Matematický časopis

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Matematický časopis, Vol. 22 (1972), No. 1, 50--58

Persistent URL: http://dml.cz/dmlcz/126798

THE LATTICE OF ALL SYSTEMS OF r-IDEALS IN A SET

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Several authors investigated the partially ordered system $\mathcal{T}$ consisting of all topologies (or of all topologies with prescribed properties) that can be defined on a given set A. E. g. H. Gaifman [2] studied the lattice of all topologies definable on an arbitrary set A. A. K. Steiner [5] proved that this lattice is complemented and P. S. Schnare [4] estimated the cardinality of the set of complements. E. S. Wolk [6] studied the system of all topologies τ defined on a partially ordered set $(A; \leqslant)$ such that τ is consistent in a certain sense with the given partial ordering on A.

In the present paper we deal with the system of all generalized topologies on a set A satisfying the following ,,finiteness condition": the closure Z_r of any set $Z \subset A$ is the set-theoretical union of all closures X_r of finite subsets X of the set Z. The study of such topologies was suggested by paper [1] of L. Fuchs concerning r-ideals in universal algebras. Our notations are as follows. The symbols $\bigcap$, $\bigcup$ and $\wedge$, $\vee$ denote the set-theoretical and lattice operations, respectively; $A \subset B$ means that A is a subset of B (equality not being excluded). If $\mathcal{S}$ is a system of sets, then by $\bigcap \mathcal{S}$ and $\bigcup \mathcal{S}$ the set $\bigcap_{X \in \mathcal{S}} X$ and $\bigcup_{X \in \mathcal{S}} X$, respectively, are meant. $\mathcal{P}(X)$, where X is a non-empty set, denotes the system of all non-empty subsets of the set X, $\mathcal{K}(X)$ the system of all finite non-empty subsets of X.

Let us have an arbitrary non-empty set A and a mapping assigning to any non-empty finite subset X of A a subset X_r of A, such that the following conditions are satisfied:

1° $X \subset X_r$;

2° $X \subset Y_r \Rightarrow X_r \subset Y_r$.

Let us extend the domain of this mapping and for infinite subsets Z of the set A put

3° $Z_r = \bigcup X_r$, where X runs over all non-empty finite subsets of Z. The range of this mapping is called a system of r-ideals in A.

In the above mentioned paper by L. Fuchs and in papers [3], [7] some results concerning the relations between a system of r-ideals in a universal algebra $(A;F)$ and algebraic operations in $(A;F)$ were derived.

Let a system of r-ideals in A be given. Let us denote $\mathscr{K}_r(X) = \{Y_r : Y \in \in \mathscr{K}(X)\}$ and let $\mathscr{P}_r(X)$ have a similar meaning. If we use this notation, the given system of r-ideals in A is in fact the system $\mathscr{P}_r(A)$ and the axiom 3° can be rewritten as follows: $Z_r = \bigcup \mathscr{K}_r(Z)$.

First let us introduce some simple consequences following from the definition of the system of r-ideals in A.

1. *Equality* $Z_r = \bigcup \mathscr{K}_r(Z)$ *holds for each set* $Z \in \mathscr{P}(A)$.

Proof. Obviously it is sufficient to prove that this equality holds for $Z \in \mathscr{K}(A)$. In this case $Z_r \in \mathscr{K}_r(Z)$, therefore $Z_r \subset \bigcup \mathscr{K}_r(Z)$ and the inverse inclusion is evident.

2. *The conditions* 1°, 2° *are fulfilled also in the case when any of the sets* X, Y *is infinite.*

Proof. Let X be an infinite set. From the relation $\bigcup \mathscr{K}(X) \subset \bigcup \mathscr{K}_r(X)$ it follows that $X \subset X_r$.

Let $X \in \mathscr{K}(A)$, Y be an infinite subset of A and let $X \subset Y_r$. Let us suppose that $X = \{x_1, \ldots, x_n\}$, $x_i \in V_r^i$, $V^i \in \mathscr{K}(Y)$. Then $X \subset V_r$, where $V = V^1 \cup \ldots \cup V^n$ and hence $X_r \subset Y_r$ follows.

Let X be an infinite and Y an arbitrary set from the system $\mathscr{P}(A)$ and let $X \subset Y_r$. Then according to the preceding results for each set $T \in \mathscr{K}(X)$ $T_r \subset Y_r$ holds, hence $X_r \subset Y_r$.

As a consequence of this statement we obtain that for any set $X \in \mathscr{P}(A)$ $X_{rr} = X_r$ holds.

Further we shall introduce a partial ordering into the set $\mathscr{E}(A)$ of all systems of r-ideals in A and we shall prove that with regard to this partial ordering $\mathscr{E}(A)$ is a complete lattice.

Let A be any non-empty set. Let $\mathscr{E}(A)$ be the set of all systems of r-ideals in A. For two systems $\mathscr{P}_{r_1}(A)$, $\mathscr{P}_{r_2}(A)$ of r-ideals in A let us put $\mathscr{P}_{r_1}(A) \leqslant \leqslant \mathscr{P}_{r_2}(A)$ iff for each set $X \in \mathscr{P}(A)$ we have $X_{r_1} \subset X_{r_2}$. The relation $\leqslant$ defined in this way is obviously a relation of partial ordering.

The following statement holds true.

3. Theorem. *With regard to the partial ordering defined above,* $\mathscr{E}(A)$ *is a lattice with the least and the greatest element.*

Proof. Let $\mathscr{P}_{r_1}(A)$, $\mathscr{P}_{r_2}(A)$ be arbitrary systems of r-ideals in A. The system $\{X_r : X_r = X_{r_1} \cap X_{r_2},\ X \in \mathscr{P}(A)\}$ is obviously a system of r-ideals in A and it is the greatest lower bound of the elements $\mathscr{P}_{r_1}(A)$, $\mathscr{P}_{r_2}(A)$ of the set $\mathscr{E}(A)$.

Let X be an arbitrary set of the system $\mathscr{P}(A)$. For every positive integer n let us define the sets X_n, X'_n by induction as follows:

1. $X_1 = X_{r_1 r_2}$, $X'_1 = X_{r_2 r_1}$;

2. if we have X_k, $X'_k (k \geqslant 1)$, then $X_{k+1} = (X_k)_{r_1 r_2}$, $X'_{k+1} = (X'_k)_{r_2 r_1}$.

$\bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} X'_n$, since for every positive integer k we have $X_k \subset X'_{k+1}$, $X'_k \subset X_{k+1}$. The system $\{X_r : X_r = \bigcup_{n=1}^{\infty} X_n,\ X \in \mathscr{P}(A)\}$ is a system of r-ideals in A. The conditions 1°, 2° from the definition of the system of r-ideals obviously hold. According to 1°, 2°, we have $\bigcup \mathscr{K}_r(Z) \subset Z_r$ for any infinite set Z from $\mathscr{P}(A)$. Conversely, let $a \in Z_r = \bigcup_{n=1}^{\infty} Z_n$. Then there exists a positive integer n such that $a \in Z_n$. By induction on n one proves that $a \in X_n$ for some set $X \in \mathscr{K}(Z)$. This system of r-ideals is the least upper bound of the elements $\mathscr{P}_{r_1}(A)$, $\mathscr{P}_{r_2}(A)$ of $\mathscr{E}(A)$. The least and the greatest element of the lattice $\mathscr{E}(A)$ is the system $\{X_{r_0} : X_{r_0} = X,\ X \in \mathscr{P}(A)\}$ and the system $\{X_{r_1} : X_{r_1} = A,\ X \in \mathscr{P}(A)\}$ of r-ideals in A, respectively.

Remark. If $\mathscr{P}_{r_1}(A)$, $\mathscr{P}_{r_2}(A)$ are systems of r-ideals in A, the system $\{X_r : X_r = X_{r_1} \cup X_{r_2}, X \in \mathscr{P}(A)\}$ need not be in general a system of r-ideals in A (differing from the system $\{X_r : X_r = X_{r_1} \cap X_{r_2}, X \in \mathscr{P}(A)\}$). To show this let A be the set of all integers, $\mathscr{P}_{r_1}(A)$ the system of r-ideals in A defined by the condition that $X_{r_1}(X \in \mathscr{P}(A))$ is the least subgroup of the additive group of all integers containing X. Further let $\mathscr{P}_{r_2}(A)$ be the system of r-ideals in A so defined that X_{r_2} is the set of all such elements x of A, for which there exists such a pair x_1, x_2 of elements of X that $x_1 \leqslant x \leqslant x_2$. Let $X = \{3\}$, $Y = \{2,4\}$. $X \subset Y_{r_1} \bigcup Y_{r_2}$ holds, but not $X_{r_1} \bigcup X_{r_2} \subset Y_{r_1} \cup Y_{r_2}$.

4. Theorem. *Let A be an arbitrary non-empty set. The set $\mathscr{E}(A)$ of all systems of r-ideals in A is a complete lattice.*

Proof. According to Theorem 3 $\mathscr{E}(A)$ is a partially ordered set, bounded below. It is sufficient to show that an arbitrary non-empty subset $\{\mathscr{P}_{r_\lambda}(A)\}_{\lambda \in \Lambda}$ of the set $\mathscr{E}(A)$ has the least upper bound in $\mathscr{E}(A)$.

To each set $X \in \mathscr{P}(A)$ let us join the set $X_r = \bigcup \mathscr{T}(X)$, where $\mathscr{T}(X) = \{X_{t_1 \ldots t_n} : \{t_i\}_{i=1}^{n} \in \mathscr{K}(\{r_\lambda\}_{\lambda \in \Lambda})\}$. Then $\{X_r : X_r = \bigcup \mathscr{T}(X), X \in \mathscr{P}(A)\}$ is a system of r-ideals in A. Let us take an arbitrary set $X \in \mathscr{P}(A)$. For each $\lambda \in \Lambda$ we have $X \subset X_{r_\lambda}$ and since $X_{r_\lambda} \in \mathscr{T}(X)$, we have $X \subset X_r$. Let $X \in \mathscr{K}(A)$, $Y \in \mathscr{P}(A)$, $X \subset Y_r$, $X = \{x_1, \ldots, x_k\}$. Then for each i $(i = 1, \ldots, k)$ there exists a set $\{t_1^i, \ldots, t_{n_i}^i\} \in \mathscr{K}(\{r_\lambda\}_{\lambda \in \Lambda})$, such that $x_i \in Y_{t_1^i \ldots t_{n_i}^i}$. Evidently for each i the following holds $Y_{t_1^i \ldots t_{n_i}^i} \subset Y_{t_1^1 \ldots t_{n_1}^1 t_1^2 \ldots t_{n_2}^2 \ldots t_1^k \ldots t_{n_k}^k}$ hence $X \subset Y_{t_1^1 \ldots t_{n_k}^k}$. If $\{t_1, \ldots, t_n\}$ is an arbitrary set from $\mathscr{K}(\{r_\lambda\}_{\lambda \in \Lambda})$, then

$X_{t_1 \dots t_n} \subset Y_{t_1^1 \dots t_{n_k}^k t_1 \dots t_n}$. Since $Y_{t_1^1 \dots t_n} \subset Y_r$, we have $X_{t_1 \dots t_n} \subset Y_r$ and from this we get $\bigcup \mathscr{T}(X) \subset Y_r$, i. e. $X_r \subset Y_r$.

Let Z be an arbitrary infinite set from $\mathscr{P}(A)$. Evidently $\bigcup \mathscr{K}_r(Z) \subset Z_r$. Now we shall prove the inverse inclusion. Let $a \in Z_r$. Then $a \in Z_{t_1 \dots t_n}$ ($\{t_1, \dots, t_n\} \in \mathscr{K}(\{r_\lambda\}_{\lambda \in \Lambda})$). It is sufficient to prove that there exists such a set $X \in \mathscr{K}(Z)$ and $\{v_1, \dots, v_p\} \in \mathscr{K}(\{r_\lambda\}_{\lambda \in \Lambda})$ that $a \in X_{v_1 \dots v_p}$. We are going to prove this by induction on n.

Let $n = 1$. Then $a \in Z_{r_\lambda}$ ($\lambda \in \Lambda$). Since $Z_{r_\lambda} = \bigcup \mathscr{K}_{r_\lambda}(Z)$ holds, there exists such a set $X \in \mathscr{K}(Z)$ that $a \in X_{r_\lambda}$.

Now let us suppose that if $a \in Z_{t_1 \dots t_{k-1}}$ ($\{t_i\}_{i=1}^{k-1} \in \mathscr{K}(\{r_\lambda\}_{\lambda \in \Lambda})$), then there exists such a set $X \in \mathscr{K}(Z)$ and $\{s_1, \dots, s_l\} \in \mathscr{K}(\{r_\lambda\}_{\lambda \in \Lambda})$ that $a \in X_{s_1 \dots s_l}$. Let $a \in Z_{t_1 \dots t_k}$. If $t_k = r_\lambda$, then from $Z_{t_1 \dots t_{k-1} r_\lambda} = \bigcup \mathscr{K}_{r_\lambda}(Z_{t_1 \dots t_{k-1}})$ we obtain $a \in S_{r_\lambda}$, where $S \in \mathscr{K}(Z_{t_1 \dots t_{k-1}})$. Let $S = \{y_1, \dots, y_m\}$. Since $y_i \in Z_{t_1 \dots t_{k-1}}$ ($i = 1, \dots, m$), we have $y_i \in X^i_{s_1^i \dots s_{l_i}^i}$, where $X^i \in \mathscr{K}(Z)$, $\{s_1^i, \dots s_{l_i}^i\} \in \mathscr{K}(\{r_\lambda\}_{\lambda \in \Lambda})$. Let us take $X = \bigcup_{i=1}^{m} X^i$. Evidently $X \in \mathscr{K}(Z)$, $S \subset X_{s_1^1 \dots s_{l_1}^1 s_1^2 \dots s_{l_2}^2 \dots s_1^m \dots s_{l_m}^m}$. Since $a \in S_{r_\lambda}$, then also $a \in X_{s_1^1 \dots s_{l_m}^m r_\lambda}$. The system $\{X_r : X_r = \bigcup \mathscr{T}(X),\ X \in \mathscr{P}(A)\}$ of r-ideals in A is the supremum of the set $\{\mathscr{P}_{r_\lambda}(A)\}_{\lambda \in \Lambda}$. Evidently $\mathscr{P}_{r_\lambda}(A) \leqslant \mathscr{P}_r(A)$ holds. Let further $\mathscr{P}_r{}'(A)$ be such a system of r-ideals in A that $\mathscr{P}'_{r'}(A) \geqslant \mathscr{P}_{r_\lambda}(A)$ for each $\lambda \in \Lambda$. From this $X_{t_1 \dots t_n} \subset X_{t_1 \dots t_{n-1} r'} \subset X_{t_1 \dots t_{n-2} r' r'} = X_{t_1 \dots t_{n-2} r'} \subset \dots \subset X_{t_1 r'} \subset X_{r' r'} = X_{r'}(\{t_1, \dots, t_n\} \in \mathscr{K}(\{r_\lambda\}_{\lambda \in \Lambda}), X \in \mathscr{P}(A))$ follows. Thus we have $\bigcup \mathscr{T}(X) \subset X_r{}'$, i. e. $X_r - X_r{}'$. Hence holds $\mathscr{P}_r(A) \leqslant \mathscr{P}_{r'}(A)$ holds.

The question arises, what the infimum of the subset $\{\mathscr{P}_{r_\lambda}(A)\}_{\lambda \in \Lambda}$ of $\mathscr{E}(A)$ in $\mathscr{E}(A)$ is. According to the considerations in section 3 we could suppose that the system $\{X_r : X_r = \bigcap_{\lambda \in \Lambda} X_{r_\lambda},\ X \in \mathscr{P}(A)\}$ will be the infimum. This conception is wrong, the mentioned system need not even be a system of r-ideals in A. To show it let $A_1 = \{a_n\}_{n=1}^{\infty}$ be an arbitrary infinite countable set and let us put $A = A_1 \bigcup \{a\}$, where $a \notin A_1$. For any positive integer n let us put $X_{r_n} = X$ if $a_n \notin X$ and $X_{r_n} = X \bigcup \{a\}$ if $a_n \in X$. Evidently $\{X_{r_n} : X \in \mathscr{P}(A)\}$ is a system of r-ideals in A. Let us put $X_r = \bigcap_{n \in N} X_{r_n}$ (N is the set of all positive integers) for every set $X \in \mathscr{P}(A)$. Then $(A_1)_r \neq \bigcup \mathscr{K}_r(A_1)$.

The construction of the infimum is given in the following statement.

5. *Let A be an arbitrary non-empty set, let $\{\mathscr{P}_{r_\lambda}(A)\}_{\lambda \in \Lambda}$ be an arbitrary set of systems of r-ideals in A. For any set $Z \in \mathscr{P}(A)$ let us put $Z_r = \bigcup_{X \in \mathscr{K}(Z)} (\bigcap_{\lambda \in \Lambda} X_{r_\lambda})$. Then $\{Z_r : Z \in \mathscr{P}(A)\}$ is the infimum of the set $\{\mathscr{P}_{r_\lambda}(A)\}_{\lambda \in \Lambda}$ in the lattice $\mathscr{E}(A)$.*

Proof. a) $\mathscr{P}_r(A)$ is a system of r-ideals in A. Thus let $Z \in \mathscr{P}(A)$. If $X \in$

$\in \mathscr{K}(Z)$, then $X \subset \bigcap_{\lambda \in \Lambda} X_{r_\lambda}$, hence $Z \subset Z_r$. Let $Z \subset U_r$ $(Z, U \in \mathscr{P}(A))$. Let us take $a \in Z_r$. Then there exists such a set $X \in \mathscr{K}(Z)$ that $a \in \bigcap_{\lambda \in \Lambda} X_{r_\lambda}$. Let $X = \{x_1, \ldots, x_n\}$. For each $i \in \{1, 2, \ldots, n\}$ there exists $T^i \in \mathscr{K}(U)$ such that $x_i \in \bigcap_{\lambda \in \Lambda} T^i_{r_\lambda}$. Let us denote $T = \bigcup_{i=1}^{n} T^i$. Evidently $T \in \mathscr{K}(U)$, $X \subset T_{r_\lambda}$ for each $\lambda \in \Lambda$. From this we obtain $X_{r_\lambda} \subset T_{r_\lambda}$ for each $\lambda \in \Lambda$. Then $a \in \bigcap_{\lambda \in \Lambda} T_{r_\lambda}$ holds and hence $a \in U_r$. The equality $Z_r = \cup \mathscr{K}_r(Z)$ for an arbitrary infinite set $Z \in \mathscr{P}(A)$ is true according to the definition of the set Z_r, since evidently for each set $X \in \mathscr{K}(A)$ there is $X_r = \bigcap_{\lambda \in \Lambda} X_{r_\lambda}$.

b) $\mathscr{P}_r(A)$ is the lower bound of the set $\{\mathscr{P}_{r_\lambda}(A)\}_{\lambda \in \Lambda}$ in the lattice $\mathscr{E}(A)$. The proof is clear.

c) $\mathscr{P}_r(A)$ is the greatest lower bound of the set $\{\mathscr{P}_{r_\lambda}(A)\}_{\lambda \in \Lambda}$ in the lattice $\mathscr{E}(A)$. Let $\mathscr{P}_{r_1}(A)$ be a system of r-ideals in A such that $\mathscr{P}_{r_1}(A) \leqslant \mathscr{P}_{r_\lambda}(A)$ for each $\lambda \in \Lambda$. Let $a \in Z_{r_1}$ $(Z \in \mathscr{P}(A))$. Then there exists a set $X \in \mathscr{K}(Z)$ such that $a \in X_{r_1}$. Then $a \in X_{r_\lambda}$ holds for each index $\lambda \in \Lambda$, hence $a \in \bigcap_{\lambda \in \Lambda} X_{r_\lambda}$ and from this we get $a \in Z_r$. Hence $\mathscr{P}_{r_1}(A) \leqslant \mathscr{P}_r(A)$.

In paper [7] the lattice $\mathscr{E}(A)$ of all systems of r-ideals in the set containing three elements is constructed. There is a table there, in which there are all systems of r-ideals in A (there are 45 of them) given and the diagram of the lattice $\mathscr{E}(A)$.

We need the following two simple lemmas.

6. *Let A_1, A_2 be arbitrary non-empty disjoint sets. Let $\mathscr{P}_{r_1}(A_1)$ and $\mathscr{P}_{r_2}(A_2)$ be systems of r-ideals in A_1 and in A_2, respectively. Then it is possible to derive from* $\mathscr{P}_{r_1}(A_1)$ and $\mathscr{P}_{r_2}(A_2)$ *a system of r-ideals in $A = A_1 \cup A_2$ as follows: For $X \in \mathscr{P}(A)$ let us put:*

1. $X_r = X_{r_1}$ *if* $X \in \mathscr{P}(A_1)$;
2. $X_r = X_{r_2}$ *if* $X \in \mathscr{P}(A_2)$;
3. $X_r = (X \cap A_1)_{r_1} \cup (X \cap A_2)_{r_2}$ *if* $X \cap A_1 \neq \emptyset$ *and simultaneously* $X \cap A_2 \neq \emptyset$. This statement is evident.

7. Definition. *Following the notations introduced in the preceding lemma we shall say that the system $\mathscr{P}_r(A)$ of r-ideals is induced by the systems $\mathscr{P}_{r_1}(A_1)$, $\mathscr{P}_{r_2}(A_2)$ of r-ideals in the set $A = A_1 \cup A_2$ and we shall denote it* comp $\{\mathscr{P}_{r_1}(A_1), \mathscr{P}_{r_2}(A_2)\}$.

8. *Let A_1, A_2 be arbitrary disjoint sets. Let $\mathscr{P}_r(A_1)$ be an arbitrary fixed system of r-ideals in A_1, let $\mathscr{E}(A_2) = \{\mathscr{P}_{r_\iota}(A_2)\}_{\iota \in I}$. Then the set* $\{$comp $\{\mathscr{P}_r(A_1), \mathscr{P}_{r_\iota}(A_2)\}\}_{\iota \in I}$ *is a sublattice of the lattice $\mathscr{E}(A)$ of all systems of r-ideals in $A = A_1 \cup A_2$ isomorphic with the lattice $\mathscr{E}(A_2)$.*

Proof. Let us take $\mathscr{P}_{ri_1}(A_2)$, $\mathscr{P}_{ri_2}(A_2) \in \mathscr{E}(A_2)$, $X \in \mathscr{P}(A)$. Distinguishing three cases: $X \subset A_1$, $X \subset A_2$, $X \cap A_1 \neq \emptyset$ & $X \cap A_2 \neq \emptyset$ one can easily show that the equalities

$$\operatorname{comp}\{\mathscr{P}_r(A_1), \mathscr{P}_{ri_1}(A_2)\} \wedge \operatorname{comp}\{\mathscr{P}_r(A_1), \mathscr{P}_{ri_2}(A_2)\} = \\ = \operatorname{comp}\{\mathscr{P}_r(A_1), \mathscr{P}_{ri_1}(A_2) \wedge \mathscr{P}_{ri_2}(A_2)\}$$

$$\operatorname{comp}\{\mathscr{P}_r(A_1), \mathscr{P}_{ri_1}(A_2)\} \vee \operatorname{comp}\{\mathscr{P}_r(A_1), \mathscr{P}_{ri_2}(A_2)\} = \\ = \operatorname{comp}\{\mathscr{P}_r(A_1), \mathscr{P}_{ri_1}(A_2) \vee \mathscr{P}_{ri_2}(A_2)\}$$

are valid.

The isomorphism is given by the mapping:

$$\mathscr{P}_{r_i}(A_2) \to \operatorname{comp}\{\mathscr{P}_r(A_1), \mathscr{P}_{r_i}(A_2)\}.$$

We are going to examine whether the lattice $\mathscr{E}(A)$ is modular and complemented.

9. Theorem. *If the set A contains at least three elements, then lattice $\mathscr{E}(A)$ is not modular.*

Proof. First of all let us suppose that the set A contains just three elements a, b, c. Let us consider the systems of r-ideals in $A = \{a, b, c\}$ described in the following table.

	r_1	r_2	r_3	r_4	r_5
$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$\{b\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{c\}$	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{a, b, c\}$	$\{a, c\}$
$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, c\}$	$\{a, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, c\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$

The following holds $\mathscr{P}_{r_1}(A) < \mathscr{P}_{r_2}(A) < \mathscr{P}_{r_3}(A) < \mathscr{P}_{r_4}(A)$, $\mathscr{P}_{r_1}(A) < \mathscr{P}_{r_5}(A) < \mathscr{P}_{r_4}(A)$. The element $\mathscr{P}_{r_5}(A)$ of the lattice $\mathscr{E}(A)$ is incomparable with the elements $\mathscr{P}_{r_2}(A)$, $\mathscr{P}_{r_3}(A)$ of the lattice $\mathscr{E}(A)$. Further evidently $\mathscr{P}_{r_5}(A) \wedge \mathscr{P}_{r_3}(A) = \mathscr{P}_{r_1}(A)$, $\mathscr{P}_{r_5}(A) \vee \mathscr{P}_{r_2}(A) = \mathscr{P}_{r_4}(A)$, hence $\mathscr{P}_{r_1}(A), \ldots \mathscr{P}_{r_5}(A)$ form the pentagonal non-modular sublattice of the lattice $\mathscr{E}(A)$.

Now let the set A contains more than three elements. Let us put $A_2 = \{a, b, c\}$, where a, b, c are arbitrary different fixed elements of the set A and $A_1 = A - A_2$. Let $\mathscr{P}_r(A_1)$ be an arbitrary fixed system of r-ideals in A_1, $\{\mathscr{P}_{r_i}(A_2)\}_{i \in I}$ the set of all systems of r-ideals in $A_2 = \{a, b, c\}$. According to the preceding part of the proof and Lemma 8, there exist indexes $i_1, \ldots, i_5 \in I$ such that $\operatorname{comp}\{\mathscr{P}_r(A_1), \mathscr{P}_{r_{i_1}}(A_2)\}, \ldots, \operatorname{comp}\{\mathscr{P}_r(A_1), \mathscr{P}_{r_{i_5}}(A_2)\}$ form the pentagonal non-modular sublattice of the lattice $\mathscr{E}(A)$.

10. Theorem. *Let A be an arbitrary set which contains at least three elements. Then the lattice $\mathscr{E}(A)$ is not complemented.*

Proof. Let a be an arbitrary fixed element of the set A. For $X \in \mathscr{P}(A)$ let us put $X_{r_1} = A$ if $a \in X$, $X \neq \{a\}$, $X_{r_1} = A - \{a\}$ if $a \notin X$ and $\{a\}_{r_1} = \{a\}$. We shall prove that the element $\mathscr{P}_{r_1}(A)$ of the lattice $\mathscr{E}(A)$ has no complement in $\mathscr{E}(A)$. Let us suppose that there exists a system $\mathscr{P}_{r_2}(A)$ of r-ideals in A such that $\mathscr{P}_{r_1}(A) \wedge \mathscr{P}_{r_2}(A)$ is the least element of the lattice $\mathscr{E}(A)$. Let us take $X \in \mathscr{P}(A)$. $X_{r_1} \cap X_{r_2} = X$ holds. If $a \in X$, $X \neq \{a\}$, then we must have $X_{r_2} = X$. Let $X = \{a\}$. If $b \in A$, $b \neq a$, we have $\{a, b\}_{r_2} = \{a, b\}$. Therefore $\{a\}_{r_2} \subset \{a, b\}$. The last inclusion holds for any element $b (\in A)$ different from a. Since A contains at least three elements, $\{a\}_{r_2} = \{a\}$. From the equalities $\{a\}_{r_1} = \{a\}_{r_2} = \{a\}$ it follows that $\mathscr{P}_{r_1}(A) \vee \mathscr{P}_{r_2}(A)$ is not the greatest element of the lattice $\mathscr{E}(A)$. Namely the mapping belonging to this system of r-ideals in A assigns to the set $\{a\}$ the same set $\{a\}$ (cf. section 3).

The following question seems to be natural: do there exist elements $\mathscr{P}_{r_1}(A)$ and $\mathscr{P}_{r_2}(A)$ in the set $\mathscr{E}(A)$ such that $\mathscr{P}_{r_1}(A)$ is a complement of $\mathscr{P}_{r_2}(A)$? The answer is positive. To show this let A be an arbitrary set which contains at least three elements. Let a be an arbitrary element of the set A. Then the system $\{X_{r_1} : X_{r_1} = X \cup \{a\}, X \in \mathscr{P}(A)\}$ of r-ideals in A is evidently a complement of the system $\{X_{r_2} : X_{r_2} = A$ if $a \in X$, $X_{r_2} = A - \{a\}$ if $a \notin X$, $X \in \in \mathscr{P}(A)\}$ of r-ideals in A in the lattice $\mathscr{E}(A)$.

11. *It is easy to verify that if the set A contains one or two elements, the lattice $\mathscr{E}(A)$ of all systems of r-ideals in A is modular and uniquely complemented.*

In the following part of the present paper we shall investigate whether the lattice $\mathscr{E}(A)$ has atoms, dual atoms and we shall prove that the lattice $\mathscr{E}(A)$ is dually atomic, when A contains at least two elements.

12. *Let A be an arbitrary non-empty set. Let $\mathscr{P}_r(A)$ be a system of r-ideals in A. If there exists such a set $X^\circ \in \mathscr{P}(A)$ that $A - X^\circ$ contains at least two elements and $X^\circ \neq X^\circ_r$, then $\mathscr{P}_r(A)$ is not an atom in the lattice $\mathscr{E}(A)$ of all systems of r-ideals in A.*

Proof. Let us take an arbitrary set $X \in \mathscr{P}(A)$. If $X \subset X^\circ$, let us put $X_{r_1} = X$ and if $X \not\subset X^\circ$, let us put $X_{r_1} = X_r$. Evidently $\{X_{r_1} : X \in \mathscr{P}(A)\}$ is a system of r-ideals in A and $\mathscr{P}_{r_1}(A) < \mathscr{P}_r(A)$. $\mathscr{P}_{r_1}(A)$ is different from the least element of the lattice $\mathscr{E}(A)$, because there exists such a set $Y \in \mathscr{P}(A)$ that $X^\circ \subsetneqq Y$, $a \notin Y$, where a is some element of the set $X^\circ_r - X^\circ$. Then $a \in Y_{r_1} - Y$.

From this lemma we obtain as an immediate consequence the following statement.

13. *If $\mathscr{P}_r(A)$ is an atom in the lattice $\mathscr{E}(A)$, then for each set $X \in \mathscr{P}(A)$ such that $A - X$ contains at least two elements $X = X_r$ holds true.*

14. Theorem. *If the set A is infinite, then the lattice $\mathscr{E}(A)$ has no atom. If the set A has n elements, where $n \geqslant 2$, then the lattice $\mathscr{E}(A)$ has n atoms.*

Proof. Let us suppose first of all that the set A is infinite and that the system $\mathscr{P}_r(A)$ of r-ideals in A is an atom in the lattice $\mathscr{E}(A)$. According to the statement 13 there exists such an element a of the set A that $(A - \{a\})_r = A$. Let us denote $Z = A - \{a\}$. Z is an infinite set, hence $Z_r = \cup \mathscr{K}_r(Z)$ holds. On the other hand for each set $X \in \mathscr{K}(Z)$ we have $X_r = X$ (again according to the statement 13), hence $\cup \mathscr{K}_r(Z) = Z \neq Z_r$, which is a contradiction.

Let the set A contains n elements ($n \geqslant 2$). Let us take an arbitrary element a of the set A and let us put $(A - \{a\})_{r_a} = A$, $X_{r_a} = X$ if $X \neq A - \{a\}$. It can be readily seen that the system $\{X_{r_a} : X \in \mathscr{P}(A)\}$ is a system of r-ideals in A and an atom in the lattice $\mathscr{E}(A)$. If a will run over the whole set A, we shall obtain n different atoms and those are already all atoms of the lattice $\mathscr{E}(A)$.

15. Theorem. *Let A be an arbitrary set which contains at least two elements. Let us denote by $\overline{\overline{A}}$ the cardinality of the set A. The lattice $\mathscr{E}(A)$ of all systems of r-ideals in A has $2^{\overline{\overline{A}}}-2$ dual atoms.*

Proof. Let X° be an arbitrary non-empty fixed proper subset of the set A. Let $X \in \mathscr{P}(A)$. Let us put $X_r = A - X^\circ$ if $X \cap X^\circ = \emptyset$ and $X_r = A$ if $X \cap X^\circ \neq \emptyset$. It is easy to show that $\{X_r : X \in \mathscr{P}(A)\}$ is a system of r-ideals in A. Evidently $\mathscr{P}_r(A)$ is not the greatest element of the lattice $\mathscr{E}(A)$. Let $\mathscr{P}_{r_1}(A)$ be an arbitrary system of r-ideals in A such that $\mathscr{P}_{r_1}(A) > \mathscr{P}_r(A)$. Then there exists a set $X^1 \in \mathscr{P}(A)$ such that $X^1_{r_1} \supsetneqq X^1_r$. Obviously $X^1 \cap X^\circ = \emptyset$, $X^1_r = A - X^\circ$. Therefore there exists an element $x^\circ \in X^\circ$ such that $x^\circ \in X^1_{r_1}$. Since $\{x^\circ\}_r \subset \{x^\circ\}_{r_1}$ and $\{x^\circ\}_r = A$, the following holds $X^1_{r_1} = A$. Let Y be an arbitrary set from the system $\mathscr{P}(A)$ such that $Y_r = A - X^\circ$. From the relation $X^1 \subset Y_{r_1}$ it follows that $Y_{r_1} = A$, hence $\mathscr{P}_{r_1}(A)$ is the greatest element of the lattice $\mathscr{E}(A)$. In this way it is proved that $\mathscr{P}_r(A)$ is a dual atom in the lattice $\mathscr{E}(A)$. Evidently for different non-empty proper subsets X° of the set A we obtain different systems of r-ideals in A. Let $\mathscr{P}_{r_1}(A)$ be an element of the lattice $\mathscr{E}(A)$ different from the greatest. Then there exists a set $X \in \mathscr{P}(A)$ such that $X_{r_1} \subsetneqq A$. Let us put $X^\circ = A - X_{r_1}$ and let us take the corresponding dual atom of the lattice $\mathscr{E}(A)$, i. e. the system of r-ideals in A constructed by the method described at the beginning of the proof. Let. $Y \in \mathscr{P}(A)$. If $X^\circ \cap Y \neq \emptyset$, then $Y_r = A$ and evidently $Y_{r_1} \subset Y_r$. Let $X^\circ \cap Y = \emptyset$. Then $Y \subset A - X^\circ$ and since $A - X^\circ = X_{r_1}$, $Y_{r_1} \subset \subset X_{r_1} = A - X^\circ = Y_r$. In this way it is proved that $\mathscr{P}_{r_1}(A) \leqslant \mathscr{P}_r(A)$. Therefore the lattice $\mathscr{E}(A)$ has as many dual atoms as there are non-empty proper subsets in the set A, i. e. $2^{\overline{\overline{A}}} - 2$.

16. Definition. *A lattice L with the greatest element* 1 *is called dually atomic iff each its element* $x \neq 1$ *is the meet of some dual atoms of the lattice L.*

17. Theorem. *The lattice* $\mathscr{E}(A)$ *of all systems of r-ideals in A is dually atomic.*

Proof. Let $\mathscr{P}_r(A)$ be a given arbitrary system of r-ideals in A different from the greatest. Let us denote $\mathscr{P}'(A)$ the system consisting of those sets X of the system $\mathscr{P}(A)$, for which $X_r \neq A$. Further let $\mathscr{P}_{r_X}(A)$ be a dual atom of the lattice $\mathscr{E}(A)$ such that $X^\circ = A - X_r$, $X \in \mathscr{P}'(A)$ (cf. Theorem 15), i. e. for the set $Y \in \mathscr{P}(A)$ $Y_{r_X} = X_r$ if $Y \subset X_r$ and $Y_{r_X} = A$ if $Y \not\subset X_r$. Let us denote $\mathscr{P}_{r_1}(A) = \bigwedge_{X \in \mathscr{P}'(A)} \mathscr{P}_{r_X}(A)$.

Let $Y \in \mathscr{K}(A) \cap \mathscr{P}'(A)$. Then (cf. statement 5) $Y_{r_1} = \bigcap_{X \in \mathscr{P}'(A)} Y_{r_X} = \bigcap_{\substack{X \in \mathscr{P}'(A) \\ X_r \supset Y}} Y_{r_X} = \bigcap_{\substack{X \in \mathscr{P}'(A) \\ X_r \supset Y}} X_r$. Evidently $Y_r \subset \bigcap_{\substack{X \in \mathscr{P}'(A) \\ X_r \supset Y}} X_r$ holds and since $Y \in \mathscr{P}'(A)$, $Y_r \supset Y$, the inverse inclusion is valid too. Therefore $Y_{r_1} = Y_r$.

Let $Y \in \mathscr{K}(A)$, $Y_r = A$. If $X \in \mathscr{P}'(A)$, then $Y \not\subset X_r$ (because otherwise it would have to be $Y_r \subset X_r$), hence $Y_{r_X} = A$. Then $Y_{r_1} = A$ and hence again $Y_{r_1} = Y_r$.

Let Z be an infinite set from the system $\mathscr{P}(A)$. Then the following holds $Z_{r_1} = \cup \mathscr{K}_{r_1}(Z) = \cup \mathscr{K}_r(Z) = Z_r$.

Therefore the equality $\mathscr{P}_{r_1}(A) = \mathscr{P}_r(A)$ is true.

REFERENCES

[1] FUCHS, L.: On partially ordered algebras II. Acta scient. math. Szeged, *26*, 1965, 35–41.

[2] GAIFMAN, H.: The lattice of all topologies on a denumerable set (abstract). Amer. math. Soc. Notices, *8*, 1961, 356.

[3] JAKUBÍK, J.: Lattice ordered algebras generated by a system of ideals. Colloq. Math., *20*, 1969, 31–44.

[4] SCHNARE, P. S.: Infinite complementation in the lattice of topologies. Fundam. math., *64*, 1969, 249–255.

[5] STEINER, A. K.: The lattice of topologies: structure and complementation. Trans. Amer. Math. Soc., *122*, 1966, 379–398.

[6] WOLK, E. S.: Order compatible topologies on a partially ordered set. Proc. Amer. Math. Soc., 1958, 524–529.

[7] ČORNAYOVÁ, J.: Systémy ideálov v univerzálnych algebrách. [Diplomová práca.] 1968.

Received March 3, 1970

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