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Matematický časopis, Vol. 23 (1973), No. 3, 285--289

Persistent URL: http://dml.cz/dmlcz/126884

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ON THE MEASURABILITY OF FUNCTIONS OF TWO VARIABLES

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Introduction

It is well known that when a real-valued function f of two real variables x, y is Lebesgue measurable in each variable separately it need not be measurable in (x, y), and that when f is continuous in each variable separately it need not be continuous in (x, y). However in the latter case f must be measurable: indeed Ursell proved [9] that if f is continuous in x for each y and measurable in y for each x, then it must be measurable in (x, y). (Marczewski and Ryll-Nardzewski [5] and Neubrunn [7] gave generalizations with x running over a separable metric space.) This was extended by Michael and Rennie [6] to the following: if f is measurable in y for almost all x, is equal to zero outside a certain measurable set E, and on E is continuous in x with respect to E for almost all y, then f must be plane measurable. One of us recently showed [2] that this theorem, with a similar proof, applies in products of more general topological measure spaces. Here we go further, replacing R^2 (R — the real line) by a product $X \times Y$ of general σ-finite measure spaces of which only X is (second-countable) topological. The method of proof is necessarily different from that in [6], which made use of the topology of R^2; in fact it turns out to be somewhat simpler. After stating and proving our theorem we show that the second-countability of X cannot be dropped from the hypotheses.

Main theorem

Theorem 1. *Let (X,μ) be a σ-finite second-countable topological measure space*(1) *and let (Y, ν) be any σ-finite measure space. If f: $X \times Y \to R$ is $\bar{\nu}$-measurable in y for $\bar{\mu}$-almost all x, is $\overline{\mu \times \nu}$-measurable on the complement of a certain $\overline{\mu \times \nu}$-*

(1) That is, the σ-algebra of subsets of X on which μ is defined includes the Borel sets.

-measurable set E, and on E is continuous in x with respect to E for $\bar{\nu}$-almost all y, then f must be $\overline{\mu \times \nu}$-measurable.

Proof. Without loss of generality we may suppose that $\mu(X) < \infty$ and $\nu(Y) < \infty$. Since the completion of $\mu \times \nu$ is the same as that of $\bar{\mu} \times \bar{\nu}$, by μ and ν we may denote the already completed measures $\bar{\mu}$ and $\bar{\nu}$, respectively. On the other hand, sets of measure zero do not affect the conclusion of the theorem, and hence we may assume that the section E^y is μ-measurable and the section $f^y : E^y \to R$ continuous for all y, and that E_x is ν-measurable and $f_x : E_x \to R$ ν-measurable for all x. Further we may suppose that $0 \leq f(x, y) \leq 1$ on E, since every real-valued function can be written (preserving continuity and measurability) as a difference of two non-negative ones and each non-negative function g is equal to $\lim_n n \,.\, g_n$, where for g_n defined by $g_n(x, y) = n^{-1} \,.\, \inf\{n, g(x, y)\}$ we have in fact $0 \leq g_n \leq 1$. We must show that f is $\overline{\mu \times \nu}$ -measurable on E.

Let $G_1, G_2, \ldots$ be a countable basis for the non-empty open sets in X. Given any n, define points $x_{n1}, x_{n2}, \ldots \in G_n$ by induction as follows: let

$$k_{ns} = \sup\{\nu(E_x \setminus \bigcup_{r<s} E_{x_{nr}});\ x \in G_n\},$$

and select $x_{ns} \in G_n$ with

$$\nu(E_{x_{ns}} \setminus \bigcup_{r<s} E_{x_{nr}}) \geq \frac{1}{2} k_{ns}.$$

Denote by F_n the set $\bigcup_{s=1}^{\infty} E_{x_{ns}}$, and by H_n the set $G_n \times F_n$.

Assertion I. $(\overline{\mu \times \nu})\,[E \cap (G_n \times Y) \setminus H_n] = 0.$

Proof of Assertion I. Observe first that $G_n \times Y$ and H_n are $\overline{\mu \times \nu}$-measurable, and therefore so is the set $K_n = E \cap (G_n \times Y) \setminus H_n$. Hence in view of Fubini's theorem it will be suficient to show that $\nu[(K_n)_x] = 0$ for all $x \in G_n$. Now for $x \in G_n$ we have $(K_n)_x = E_x \setminus \bigcup_{s=1}^{\infty} E_{x_{ns}}$. Consequently, if $\nu[(K_n)_x] = d > 0$, then $k_{ns} \geq d$ for all $s = 1, 2, \ldots$, and

$$\nu(Y) \geq \nu(\bigcup_{s=1}^{\infty} E_{x_{ns}}) = \sum_{s=1}^{\infty} \nu(E_{x_{ns}} \setminus \bigcup_{r<s} E_{x_{nr}}) = \infty,$$

a contradiction. Our assertion is proved.

From Assertion I it follows that the set

$$Z = \bigcup_{n=1}^{\infty} [E \cap (G_n \times Y) \setminus H_n]$$

has $\overline{\mu \times \nu}$-measure zero, and it will be sufficient to prove that $f|(E \setminus Z)$ is $\overline{\mu \times \nu}$-measurable. Let

$$D = \{x_{ns};\, n = 1, 2, \ldots, s = 1, 2, \ldots\}.$$

For each n define a function $f_n\colon E \setminus Z \to R$ as follows:

if $(x, y) \in (E \setminus Z) \setminus (G_n \times Y)$ then $f_n(x, y) = 1$;

if $(x, y) \in (E \setminus Z) \cap (G_n \times Y)$ then $f_n(x, y) = \sup\{f(w, y);$

$$w \in D \cap G_n \textit{ and } (w, y) \in E\}.$$

Observe that if $(x, y) \in (E \setminus Z) \cap (G_n \times Y)$ then $(x, y) \in G_n \times F_n$, so $x \in G_n$ and $y \in E_{x_{ns}}$ for some s; hence $y \in E_w$ for some $w \in D \cap G_n$, that is, $w \in D \cap G_n$ and $(w, y) \in E$ for some w, so the supremum is over a non-empty set. Since f_n is obviously $\overline{\mu \times \nu}$-measurable, it will be enough to prove the following.

Assertion II. *On $E \setminus Z$ we have $f = \inf_n f_n$.*

Proof of Assertion II. (a) To show that $f(x, y) \leq \inf_n f_n(x, y)$ on $E \setminus Z$, we must show that $f(x, y) \leq f_n(x, y)$ for all n. This is obvious if $(x, y) \in (E \setminus \setminus Z) \setminus (G_n \times Y)$, because then $f(x, y) \leq 1 = f_n(x, y)$. Hence we may suppose that $(x, y) \in (E \setminus Z) \cap (G_n \times Y)$; in particular $x \in G_n$. It will be enough to show that $f(x, y) - \varepsilon \leq f_n(x, y)$ for every $\varepsilon > 0$.

In view of the continuity of f^y, there is an open set G containing x such that $f(z, y) \geq f(x, y) - \varepsilon$ for all $z \in G \cap E^y$. For some m we have $x \in G_m \subset G \cap G_n$. Then $(x, y) \in (E \setminus Z) \cap (G_m \times Y)$, and as observed earlier there exists $w \in D \cap G_m$ with $(w, y) \in E$. Then $f(w, y) \geq f(x, y) - \varepsilon$ and, since $w \in D \cap G_n$, $f_n(x, y) \geq f(w, y) \geq f(x, y) - \varepsilon$ as required.

(b) Finally we show that $f(x, y) \geq \inf_n f_n(x, y)$ on $E \setminus Z$; that is, given $\varepsilon > 0$ we have $f(x, y) + \varepsilon \geq f_m(x, y)$ for some m. As above, there is an open set G containing x such that $f(z, y) \leq f(x, y) + \varepsilon$ for all $z \in G \cap E^y$. For some m we have $x \in G_m \subset G$. Then $(x, y) \in (E \setminus Z) \cap (G_m \times Y)$, and for every $w \in D \cap G_m$ with $(w, y) \in E$ we certainly have $w \in G \cap E^y$ and therefore $f(w, y) \leq f(x, y) + \varepsilon$. Hence $f_m(x, y) \leq f(x, y) + \varepsilon$, as required.

A counter-example

Our proof that the second-countability hypothesis is essential in Theorem 1 will be based on two key notions: Sierpiński's paradoxical decomposition of R^2 [8] and the density topology on R (see [3]).

Theorem 2. *There exists a σ-finite topological measure space (X, μ), a σ-finite measure space (Y, ν), and a function $f\colon X \times Y \to R$ such that f_x is ν-measurable for all x and f^y is continuous for all y, but f is not $\overline{\mu \times \nu}$-measurable.*

Proof. Let $\aleph_\alpha$ be the least possible cardinality for a subset of (0, 1) having positive outer Lebesgue measure, and choose a set $S \subset (0, 1)$ of cardinality $\aleph_\alpha$ with $m^*(S) > 0$. Let (S, μ) be the measure space in which the σ-algebra consists of the intersections with S of the Lebesgue measurable subsets of (0, 1), and in which μ is outer Lebesgue measure on this σ-algebra. We consider $(S \times S, \mu \times \mu)$, the first factor being endowed with the topology induced on S by the density topology on R.

Let $\prec$ be a well ordering of S of type ω_α. Define $M = \{(x, y); x \prec y\}$ and observe that $(S \times S \setminus M)_x$ has measure zero for all $x \in S$ and M^y has measure zero for all $y \in S$. In particular M^y is a closed set with respect to the density topology on R. We can choose a set $K = K(y)$ in $R \setminus M^y$ which is closed in the ordinary topology, such that $K \cap S$ has positive μ-measure. By the Remark after Theorem 3 of [3] there is a function f^y from (0, 1) to $\langle 0, 1\rangle$ which is continuous with respect to the density topology, such that $f^y(x) = 1$ on M^y and $f^y(x) = 0$ on $K(y)$.

Let $f\colon S \times S \to \langle 0, 1\rangle$ be defined by $f(x, y) = f^y(x)$ for $(x, y) \in S \times S$. For each fixed x, f_x differs from the characteristic function of M_x on a set of measure zero only, and so

$$\int_S f(x, y)d\mu(y) = \mu(M_x) = \mu(S),$$

while

$$\int_S f(x, y)d\mu(x) \leq \mu(S) - \mu[K(y) \cap S] < \mu(S),$$

an application of Fubini's theorem yields the desired non-measurability of f.

Remarks

In view of our results, it is natural to ask whether if (X, μ) is an arbitrary σ-finite topological measure space and $f\colon X \times X \to R$ is continuous in y for all x and continuous in x for all y, the function f is necessarily $\overline{\mu \times \nu}$-measurable. One of us has shown [1] that the answer is negative, assuming the existence of a non-measurable cardinal, but that the answer is positive in the special case when X is R with the density topology and μ is Lebesgue measure. The latter result resolves a problem of Mišík recently quoted by Lipiński [4].

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Received June 20, 1972

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