Ladislav Györffy; Beloslav Riečan

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ON THE EXTENSION OF MEASURES IN RELATIVELY COMPLEMENTED LATTICES

LADISLAV GYÖRFFY and BELOSLAV RIEČAN, Bratislava

In the paper we extend the main result of paper [1] for relatively complemented lattices. Theorem 2 belongs to the first author, Theorem 3 and Lemma 2 to the second author. Lemma 1 was proved by Prof. M. Kolibiar. Theorem 1 is a consequence of the lemma.

First some notations and terminology. A lattice is called σ-continuous if it is σ-complete and $x_n \nearrow x$, $y_n \nearrow y$ (resp. $x_n \searrow x$, $y_n \searrow y$) implies $x_n \cap y_n \nearrow x \cap y$ (resp. $x_n \cup y_n \searrow x \cup y$). A measure is any function $\gamma : R \to \langle 0, \infty)$ defined on a lattice R with the least element 0 and satisfying the following conditions: 1. $\gamma(0) = 0$. 2. $\gamma(x) + \gamma(y) = \gamma(x \cup y) + \gamma(x \cap y)$ for any $x, y \in R$. 3. If $x_n \nearrow x$, $x_n \in R$ $(n = 1, 2, \ldots)$, $x \in R$, then $\gamma(x_n) \nearrow \gamma(x)$. A subset M of a σ-complete lattice H is called monotone, if $x_n \in M$ $(n = 1, 2, \ldots)$, $x_n \nearrow x$ resp. $x_n \searrow x$, implies $x \in M$. If $D \subset H$, then by $M(D)$ we denote the least monotone set over D.

Theorem 1. *Let H be a σ-continuous, modular, complemented lattice, R be such a sublattice of H that $a \cap b' \in R$ for any $a \in R$ and any complement b' of any $b \in R$. Let γ be a σ-finite measure $\gamma : R \to \langle 0, \infty)$. Then there is just one measure $\gamma : M(R) \to \langle 0, \infty)$ which is an extension of γ.*

Proof. The assumptions of the main theorem of [1] are the same as those of Theorem 1 except of the following one:

(H) To any $x, y, z \in H$ such that $x \leqq y \leqq z$ and any complements x' resp. z' of x resp. z such that $x' \geqq z'$ there is a complement y' of y such that $x' \geqq y' \geqq z'$.

Hence Theorem 1 will be proved if we prove that the condition (H) is satisfied in any modular complemented lattice.

Lemma 1.*) *In any modular complemented lattice the condition (H) is satisfied.*

Proof. Put $t = (y \cup z') \cap x'$. Evidently $z' \leqq t \leqq x'$. Let u be the relative complement of t in $[z', x']$, i.e. $t \cap u = z'$, $t \cup u = x'$. Then

*) Lemma 1 gives the answer to a problem stated in [1] and simultaneously in Čas. pěst. mat., 93 (1968), p. 236.

$$0 = z' \cap z = u \cap t \cap z = u \cap (y \cup z') \cap x' \cap z =$$
$$= u \cap (y \cup z') \cap z = u \cap [y \cup (z' \cap z)] = u \cap y,$$

and

$$1 = x' \cup x = u \cup t \cup x = u \cup x \cup [(y \cup z') \cap x'] =$$
$$= u \cup [(y \cup z') \cap (x \cup x')] = u \cup y \cup z' = u \cup y.$$

Hence $u = y'$ is a complement of y and $z' \leqq y' \leqq x'$.

Let H be now a relatively complemented lattice with the zero element 0. By $a - b$ we denote the set of all complements of $a \cap b$ with respect to $\langle 0, a \rangle$, i.e. $a - b = \{x : x \cap a \cap b = 0,\ x \cup (a \cap b) = a\}$. A sublattice R of H will be called a lattice ring if $a - b \subset R$ for any $a, b \in R$. A lattice σ-ring is a σ-complete lattice ring.

Theorem 2. *Let H be a relatively complemented, modular, σ-continuous lattice with the least element, $R \subset H$ be a lattice ring, γ be a σ-finite measure on R. Then there is just one measure $\bar{\gamma}$ on $M(R)$ that is an extension of γ; the measure $\bar{\gamma}$ is σ-finite.*

Proof. For any $c \in R$ put $R_c = \{x \in R; x \leqq c\}$, $H_c = \{x \in H; x \leqq c\}$ and define $\gamma_c : R_c \to \langle 0, \infty)$ by the formula $\gamma_c(x) = \gamma(x)$. Then H_c, R_c, γ_c satisfy all the assumptions of Theorem 1, therefore there exists just one measure $\bar{\gamma}_c$ on $M(R_c)$ that is an extension of γ_c.

Further denote by B the set of all elements b of the form $b = \bigcup_{n=1}^{\infty} c_n$, $c_n \in R$. As before put $R_b = \{x \in R; x \leqq b\}$. First we prove: If $c \leqq b$, $c \in R$ and $x \in M(R_b)$, $x \leqq c$, then $x \in M(R_c)$. Indeed, the set $K = \{x \in M(R_b);\ x \cap c \in M(R_c)\}$ is monotone and $K \supset R_b$, therefore $K \supset M(R_b)$.

Let $b \in B$, $b = \bigcup_{n=1}^{\infty} c_n$, $c_n \in R$. We can assume $c_n \leqq c_{n+1}$ $(n = 1, 2, \ldots)$. Let $x \in M(R_b)$. Then we put

$$\bar{\gamma}(x) = \lim_{n\to\infty} \bar{\gamma}_{c_n}(x \cap c_n)\,.$$

Of course, we must prove that $\gamma(x)$ does not depend on the choice of the sequence $\{c_n\}_{n=1}^{\infty}$. First, if $y \leqq u \leqq v$, $u, v \in R$, $y \in M(R_v)$, then $\bar{\gamma}_v$ is an extension of γ_u, hence $\bar{\gamma}_v(y) = \bar{\gamma}_u(y)$. Hence, if $x \in M(R_d)$, $d_n \in R$, $d_n \leqq d_{n+1}$ $(n = 1, 2, \ldots)$, $\bigcup_{n=1}^{\infty} d_n = d$, then

$$\bar{\gamma}_{c_n}(x \cap c_n) = \lim_{m\to\infty} \bar{\gamma}_{c_n}(x \cap c_n \cap d_m) =$$

$$= \lim_{m\to\infty} \bar{\gamma}_{c_n \cap d_m}(x \cap c_n \cap d_m) =$$

$$= \lim_{m\to\infty} \bar{\gamma}_{d_m}(x \cap c_n \cap d_m) \leqq \lim_{m\to\infty} \bar{\gamma}_{d_m}(x \cap d_m)$$

and therefore

$$\lim_{n\to\infty} \bar{\gamma}_{c_n}(x \cap c_n) = \lim_{m\to\infty} \bar{\gamma}_{d_m}(x \cap d_m) .$$

To prove that $\bar{\gamma}$ is a measure put $x_k \in M(R_b)$ $(k = 1, 2, \ldots)$, $x_k \nearrow x$. Then evidently $\bar{\gamma}(x) \geqq \lim_{k\to\infty} \bar{\gamma}(x_k)$. On the other hand $\bar{\gamma}(x_k) \geqq \bar{\gamma}_{c_n}(x_k \cap c_n)$, therefore

$$\bar{\gamma}(x) = \lim_{n\to\infty} \bar{\gamma}_{c_n}(x \cap c_n) = \lim_{n\to\infty} \lim_{k\to\infty} \bar{\gamma}_{c_n}(x_k \cap c_n) \leqq \lim_{n\to\infty} \lim_{k\to\infty} \bar{\gamma}(x_k) = \lim_{k\to\infty} \bar{\gamma}(x_k) ,$$

hence

$$\bar{\gamma}(x) = \lim_{k\to\infty} \bar{\gamma}(x_k) .$$

Finally let $x, y \in M(R_b)$, then

$$\bar{\gamma}(x) + \bar{\gamma}(y) = \lim_{n\to\infty} \bar{\gamma}_{c_n}(x \cap c_n) + \lim_{n\to\infty} \bar{\gamma}_{c_n}(y \cap c_n) =$$

$$= \lim_{n\to\infty} \bar{\gamma}_{c_n}((x \cap c_n) \cup (y \cap c_n)) + \lim_{n\to\infty} \bar{\gamma}_{c_n}(x \cap y \cap c_n) =$$

$$= \bar{\gamma}(x \cup y) + \bar{\gamma}(x \cap y)$$

(since $(x \cap c_n) \cup (y \cap c_n) \nearrow x \cup y$, $x \cap y \cap c_n \nearrow x \cap y$).

We have proved that $\bar{\gamma}$ is a measure on the set $M = \bigcup_{b\in B} M(R_b)$. Since $R_b \subset R$ for every b, we have $M(R_b) \subset M(R)$, hence $M \subset M(R)$. But M is a monotone set, $M \supset R$. Therefore $M \supset M(R)$ and $\bar{\gamma}$ is a measure on $M(R)$.

Now we prove that $\bar{\gamma}$ is unique. Let τ be an extension of γ, $\tau : M(R) \to \to \langle 0, \infty)$. If $c \in R$, $x \in M(R_c)$ then $\tau(x) = \bar{\gamma}(x)$, since $\bar{\gamma}, \tau$ are extensions of γ_c on $M(R_c)$. Let $x \in M(R)$ i.e. $x \leqq b$, $b \in B$, $c_n \nearrow b$, $c_n \in R$. Then

$$\tau(x) = \lim_{n\to\infty} \tau(c_n \cap x) = \lim_{n\to\infty} \bar{\gamma}(c_n \cap x) = \bar{\gamma}(x).$$

The measure γ is σ-finite, since the set $N = \{x \in M(R) ; x \leqq \bigcup_{n=1}^{\sigma} c_n \, . \, \gamma(c_n) < \infty \}$ is monotone and contains R.

Let $S(R)$ be the lattice σ-ring generated by R. Finally we prove:

Theorem 3. *If R is a lattice ring in a σ-continuous, modular, relatively complemented lattice with the least element, then $M(R) = S(R)$.*

In the proof of Theorem 3 we need the following lemma:

Lemma 2. *Let H be a modular, relatively complemented lattice with the least* *element. Let $a, b, c \in H$, $a \leqq c$. Then to any $x \in a - b$ there is $y \in c - b$ such that $x \leqq y$.*

Proof. Since $x \in a - b$, we have $x \cap a \cap b = 0$, $x \cup (a \cap b) = a$. Let y be a relative complement of $a \cup (b \cap c)$ in the interval $\langle x, c \rangle$, i.e. $y \cap [a \cup (b \cap c)] = x$ and $y \cup a \cup (b \cap c) = c$. Evidently $a \cap y = x$. Further

$$c = y \cup a \cup (b \cap c) = y \cup (a \cap b) \cup x \cup (b \cap c) =$$
$$= [(a \cap b) \cup (c \cap b)] \cup (x \cup y) = (b \cap c) \cup y,$$

hence

$$(1) \qquad (b \cap c) \cup y = c.$$

The proof of the relation $(b \cap c) \cap y = 0$ is a little more complicated. First we have

$$(a \cup y) \cap [a \cup (b \cap c)] = a \cup (y \cap [a \cup (b \cap c)]) =$$
$$= a \cup x = a,$$

hence

$$\begin{aligned} x \cup (b \cap c) &= (a \cap y) \cup (b \cap c) = \\ &= (\{(a \cup y) \cap [a \cup (b \cap c)]\} \cap y) \cup (b \cap c) = \\ &= (y \cap [a \cup (b \cap c)]) \cup (b \cap c) = \\ &= [y \cup (b \cap c)] \cap [a \cup (b \cap c)] = \\ &= c \cap [a \cup (b \cap c)] = \\ &= a \cup (b \cap c). \end{aligned}$$

Finally

$$\begin{aligned} 0 &= a \cap b \cap x = a \cap b \cap a \cap y = a \cap b \cap y = \\ &= \{[a \cup (b \cap c)] \cap (a \cup y)\} \cap b \cap y = \\ &= [a \cup (b \cap c)] \cap b \cap y = \\ &= [x \cup (b \cap c)] \cap b \cap y = \\ &= (x \cap b \cap y) \cup (b \cap c \cap y) = b \cap c \cap y, \end{aligned}$$

hence

$$(2) \qquad 0 = (b \cap c) \cap y.$$

From (1) and (2) we get that $y \in c - b$. Moreover $y \geqq x$, hence the proof is complete.

Proof of Theorem 3. Since $S(R)$ is a monotone set, evidently $M(R) \subset S(R)$. It is sufficient to prove that $M(R)$ is a lattice ring. Indeed then $M(R)$ is a lattice σ-ring, hence $M(R) \supset S(R)$.

It is not difficult to prove that $M(R)$ is a lattice. The only difficulty is in proving that $a, b \in M(R)$, $x \in a - b$ imply $x \in M(R)$.

First let $b \in R$ be a fixed element and put

$$K = \{a \in M(R);\ x \in a - b \Rightarrow x \in M(R)\}\,.$$

Evidently $K \supset R$. We prove that K is monotone. Hence let $a_n \in K$ $(n = 1, 2, \ldots)$, $a_n \nearrow a$, $x \in a - b$. Since H is σ-continuous, we have $a_n \cap b \nearrow a \cap b$. According to Lemma 1 there are $x_n \in a - a_n \cap b$ such that $x_n \searrow x$. But $x_n \cap a_n \in a_n - a_n \cap b = a_n - b$, since $x_n \cap a_n \cap a_n \cap b = 0$ and $(x_n \cap a_n) \cup (a_n \cap b) = [x_n \cup (a_n \cap b)] \cap a_n = a \cap a_n = a_n$. Thus $x_n \cap a_n \in M(R)$. Since $M(R)$ is a lattice, also $x_m \cap a_n = \bigcap_{i=n}^{m} (x_i \cap a_i) \in M(R)$ for every $m \geqq n$. Hence $x \cap a_n = \bigcap_{m=n}^{\infty} x_m \cap a_n \in M(R)$ $(n = 1, 2, \ldots)$ and therefore $x = x \cap a = \bigcup_{n=1}^{\infty} (x \cap a_n) \in M(R)$. We have proved that K is closed under limits of non-decreasing sequences.

Now let $a_n \in K$ $(n = 1, 2, \ldots)$, $a_n \searrow a$, $x \in a - b$. According to Lemma 2 there are $y_n \in a_n - b$ such that $y_n \geqq x$. Since $a_n \in K$, we have $y_n \in M(R)$ and also $y = \bigcap_{n=1}^{\infty} y_n = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{n} y_i \in M(R)$. We assert that $y = x$. Indeed, first

$$y \cap (a \cap b) \leqq y_n \cap a_n \cap b = 0; \tag{3}$$

further

$$y \cup (a \cap b) \geqq x \cup (a \cap b) = a$$

and

$$y \cup (a \cap b) \leqq \bigcap_{n=1}^{\infty} (y_n \cup (a \cap b)) \leqq \bigcap_{n=1}^{\infty} (y_n \cup (a_n \cap b)) =$$

$$= \bigcap_{n=1}^{\infty} a_n = a\,,$$

hence

$$y \cup (a \cap b) = a\,. \tag{4}$$

The relations (3) and (4) with $y \geqq x$ give $y = x$. Hence $x \in M(R)$, therefore $a \in K$.

Now let $a \in M(R)$ be a fixed element. Put

$$L = \{b \in M(R);\ x \in M(R) \text{ for every } x \in a - b\}\,.$$

We have $L \supset R$. Now with the help of Lemma 1 it is not difficult to prove that L is a monotone set. Hence $L \supset M(R)$ and $x \in M(R)$ for every $a, b \in$

$\in M(R)$, $x \in a - b$. Since $M(R)$ is now evidently a lattice σ-ring, the proof is complete.

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Katedra matematiky
Vysokej školy ekonomickej
Bratislava

Katedra numerickej matematiky
a matematickej štatistiky
Prírodovedeckej fakulty Univerzity Komenského
Bratislava