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THE CANTOR EXTENSION OF A LEXICOGRAPHIC PRODUCT OF l-GROUPS

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Lexicographic products of linearly ordered groups and l-groups were considered by Malcev [3] and Fuchs [2]. Let G be an Abelian lattice ordered group. The Cantor extension of G will be denoted by G_c. Assume that G is isomorphic with the lexicographic product

$$^l\Pi\, A_i(i \in I),$$

where I is a linearly ordered set. In this Note we prove that if I has no greatest element, then G_c is isomorphic with G. Further we show that if i_0 is the greatest element of I, then G_c is isomorphic with the lexicographic product $^l\Pi\, B_i(i \in I)$ such that $B_i = A_i$ for each $i \in I$, $i \neq i_0$ and $B_{i_0} = (A_{i_0})_c$.

1. Let us recall the definition and some properties of the lexicographic product of partially ordered groups (cf. Fuchs [2], p. 40).

Let $I \neq \emptyset$ be a linearly ordered set and let $A_i(i \in I)$ be a set of partially ordered groups. Denote by $^l\Pi\, A_i(i \in I)$ the set of all functions $f: I \to \cup A_i$ $(i \in I)$ satisfying the following two conditions:

(a) $f(i) \in A_i$ for each $i \in I$,

(b) $\sigma(f) = \{i \in I | f(i) \neq 0\}$ is a well ordered set (in the order of I) for each $f \in {}^l\Pi\, A_i$ $(i \in I)$.

If we put for each $f, g \in {}^l\Pi\, A_i$ $(i \in I)$

(a$_1$) $(f + g)(i) = f(i) + g(i)$ for each $i \in I$,

(b$_1$) $f > 0$ if and only if $f(i^*) > 0$, where i^* is the least element of $\sigma(f)$. then $^l\Pi\, A_i$ $(i \in I)$ is a partially ordered group which will be called the lexicographic product of the partially ordered groups A_i $(i \in I)$.

If $I = \{1, 2\}$ (with the natural order), then the lexicographic product of partially ordered groups A_i $(i \in I)$ is denoted by $A_1 \circ A_2$. The following assertions are easy to verify:

(i) $^l\Pi\, A_i$ $(i \in I)$ is a linearly ordered group if and only if A_i $(i \in I)$ are linearly ordered groups.

(ii) If I has no greatest element, then $^l\Pi\, A_i$ $(i \in I)$ is an l-group if and only if A_i $(i \in I)$ are linearly ordered groups.

(iii) If there exists the greatest element i_0 in I, then

(a) ${}^l\Pi A_i$ $(i \in I)$ is an l-group if and only if $A_i (i \in I \setminus \{i_0\})$ are linearly ordered groups and A_{i_0} is an l-group.

(b) The set $\bar{A}_{i_0} = \{f \in {}^l\Pi A_i\ (i \in I) \mid f(i) = 0$ for each $i \in I,\ i \neq i_0\}$ is convex in ${}^l\Pi A_i\ (i \in I)$.

In the whole paper we assume that G is an Abelian l-group. By the symbol $\simeq$ we denote an isomorphism of l-groups.

2. Now we describe the method for constructing the Cantor completion of an Abelian l-group G (the proofs are omitted, cf. Everett [1] and Fuchs [2] p. 149). We may use (see [1]) ordinary sequences (x_n) $(n = 1, 2, \ldots)$. Denote by N the set of all positive integers.

If (t_n) $((t'_n))$ is a descending (increasing)(1) sequence of elements of G and if there is $t = \wedge t_n\ (n \in N)$ $(t' = \vee t'_n\ (n \in N))$ in G, then we write $t_n \downarrow t$ $(t'_n \uparrow t')$. We write $x_n \to x$ (x_n o-converges to x or x is o-limit of x_n) if there exist monotone sequences (t_n) and (t'_n) such that $t_n \downarrow x$, $t'_n \uparrow x$ and $t'_n \leq x_n \leq t_n$ for each $n \in N$. A sequence (x_n) such that $x_n = x$ for each $n \in N$ will be denoted by (x). If $x_n \to 0$, then (x_n) is said to be a zero sequence. It is easy to verify that $x_n \to 0$ exactly if $|x_n| \leq t_n$ $(n \in N)$ for some (t_n) such that $t_n \downarrow 0$. The sequence (x_n) is fundamental if there exists a sequence (t_n) such that $t_n \downarrow 0$ and $|x_n - x_m| \leq t_n$ for each n and each $m \geq n$.

Denote by H the set of all fundamental sequences of G. If we define the operation $+$ in H in a natural way, i.e., if we put $(x_n) + (y_n) = (x_n + y_n)$ for each $(x_n), (y_n) \in H$, then H is a group. The set E of all zero sequences is an invariant subgroup of H. Put $H/E = G_c$. If $(x_n), (y_n) \in H$ then $(x_n \vee y_n) \in H$ holds. A coset of G_c containing a fundamental sequence (x_n) will be denoted by $\overline{(x_n)}$. For $\overline{(x_n)}$, $\overline{(y_n)}$ we put $\overline{(x_n)} \leq \overline{(y_n)}$ if $\overline{(x_n \vee y_n)} = \overline{(y_n)}$. Then G_c becomes an l-group. It is said to be the Cantor extension of G.

3. Let $A_1 \neq \{0\}$, $A_2 \neq \{0\}$ be partially ordered groups. Assume that there exists a mapping φ of an Abelian l-group G into $A_1 \circ A_2$ such that

$$G \simeq A_1 \circ A_2 \tag{1}$$

is true under the mapping φ. By (iii) (a), A_1 is a linearly ordered group and A_2 is an l-group. For a component of an element $x \in G$ in $A_1 (A_2)$ we shall use the symbol $\varphi(x)(1)$ $(\varphi(x)(2))$. Form the sets

$$\bar{A}_1 = \{x \in G \mid \varphi(x)(2) = 0\},$$

$$\bar{A}_2 = \{x \in G \mid \varphi(x)(1) = 0\}.$$

(1) If x_n $(n \in N)$ are elements of a partially ordered set and $x_1 \leq x_2 \leq \ldots$, then (x_n) is said to be an increasing sequence. Analogously we define a descending sequence.

It is clear that $\overline{A}_1$, $\overline{A}_2$ are subgroups of G and

$$\overline{A}_1 \simeq A_1, \; \overline{A}_2 \simeq A_2 \tag{2}$$

hold. Let ψ be a mapping of G into $\overline{A}_1 \circ \overline{A}_2$ such that $\psi(x) = (\varphi^{-1}(\varphi(x)(1), 0), \varphi^{-1}(0, \varphi(x)(2)))$ for all x in G. Then

$$G \simeq \overline{A}_1 \circ \overline{A}_2 \tag{3}$$

under the mapping ψ. For any element $x \in G$ we put $x(1)$ $(x(2))$ instead of $\psi(x)(1)$ $(\psi(x)(2))$. It is easily seen that

$$x \in \overline{A}_1 \text{ if, and only if, } x(2) = 0,$$

$$x \in \overline{A}_2 \text{ if, and only if, } x(1) = 0. \tag{*}$$

4. *If $t_n \downarrow 0 (\uparrow 0)$ in G, then there exists $n_0 \in N$ such that $t_n \in \overline{A}_2$ for each $n \in N$, $n \geq n_0$.*

Proof. Assume that $t_n \downarrow 0$. First let us prove that there exists $n_0 \in N$ such that $t_{n_0}(1) = 0$. Suppose (by way of contradiction) that $t_n(1) > 0$ for each n. Because of $\overline{A}_2 \neq \{0\}$, we can find an element $g \in G$ such that $g > 0$, $g(1) = 0$. Then $g < t_n$ for each n contrary to $\wedge t_n = 0$ and thus with respect to $(*)$ $t_{n_0} \in \overline{A}_2$ for some $n_0 \in N$. Since by (iii) (b) $\overline{A}_2$ is convex in G and $t_n \leq t_{n_0}$ whenever $n \geq n_0$, we have $t_n \in \overline{A}_2$ for each $n \geq n_0$. If $t_n \uparrow 0$, the proof is similar.

5. *If $x_n \to 0$ in G, then there exists $n_0 \in N$ such that $x_n \in \overline{A}_2$ for each $n \in N$, $n \geq n_0$.*

Proof. There exists $t_n \downarrow 0$ such that $|x_n| \leq t_n$ for each n. By 4 there exists $n_0 \in N$ such that $t_n \in \overline{A}_2$ for each $n \geq n_0$. The convexity of $\overline{A}_2$ in G implies $x_n \in \overline{A}_2$ for each $n \geq n_0$.

Let $E'(H')$ be the set of all zero (fundamental) sequences in $\overline{A}_2$. A coset of $(\overline{A}_2)_c$ containing a sequence $(a_n) \in H'$ will be denoted by $\overline{\overline{(a_n)}}$.

6. *If $(x_n) \in E$, then $(x_n(2)) \in E'$.*

Proof. If $(x_n) \in E$, then there exist $t_n \downarrow 0$, $t'_n \uparrow 0$ in G such that $t'_n \leq x_n \leq t_n$ for each n. By 4 there exist $n_1, n_2 \in N$ such that $t_n \in \overline{A}_2$ for each $n \geq n_1$ and $t'_n \in \overline{A}_2$ for each $n \geq n_2$. We have to show that there are $z_n \downarrow 0$, $z'_n \uparrow 0$ in $\overline{A}_2$ such that $z'_n \leq x_n(2) \leq z_n$ for each n. Put $z_n = x_n(2) \vee x_{n+1}(2) \vee \ldots \vee x_{n_1-1}(2) \vee t_{n_1}$ for $n = 1, 2, \ldots, n_1-1$, $z_n = t_n$ for each $n \geq n_1$, $z'_n = x_n(2) \wedge x_{n+1}(2) \wedge \ldots \wedge x_{n_2-1}(2) \wedge t'_{n_2}$ for $n = 1, 2, \ldots, n_2 - 1$. $z'_n = t'_n$ for each $n \geq n_2$. The sequences (z_n) and (z'_n) satisfy the mentioned conditions.

7. *If (x_n) is a fundamental sequence in G, then there exists $n_0 \in N$ such that $x_n(1) = x_{n_0}(1)$ for each $n \in N$, $n \geq n_0$.*

Proof. Using the definition of the fundamental sequence we get $|x_n - x_m| \leq t_n$ for some $t_n \downarrow 0$, each n and each $m \geq n$. Because of 4 there exists

$n_0 \in N$ such that $t_n \in \bar{A}_2$ for each $n \geq n_0$. The convexity of $\bar{A}_2$ in G implies $x_n - x_m \in \bar{A}_2$, thus $x_n(1) = x_{n_0}(1)$ for each $n \geq n_0$.

8. *If* $(x_n) \in H$, *then* $(x_n(2)) \in H'$.

Proof. There exists $t_n \downarrow 0$ such that $|x_n - x_m| \leq t_n$ for each n and each $m \geq n$. Using 4 and 7 we obtain that there exists $n_0 \in N$ such that $t_n = t_n(2)$ and $x_n - x_m = x_n(2) - x_m(2)$ for each $n \geq n_0$ and each $m \geq n$. We have to show that there exists $z_n \downarrow 0$ in $\bar{A}_2$ such that $|x_n(2) - x_m(2)| \leq z_n$ for each n and each $m \geq n$. In view of [2], p. 112, the property J we obtain

$$|x_{n_0-1}(2) - x_m(2)| = |(x_{n_0-1}(2) - x_{n_0}(2)) + (x_{n_0}(2) - x_m(2)) \leq$$

$$\leq |x_{n_0-1}(2) - x_{n_0}(2)| + |x_{n_0}(2) - x_m(2)| \leq |x_{n_0-1}(2) - x_{n_0}(2) + t_{n_0}$$

for each $m \geq n_0 - 1$. Thus we may put

$$z_n = |x_n(2) - x_{n+1}(2)| + \ldots + |x_{n_0-1}(2) - x_{n_0}(2)| +$$

$$+ t_{n_0} \text{ for } n = 1, 2, \ldots, n_0 - 1,$$

$$z_n = t_n \text{ for each } n \geq n_0.$$

Let (x_n), (y_n) be fundamental sequences in G.

9. $\overline{(x_n)} = \overline{(y_n)}$ *if and only if there exists* $n_0 \in N$ *such that* $x_n(1) = y_n(1)$ *for each* $n \geq n_0$ *and* $\overline{\overline{(x_n(2))}} = \overline{\overline{(y_n(2))}}$.

Proof. If $\overline{(x_n)} = \overline{(y_n)}$ or equivalently $(x_n - y_n) \in E$, then by 5 there exists $n_0 \in N$ such that $x_n(1) = y_n(1)$ for each $n \geq n_0$ and by 6 $(x_n(2) - y_n(2)) \in E'$. i. e., $\overline{\overline{(x_n(2))}} = \overline{\overline{(y_n(2))}}$. Conversely, let $\overline{\overline{(x_n(2))}} = \overline{\overline{(y_n(2))}}$ and $x_n(1) = y_n(1)$ for each $n \geq n_0$. Then $(x_n(2) - y_n(2)) = ((x_n - y_n)(2)) \in E'$. Since $(x_n - y_n)(1) = 0$, by $(*)$ we get $(x_n - y_n)(2) = x_n - y_n$ for each $n \geq n_0$. Then in a similar way as in the proof of 6 we can find sequences (t_n) and (t_n') such that $t_n \downarrow 0$, $t_n' \uparrow 0$ in G and $t_n' \leq x_n - y_n \leq t_n$, for each n. Thus $(x_n - y_n) \in E$, i. e., $\overline{(x_n)} = \overline{(y_n)}$.

10. $G_c \simeq A_1 \circ (A_2)_c$.

Proof. Let $\overline{(x_n)}$ be an arbitrary element of G_c. By 7 there exists $n_0 \in N$ such that $x_n(1) = x_{n_0}(1)$ for each $n \geq n_0$. Define a mapping α of G_c into $\bar{A}_1$ $(\bar{A}_2)_c$ by the rule $\alpha(\overline{(x_n)}) = (x_{n_0}(1), \overline{\overline{(x_n(2))}})$. In view of 8 and 9$\alpha$ is a one-to-one mapping of G_c into $\bar{A}_1 \circ (\bar{A}_2)_c$. If $(a, \overline{(b_n)}) \in \bar{A}_1 \circ (\bar{A}_2)_c$, then $((a, b_n))$ is a fundamental sequence in $\bar{A}_1 \circ \bar{A}_2$ and thus because of (3) it is clear that α is a mapping of G_c onto $\bar{A}_1 \circ (\bar{A}_2)_c$. It can be easily verified that α preserves the group operation and the lattice operations. Then (2) completes the proof.

11. Theorem 1. *Assume that a linearly ordered set (finite or infinite) has the greatest element* i_0 *and* $A_i (i \in I)$ *are partially ordered groups such that* $A_i \neq \{0\}$ *for each* $i \in I$. *If* G *is an Abelian l-group such that* $G \simeq {}^l\Pi\, A_i$ $(i \in I)$, *then* $G_c \simeq {}^l\Pi\, B_i$ $(i \in I)$, *where* $B_i = A_i$ *for each* $i \in I$, $i \neq i_0$ *and* $B_{i_0} = (A_{i_0})_c$.

Proof. From the assumption we get $G \simeq A \circ A_{i_0}$, where $A = {}^l\Pi A_i$ $(i \in I \setminus \{i_0\})$ with respect to (i) is a linearly ordered group. By 10 we conclude $G_c \simeq A \circ (A_{i_0})_c$, which completes the proof.

12. Now assume that a linearly ordered set $I \neq \emptyset$ has no greatest element and A_i $(i \in I)$ are partially ordered groups such that $A_i \neq \{0\}$ for any $i \in I$. Let there exist a mapping φ of an Abelian l-group G into ${}^l\Pi A_i$ $(i \in I)$ such that

$$G \simeq {}^l\Pi A_i \ (i \in I) \tag{4}$$

under the mapping φ. Let $i \in I$ be fixed and let us put

$$\overline{A}_i = \{x \in G \mid \varphi(x)(j) = 0 \quad \text{for each} \quad j \in I,\ j \neq i\}.$$

$\overline{A}_i$ is a subgroup of G and $\overline{A}_i \simeq A_i$ for each $i \in I$. Then

$$G \simeq {}^l\Pi \overline{A}_i \ (i \in I). \tag{5}$$

If $x \in G$ and if under the isomorphism (5) $x \to f$, then we denote $x(i) = f(i)$.

Since I has no greatest element, for a fixed element $i \in I$ there exists $j \in I$, $j > i$. If we denote

$$A^i = {}^l\Pi \overline{A}_j \ (j \in I, j \leq i), \quad A'^i = {}^l\Pi \overline{A}_j (j \in I,\ j > i),$$

then

$$G \simeq A^i \circ A'^i. \tag{6}$$

Let $t_n \downarrow 0$ in G and let i_n denote the least element of $\sigma(t_n)$. Then $t_n(i_n) > 0$ holds. The sequence (i_n) is increasing, since the sequence (t_n) is descending.

With respect to (6) and 4, 5, 7 we get the following assertions:

13. *For each $i \in I$ there exists $n_i \in N$ such that $i_n > i$ for each $n \in N$, $n \geq n_i$.*

14. *If $(x_n) \in E$, then for each $i \in I$ there exists $n_i \in N$ such that $x_n(i) = 0$ for each $n \in N$, $n \geq n_i$.*

15. *If $(x_n) \in H$, then for each $i \in I$ there exists $n_i \in N$ such that $x_n(i) = x_{n_i}(i)$ for each $n \in N$, $n \geq n_i$.*

Let $(x_n) \in H$ and for any $i \in I$ let $n_i \in N$ be as in 15. Put $x_i^* = x_{n_i}(i)$ for each $i \in I$. With this denotation we have:

16. *There exists an element $x \in G$ such that $x(i) = x_i^*$ for each $i \in I$.*

Proof. Since $x_i^* \in \overline{A}_i$ for each $i \in I$, we have only to prove that the set $A = \{i \in I \mid x_i^* \neq 0\}$ is well ordered. To show this pick out any set $I_1 \neq \emptyset$, $I_1 \subseteq A$ and any element $i_0 \in I_1$. If i_0 is not the least element of I_1, then $I_2 = \{i \in I_1 \mid i < i_0\} \neq \emptyset$ holds. According to 13 for i_0 there exists $n_0 \in N$ such that $i_{n_0} > i_0$. Then we have $t_{n_0}(i) = 0$ for each $i \in I$, $i \leq i_0$. This implies $x_n(i) = x_{n_0}(i)$ for each $n \geq n_0$ and each $i \in I$, $i \leq i_0$. Thus $x_{n_0}(i) = x_i^*$ for each $i \in I$, $i \leq i_0$. We infer $x_{n_0}(i) \neq 0$ for each $i \in I_2$, and so $I_2 \subseteq \sigma(x_{n_0})$. Since the set $\sigma(x_{n_0})$ is well ordered, the set I_2 is well ordered, too, and so I_2 has the least element i^*. Then i^* is the least element in I_1, too.

17. *Suppose that* (x_n), $(y_n) \in H$ *and* $x, y \in G$ *such that* $x(i) = x_i^*$, $y(i) = y_i^*$ *for each* $i \in I$. *Then* $\overline{(x_n)} = \overline{(y_n)}$ *if and only if* $x = y$.

Proof. Let $\overline{(x_n)} = \overline{(y_n)}$, that is, $(x_n - y_n) \in E$. By 14 and 15 for each $i \in I$ there exists $n_i \in N$ such that $(x_n - y_n)(i) = 0$ and $x_n(i) = x_i^*$, $y_n(i) = y_i^*$ for each $n \in N$, $n \geq n_i$. Thus $x = y$. The converse is obvious.

18. Corollary. $\overline{(x_n)} = \overline{(x)}$ *where* $x \in G$ *such that* $x(i) = x_i^*$ *for each* $i \in I$.

19. $G \simeq G_c$.

Proof. Define a mapping α of G into G_c by the rule $\alpha(g) = \overline{(g)}$ for any $g \in G$. By 17 and 18 α is a one-to-one mapping of G onto G_c. We can easily verify that α preserves the group operation and the lattice operations, thus $G \simeq G_c$.

We have arrived at

Theorem 2. *Let a linearly ordered set* $I \neq \emptyset$ *have no greatest element and let* $A_i (i \in I)$ *be partially ordered groups such that* $A_i \neq \{0\}$ *for each* $i \in I$. *If* G *is an Abelian l-group such that* $G \simeq {}^l\Pi\, A_i\ (i \in I)$, *then* $G_c \simeq G$.

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