alpha_{i1}} f_{\alpha_{i2}} \cdots f_{\alpha_{ik(i)}}$. Then define $(\sigma(\psi))(f) = \Sigma_{i=1}^n \psi_{\alpha_{i1}}(f_{\alpha_{i1}}) \cdots \psi_{\alpha_{ik(i)}}(f_{\alpha_{ik(i)}})$. $\sigma(\psi)$ is then well defined as a function on A. In fact, $\sigma(\psi)$ is a non--zero homomorphism on A with the additional property that $f \geqslant 0$ implies $(\sigma(\psi))(f) \geqslant 0$. It follows that

$$|(\sigma(\psi))(f)| \leqslant ||f|| \text{ for all } f \in A .$$

If $f \in B$, choose $(f_n) \in A$ such that $||f - f_n|| \to 0$. Then

$$|(\sigma(\psi))(f_n) - (\sigma(\psi))(f_m)| \leqslant ||f_n - f_m|| \to 0$$

as $m, n \to \infty$. We may now define

$$(\sigma(\psi))(f) = \lim_{n\to\infty} (\sigma(\psi))(f_n) .$$

It is immediately seen that $\sigma(\psi)f \in E^{\wedge}$.

Conversely, if $\psi \in E^{\wedge}$, we define $\tau(\psi) \in \mathbf{X}_{\alpha\in I} E_\alpha^{\wedge}$. For $f_\alpha \in B_\alpha$, let

$$(\tau(\psi))_\alpha(f_\alpha) = \psi(f_\alpha) .$$

σ and τ are bijections because we notice that $\sigma\tau$ and $\tau\sigma$ are the identities on $E^{\wedge}$ and $\mathbf{X}_{\alpha\in I} E_\alpha^{\wedge}$, respectively.

Lemma 2. *The weak topology on* $E = \mathbf{X}_{\alpha\in I} E_\alpha$ *generated by* $B = \overline{\bigotimes}_{\alpha\in I} B_\alpha$ *is the product topology.*

Proof. Let U be an open set in E under the product topology. Then there is an open set (in the product topology) $V \subseteq U$ where $V = \mathbf{X}_{\alpha\in I} V_\alpha$, where V_α is open in E_α and $V_\alpha = E_\alpha$ for all but a finite number of α. Choose $a_1, \alpha_2, \ldots, a_n \in I$ so that $V_\alpha \underset{\neq}{\subset} E_\alpha$ implies $\alpha = \alpha_j$ for some j, $1 \leqslant j \leqslant n$. We may then choose $f_{\alpha_j} \in B_{\alpha_j}$ such that

$$\{x_{\alpha_j} : x_{\alpha_j} \in E_{\alpha_j} \text{ and } F_{\alpha_j}(x_{\alpha_j}) \neq 0\} \subseteq V_{\alpha_j} .$$

Let $f = f_{\alpha_1} f_{\alpha_2} \ldots f_{\alpha_n}$. Then $\{x : x \in E \text{ and } f(x) \neq 0\} \subseteq V \subseteq U$. Since $f \in B$, the weak topology generated by B is finer than the product topology.

On the other hand, each $f \in A$ is seen to be continuous in the product topology. Hence this statement holds for each $f \in B$. It follows that the weak topology is coarser than the product topology. Hence the result.

Lemma 1 shows that we may identify the sets $E^{\wedge}$ and $\mathbf{X}_{\alpha\in I} E_\alpha^{\wedge}$. This is what we do in future. Then $B^{\wedge}$ and $\overline{\bigotimes}_{\alpha\in I} B_\alpha^{\wedge}$ are function algebras on $E^{\wedge} = \mathbf{X}_{\alpha\in I} E_\alpha^{\wedge}$.

Lemma 3. $B^{\wedge} = \overline{\bigotimes}_{\alpha\in I} B_\alpha^{\wedge}$.

Proof. Let $f = \sum_1^n f_{\alpha_{i1}} f_{\alpha_{i2}} \ldots f_{\alpha_{ik(j)}} \in A$.
Let $\psi \in E^{\wedge}$. Then

$$\begin{aligned} f^{\wedge}(\psi) = \psi(f) &= \psi(\sum_1^n f_{\alpha_{i1}} f_{\alpha_{i2}} \cdots f_{\alpha_{ik(j)}}) \\ &= \sum_1^n f_{\alpha_{i1}}^{\wedge}(\psi_{\alpha_{i1}}) \cdots f_{\alpha_{ik(j)}}^{\wedge}(\psi_{\alpha_{ik(j)}}) \\ &= (\sum_1^n f_{\alpha_{i1}}^{\wedge} \cdots f_{\alpha_{ik(j)}}^{\wedge})(\psi) . \end{aligned}$$

So $f^{\wedge} = \Sigma_1^n f^{\wedge}_{\alpha_{i1}} f^{\wedge}_{\alpha_{i2}} \dots f^{\wedge}_{\alpha_{ik(i)}}$. In this way we have $A^{\wedge} = \overline{\bigotimes_{\alpha\in I}} B^{\wedge}_{\alpha}$.

Now if $f \in B$ chose $(g_n) \in A$ such that $\|f - g_n\| \leqslant 1/n$. Then $\|f^{\wedge} - g^{\wedge}_n\| \leqslant 1/n$, so that $f^{\wedge}$ is in the closure of $A^{\wedge} = \bigotimes_{\alpha\in I} B^{\wedge}_{\alpha}$. Hence $B^{\wedge} \subseteq \overline{\bigotimes_{\alpha\in I}} B^{\wedge}_{\alpha}$, and the reverse inclusion is clear.

Lemma 2 and Lemma 3 combine to give

Lemma 4. *As topological spaces,* $E^{\wedge} = \mathsf{X}_{\alpha\in I} E^{\wedge}_{\alpha}$.

Theorem 1. *Let* $(E_{\alpha})_{\alpha\in I}$ *be a family of sets and let* B_{α} *be a function algebra on* E_{α} *with* B_{α}*-compactification* $E^{\wedge}_{\alpha}$. *Let* $E = \mathsf{X}_{\alpha\in I} E_{\alpha}$ *and let* $B = \overline{\bigotimes_{\alpha\in I}} B_{\alpha}$ *be the closed (tensor) product algebra on* E. *Let* B_1 *be a function algebra on* E *with* B_1*-compactification* $E^{\wedge}_1$. *Then* $E^{\wedge}_1 = \mathsf{X}_{\alpha\in I} E^{\wedge}_{\alpha}$ *if and only if* $B_1 = B = \overline{\bigotimes_{\alpha\in I}} B_{\alpha}$.

Proof. If $B_1 = B$, Lemma 4 gives the result. Conversely, $E^{\wedge}_1 = \mathsf{X}_{\alpha\in I} E^{\wedge}_{\alpha}$ implies $B^{\wedge}_1 = B^{\wedge}$, by Lemma 4. Hence $B_1 = B$.

Now let $(E_{\alpha})_{\alpha\in I}$ be a family of completely regular spaces, and let our function algebra B_{α} be $C(E_{\alpha})$. Then the B_{α}-compactification $E^{\wedge}_{\alpha}$ is simply the Stone-Čech compactification βE_{α} of E_{α}. By Glicksberg's theorem (see introduction) we may deduce

Theorem 2. *If* $\mathsf{X}_{\substack{\alpha\neq\alpha_0\\ \alpha\in I}} E_{\alpha}$ *is infinite for each* $\alpha_0 \in I$ *we have:* $E = \mathsf{X}_{\alpha\in I} E_{\alpha}$ *is a pseudo--compact if and only if* $C(E) = \overline{\bigotimes_{\alpha\in I}} C(E_{\alpha})$.

For the case where the index set I consists of two elements, we state Theorem 2 as follows:

Let E_1 and E_2 be infinite completely regular spaces. Let $E = E_1 \times E_2$. Then $E = E_1 \times E_2$ is pseudo-compact if and only if for each $f \in C(E)$ and $\varepsilon > 0$, there exist $f_1, f_2, \dots, f_n \in C(E_1)$ and $g_1, g_2, \dots, g_n \in C(E_2)$ such that

$$\|f - \sum_1^n f_i g_i\| < \varepsilon .$$

Lemma 4 also enables us to prove the

Tychonoff Theorem. *The product of compact spaces is compact.*

For in Lemma 4 let E_{α} be compact. Then $\beta E_{\alpha} = E_{\alpha}$ and we have $E^{\wedge} = \mathsf{X}_{\alpha\in I} \beta E_{\alpha} = \mathsf{X}_{\alpha\in I} E_{\alpha}$ and is compact.

COMPACTIFICATION OF GROUPOIDS

Here (E, S) shall denote a groupoid i. e., E is a set and $S : E \times E \to E$ is a map. B is a function algebra on E and $E^\wedge$ is the resulting B-compactification. We define the triple (E, S, B) to be extendible if and only if $B \circ S \subseteq B \overline{\otimes} B$. When this is the case, S is continuous in the B-topology on E.

Theorem 3. *$E^\wedge$ can be given the structure of a topological groupoid such that (E, S) is a topological subgroupoid if and only if (E, S, B) is extendible.*

Proof. If (E, S, B) is extendible, Theorems B and 2 show that S has a continuous extension $S^\wedge$, where $S^\wedge : E^\wedge \times E^\wedge \to E^\wedge$. Theorems B and 2 also imply the converse.

When (E, S, B) is extendible, $S^\wedge : E^\wedge \times E^\wedge \to E^\wedge$ will denote the unique continuous extension of $S : E \times E \to E$ given by Theorem 3.

Theorem 4. *Let (E, S, B) be extendible. Then the following hold*

(1) *If (E, S) is associative, so too is $(E^\wedge, S^\wedge)$.*

(2) *If (E, S) is commutative, so too is $(E^\wedge, S^\wedge)$.*

(3) *If (E, S) has a left identity element e, e is also a left identity for $(E^\wedge, S^\wedge)$. Similarly for a right identity.*

Proof. We prove (1), the others being analogous. Consider the maps ψ_1 and ψ_2 from $E^\wedge \times E^\wedge \times E^\wedge$ given by $\psi_1(x, y, z) = S^\wedge(S^\wedge(x, y), z)$ and $\psi_2(x, y, z) = S^\wedge(x, S^\wedge(y, z))$. Then ψ_1 and ψ_2 are clearly continuous, so that $\{(x, y, z) : \psi_1(x, y, z) = \psi_2(x, y, z)\}$ is a closed set containing $E \times E \times E$ and hence is the whole of $E^\wedge \times E^\wedge \times E^\wedge$.

Theorem 5. *Let (E, S) be a semigroup and suppose that (E, S, B) is extendible. Then the groupoid $(E^\wedge, S^\wedge)$ is also a semigroup.*

Proof. Theorems 3 and 4 (1).

Lemma 5. *Suppose that (E, S, B) is extendible and that (E, S) has an identity e. Define $I = \{x : x \in E^\wedge$ and there exists $x^{-1} \in E^\wedge$ such that $S^\wedge(x, x^{-1}) = e\}$. Then I is closed.*

Proof. If I is not closed, choose $z \in E^\wedge - I$ such that a net (z_α) of elements of I converges to z. $E^\wedge$ is compact, so the net (z_α^{-1}) has a subnet converging to a point $y \in E^\wedge$. (Kelley [5], p. 136). Hence we may assume that (z_α) converges to z and (z_α^{-1}) converges to y. Continuity of $S^\wedge$ now gives : $S^\wedge(z_\alpha, z_\alpha^{-1})$ converges to $S^\wedge(z, y)$ as $S^\wedge(z_\alpha, z_\alpha^{-1}) = e$ for each α, we have that $S^\wedge(z, y) = e$, a contradiction since $z \in I$.

Theorem 6. *Suppose that (E, S) is a group. Then $E^\wedge$ can be given the structure of a topological group of which (E, S) is a dense subgroup if and only if (E, S, B) is extendible.*

Proof. If $E^\wedge$ is such a group, Theorem 3 gives that (E, S, B) is extendible. Conversely, apply theorem 3 to deduce that the groupoid structure of (E, S) can be extended to $(E^\wedge, S^\wedge)$. $(E^\wedge, S^\wedge)$ is a semigroup by Theorem 5. Theorem 4

(3) now implies that the identity for (E, S) is an identity for $(E^{\wedge}, S^{\wedge})$. Lemma 5 now shows that each element of $E^{\wedge}$ has a right inverse. We deduce that $(E^{\wedge}, S^{\wedge})$ is a group.

To complete the proof we need only show that inversion is continuous. To do this, let (x_α) be a net in $E^{\wedge}$ which converges to the point $x \in E^{\wedge}$. Then some subnet of (x_α^{-1}) converges to a point $y \in E^{\wedge}$. As $S^{\wedge}$ is continuous, we deduce that $S^{\wedge}(x, y) = e$. i. e., $y = x^{-1}$. Since inverses are unique, (x_α^{-1}) has exactly one cluster point in $E^{\wedge}$. Together with the fact that any net in $E^{\wedge}$ has a convergent subnet, this implies that (x_α^{-1}) converges to x^{-1}.

Theorem 7. *Let (E, S) be a group. For $i = 1,2$ let B_i be a function algebra on E such that (E, S, B_i) is extendible. Then the B_1-topology coincides with the B_2-topology if and only if $B_1 = B_2$.*

Proof. If $B_1 = B_2$ we have the result. Convesely we apply theorem 6, $(E_1^{\wedge}, S_1^{\wedge})$ and $(E_2^{\wedge}, S_2^{\wedge})$ respectively denote the group compactifications of (E, S) with respect to B_1 and B_2. Since the B_1 and B_2 topologies coincide on E, we see that in this topology E is a dense topological subgroup of each of the compact groups $E_1^{\wedge}$ and $E_2^{\wedge}$. Being compact, we see that $G_1^{\wedge}$ and $G_2^{\wedge}$ are completions of G in the two sided (or left, or right) uniformity. By the uniqueness theorem for the completion of uniform spaces, there is a uniform isomorphism ψ from $E_1^{\wedge}$ onto $E_2^{\wedge}$ which leaves E pointwise fixed. (Kelley [5], p. 197). Hence $B_2^{\wedge} \circ \psi \subseteq B_1^{\wedge}$. i. e., if $f_2 \in B_2$ there is $f_1 \in B_1$ such that $f_2^{\wedge} \circ \psi = = f_1^{\wedge}$. Considering restrictions to E, we have $f_1 = f_2$. So $f_2 \in B_1$. i. e., $B_2 \subseteq B_1$ and likewise $B_1 \subseteq B_2$.

Theorem 8. *Suppose that (E, S) is a group and that (E, S, B) is extendible. Then in the B-topology on E, either E is compact or E is not locally compact.*

Proof. By Theorem 6, E is a dense subgroup of the compact group $E^{\wedge}$. By theorem 5.11 (p. 35) of Hewitt and Ross [4], if E were locally compact in the B-topology, then E would be closed in $E^{\wedge}$. E would then be the compact group $E^{\wedge}$.

Now suppose that (E, S) denotes a locally compact abelian group. Let Γ be its character group. We define a complex valued function f on E to be almost periodic if, to each $\varepsilon > 0$, there correspond $\lambda_1, \ldots, \lambda_n \in \Gamma$ and complex numbers $C_1, \ldots, C_n$ such that $\|f - \sum_{i=1}^{n} C_i \lambda_i\| < \varepsilon$. $AP(E)$ shall denote the set (algebra) of all almost periodic functions on E. We define B to consist of those functions in $AP(E)$ whose values are real. B is easily seen to be a function algebra on E. (Γ separates points of E). We also see that $AP(E) = \{f + ig : f, g \in B\}$.

Lemma 6. $AP(E) \circ S \subseteq AP(E) \overline{\otimes} AP(E)$.

Proof. Let $h \in AP(E)$, $\varepsilon > 0$. Choose $\lambda_1, \ldots, \lambda_n \in \Gamma$ and $C_1, \ldots, C_n$ such that $\|h - \sum_{i=1}^{n} C_i \lambda_i\| < \varepsilon$. Then for $x, y \in E$ and all i we have $\lambda_i(S(x, y)) = \lambda_i(x) \lambda_i(y)$. Hence for all $x, y \in E$ we have $|h(S(x, y)) - \sum_{i=1}^{n} C_i \lambda_i(x) \lambda_i(y)| < \varepsilon$. This gives $h \circ S \in AP(E) \overline{\otimes} AP(E)$, as $\Gamma \subseteq AP(E)$. Hence the result.

Theorem 9. *(E, S, B) is extendible.*

Proof. Let $f \in B$. Let $\varepsilon > 0$ and use Lemma 6 to choose $f_1, \ldots, f_n$ and $g_1, \ldots, g_n \in AP(E)$ such that $\|f \circ S - \sum_{k=1}^{n} f_k g_k\| < \varepsilon$. Let $f_k = p_k + iq_k$ and $g_k = p_k' + iq_k'$ where $p_k, p_k', q_k, q_k' \in B$. We then deduce that $\|f \circ S - \sum_{k=1}^{n} (p_k p_k' - q_k q_k')\| < \varepsilon$. i. e., $f \circ S \in B \overline{\otimes} B$, true for each $f \in B$. Hence $B \circ S \subseteq B \overline{\otimes} B$, as required.

In view of Theorem 6, we could express Theorem 9 by saying that a locally compact abelian group E has a Bohr compactification, which is obtained by compactifying E using the real valued almost periodic functions. Theorem 8 then indicates that if E is not compact, it is not locally compact in the weak topology inherited from the almost periodic functions, although it is a topological group in this topology.

Our discussion now enables us to give an alternative proof of a result of Comfort and Ross. We consider a completely regular topological group G and use

Lemma 7. (*Comfort and Ross* [1], *p.* 494). *If G is pseudocompact, so too is the product group $G \times G$.*

Theorem 10. (*Comfort and Ross* [1], p. 494). *If G is pseudocompact, then the Stone-Čech compactification βG of G admits a compact topological group structure relative to which G is a dense subgroup.*

Proof. By Theorem 6, we need only show that $(G, S, C(G))$ is extendible, S being the group operation. If G is finite there is nothing to prove. If G is infinite, Theorem 2 and Lemma 7 give $C(G) \circ S \subseteq C(G) \overline{\otimes} C(G)$ and we have the result.

Using the fact that a continuous real valued function on a compact space is uniformly continuous, Theorem 10 readily implies that a continuous real valued function f on a pseudocompact group G is such that $\{f_a : a \in G\}$ is precompact in the uniform metric.

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