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Matematický časopis, Vol. 25 (1975), No. 1, 63--81

Persistent URL: http://dml.cz/dmlcz/127040

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APPLICATION OF ROTHE'S METHOD TO NONLINEAR EVOLUTION EQUATIONS

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This paper deals with the initial boundary value problem for abstract nonlinear evolution equations of the form

$$\frac{\mathrm{d}\, u(t)}{\mathrm{d}t} + A(t)\, u(t) = f(t), \quad u(0) = u_0, \quad 0 \leqslant t \leqslant T < \infty, \tag{1}$$

where $A(t)$ is for every $t \in \langle 0, T\rangle$ a nonlinear operator. Using Rothe's method, the author proved in [1] the existence of a weak solution for some class of nonlinear differential equations of the form (1). Using this method and following some technics used by J. Nečas in [2] we can generalize and strengthen the results of [1] (part II). Deriving a priori estimates we use some results of P. P. Mosolov [3].

The method of Rothe consists in the following idea: Successively, for $j = 1, 2, \ldots, n$ we solve (see the definition 4) the equations

$$\frac{z_j - z_{j-1}}{h} + A(t_j)z_j = f(t_j), \tag{1a}$$

where $\{t_j\}$ $(j = 0, 1, \ldots, n)$ is an equidistant partition of the interval $\langle 0, T\rangle$, $h = Tn^{-1}$ and $t_j = jh \,.\, z_0 \equiv u_0$, where u_0 is from (1). Then, under certain assumptions, Rothe's function

$$z^n(t) = z_{j-1} + (t - t_{j-1})\, h^{-1}\, (z_j - z_{j-1}) \quad \text{for} \quad t_{j-1} \leqslant t \leqslant t_j, \tag{*}$$

j 1, 2, ..., n converges toward the solution of (1). This method, introduced by E. Rothe in [4], has been used by many authors — for this purpose see references [1]—[8].

Assumptions

Let V be a real reflexive Banach space and V' its dual space. The duality between V and V' we denote by $[.\,,.]$. Let H be a real Hilbert space with scalar product $(.\,,.)$ and the norm $\|.\|$. The norm in V, V' we denote by $\|.\|_V$, $\|.\|_{V'}$. We assume that $V \cap H$ is a dense set in both V and H with the corresponding norms. $A(t)$, $t \in \langle 0, T\rangle$ is a system of operators satisfying

$$(2) \qquad A(t) : V \to V' \quad \text{is continuous for each} \quad t \in \langle 0, T\rangle$$

$$(3) \qquad [A(t)\, u - A(t)\, v, u - v] \geqslant 0 \quad \text{for all} \quad u, v \in V, t \in \langle 0, T\rangle$$

$$(4) \qquad [A(t)\, u, u] \geqslant \|u\|_V\, r(\|u\|_V) \quad \text{for all} \quad u \in V, t \in \langle 0, T\rangle,$$

where the function $r(s)$ is nondecreasing for $s \geqslant s_0 > 0$, bounded in $\langle 0, s_0\rangle$ and satisfying $\lim_{s\to\infty} r(s) = \infty$.

$$(5) \qquad A(t)\, u = \operatorname{grad} \Phi(t, u) \quad \text{for} \quad t \in \langle 0, T\rangle,\ u \in V,$$

where $\Phi(t, u)$ is a functional defined on V, i.e., $A(t)$ are potential operators.

There exist derivatives $A'(t)\, u$, $A''(t)\, u$ of $A(t)\, u$ in V' with respect to $t \in (0, T)$ and

$$(6) \qquad \|A'(t)u\|_{V'} + \|A''(t)\, u\|_{V'} \leqslant C_1 + C_2 r(\|u\|_V)\,.$$

We shall assume that $f(t)$ is Lipschitz continuous from $\langle 0, T\rangle$ into H, i.e.,

$$(7) \qquad \|f(t) - f(t')\| \leqslant L|t - t'| \quad \text{for all} \quad t, t' \in \langle 0, T\rangle.$$

Remark 1. If $V \equiv W_p^k$ (Sobolev space) with $p > 1$, then $r(s) = C_1\, s^{p-1} - C_2$.

Remark 2. In Remark 4 we point out that the conditions (4) and (6) can be substituted by (4′) and (6′), which are more general in some sense:

$$(4') \qquad (\|u\|_V)^{-1}\, [A(t)\, u, u] \to \infty \quad \text{for} \quad \|u\|_V \to \infty$$

uniformly in $t \in \langle 0, T\rangle$.

$$(6') \quad \text{i)} \left|\frac{\partial}{\partial t}\, \Phi(t, u)\right| + \left|\frac{\partial^2}{\partial t^2}\, \Phi(t, u)\right| \leqslant C_1 + C_2\, |\Phi(t, u)|$$

$$\text{ii)} \ \|A'(t)\, u\|_{V'} < \infty \quad \text{for all} \quad t \in \langle 0, T\rangle, \quad u \in V$$

$$\text{iii)} \ |\Phi(t, u)| \leqslant C_1 + C_2\, [A(t)\, u, u]\,.$$

Remark 3. In (6) or (6′) it suffices to consider the difference quotient of the first and second order in the place of corresponding derivatives of $A(t)$ and $\Phi(t, u)$.

Definition 1. *$u(t) \in C_w^1(\langle 0, T\rangle, H)$, iff $(u(t), v) \in C^1(\langle 0, T\rangle)$ for all $v \in H$. If $u(t) \in C_w^1(\langle 0, T\rangle, H)$, then $\dfrac{u(t+h) - u(t)}{h}$ is weakly convergent in H for*

$h \to 0$ *and we denote by* $\dfrac{\mathrm{d}\, u(t)}{\mathrm{d}\, t}$ *this weak limit.*

Definition 2. *Under the solution of the problem* (1) *we understand a strongly continuous function* $u(t) : \langle 0, T\rangle \to H$ *such that* $u(t) \in C^1_w(\langle 0, T\rangle, H)$, $u(t) \in V \cap H$ *for* $t \in \langle 0, T\rangle$, $u(0) = u_0$ *and* $u(t)$ *satisfies* (1) *for all* $t \in (0, T)$.

Let X be a Banach space with the norm $\|.\|_X$.

Definition 3. *By* $L_\infty(\langle 0, T\rangle, X)$ *we denote the set of all measurable functions* (*see* [9]) $u(t) : \langle 0, T\rangle \to X$ *with* $\|u\|_{L_\infty(<0,T>,X)} = \sup\limits_{t\, <0,T>} \text{ess}\, \|u(t)\|_X < \infty$.

The space $V \cap H$ with the norm $\|.\|_{V\cap H} = \|.\| + \|.\|_V$ is a reflexive Banach space. We denote the weak convergence by $\rightharpoonup$ and the strong convergence by $\to$. $u(t)$ is weakly continuous in $V \cap H$ with respect to $t \in \langle 0, T\rangle$, iff $u(t) \rightharpoonup u(t_0)$ for $t \to t_0$ holds for each $t_0 \in \langle 0, T\rangle$, where $t \in \langle 0, T\rangle$.

The positive constants will be denoted by C and the dependence of C on the parameter ε will be denoted by $C(\varepsilon)$. C and $C(\varepsilon)$ will denote even different constants in the same consideration.

Let us denote by $x^n(t)$ the step function

(**) $$x^n(t) = z_j \quad \text{for} \quad t_{j-1} < t \leqslant t_j, \quad j = 1, 2, \dots n$$

and $x^n(0) = u_0$, where $z_j \in V \cap H$ ($j = 1, 2, \dots n$) are the solutions of the equations (1*a*) and $u_0 \in V \cap H$ is from (1).

Theorem. *Let us assume that* (2)—(7) *are fulfilled. If* $u_0 \in V \cap H$ *and* $A(0)\, u_0 \in H$, *then there exists a unique solution* $u(t)$ *of* (1) *with the following properties*:

a) $u(t)$ *is Lipschitz continuous from* $\langle 0, T\rangle$ *into* H

b) $u(t) \in L_\infty(\langle 0, T\rangle,\ V \cap H)$ *and* $u(t)$ *is weakly continuous in* $V \cap H$ *with respect to* $t \in \langle 0, T\rangle$.

c) $A(t)\, u(t)$ *is weakly continuous in* H *with respect to* $t \in \langle 0, T\rangle$.

d) $u(t) \in C^1_w(\langle 0, T\rangle, H)$ *and* $\dfrac{\mathrm{d}\, u(t)}{\mathrm{d}\, t} \in L_\infty(\langle 0, T\rangle,\ H)$

e) $\max\limits_{0<t<T} \|z^n(t) - u(t)\|^2 \leqslant C(u_0, f) n^{-1}$

f) $\max\limits_{0<t<T} \|z^n(t) - x^n(t)\| \leqslant C(u_0, f)\, n^{-1}$

g) $z^n(t) \rightharpoonup u(t)$, $x^n(t) \rightharpoonup u(t)$ *in* $V \cap H$ *for* $n \to \infty$ *and for each* $t \in \langle 0, T\rangle$

h) *If* u_i ($i = 1, 2$) *is a solution of the problem* (1) *corresponding to the right-hand side* f_i *and the initial condition* u_{0i}, *then*

$$\max_{0<t<T} \|u_1(t) - u_2(t)\| \leqslant 2 \int_0^T \|f_1(t) - f_2(t)\|\, dt + \|u_{01} - u_{02}\|.$$

First, in several assertions we obtain a priori estimates and deduce some consequences. Then, we prove the theorem.

For simplicity we denote $A(j) \equiv A(t_j)$ and $f(j) \equiv f(t_j)$ $(j = 1, 2, \ldots n)$.

Successively, for $j = 1, 2, \ldots n$ let us solve the equations

$$(z_j - z_{j-1})h^{-1} + A(j)z_j = f(j)$$

where $z_0 \equiv u_0$.

Definition 4. $z_j \in V \cap H$ *is a solution of* (8), *iff*

$$\left(\frac{z_j - z_{j-1}}{h}, v\right) + [A(j)z_j, v] = (f(j), v)$$

holds for all $v \in V \cap H$.

Due to (4), the operator $A(t)\,u + \lambda u$ for $\lambda > 0$ is coercive in the space $V \cap H$ and strictly monotone. Thus, there exists a unique solution $z_j \in V \cap \cap H$ of (8) which is also a point of minimum for the coercive, strictly convex functional

$$\Phi(t_j, u) + (2h)^{-1} \|u - z_{j-1}\|^2 - (f(j), u) \equiv \Psi(t_j, u, z_{j-1}) \tag{9}$$

on the reflexive space $V \cap H$.

Assertion 1. There exist $C(u_0, f)$ and $h_0 > 0$ such that

i) $\sum_{j=1}^{n} h \|z_j\|_V \, r(\|z_j\|_V) \leqslant C(u_0, f)$ ii) $\|z_j\| \leqslant C(u_0, f)$

for each $j = 1, 2, \ldots n$ and $h \leqslant h_0$.

Proof. We have

$$[A(j)\, z_j, v] + h^{-1}(z_j - z_{j-1}, v) = (f(j), v) \tag{10}$$

for all $v \in V \cap H$, $j = 1, 2, \ldots n$. Let $1 \leqslant p \leqslant n$. Substituting $v = hz_j$ and summing (10) through $j = 1, 2, \ldots p$ we obtain

$$\sum_{j=1}^{p} h[A(j)\, z_j, z_j] + \sum_{j=1}^{p} (z_j - z_{j-1}, z_j) = h \sum_{j=1}^{p} (f(j), z_j). \tag{11}$$

The following identity

$$\sum_{j=1}^{p} 2(z_j - z_{j-1}, z_j) = \sum_{j=1}^{p} \|z_j - z_{j-1}\|^2 + \|z_p\|^2 - \|z_0\|^2 \tag{12}$$

holds.

Using Young's inequality

$$ab \leqslant 2^{-1} \varepsilon^2 a^2 + (2\varepsilon^2)^{-1} b^2 \quad (\varepsilon \neq 0) \tag{13}$$

we estimate

(14) $$|(f(j), z_j)| \leqslant \|f(j)\| \, \|z_j\| \leqslant 2^{-1} \|z_j\|^2 + 2^{-1} \|f(j)\|^2.$$

Due to (4), we deduce that there exists a C such that

$$[A(j) z_j, z_j] \geqslant -C \quad \text{for each } n \text{ and } \quad j = 1, 2, \ldots n.$$

From this estimate, (7), (11), (12) and (14) we obtain

$$\|z_p\|^2 \leqslant C + \|u_0\|^2 + \sum_{j=1}^{p} h \, \|f(j)\|^2 + \sum_{j=1}^{p} h \|z_j\|^2 \leqslant C(u_0, f) + \sum_{j=1}^{p} h \|z_j\|^2.$$

From this inequality for $h \leqslant h_0 < 1$ we successively deduce

$$\|z_1\|^2 \leqslant C(u_0, f) \, (1 - h)^{-1} \quad (\text{for } p = 1),$$

$$\|z_2\|^2 \leqslant C(u_0, f) \, (1 - h)^{-1} \left(1 + \frac{h}{1 - h}\right)$$

and

(15) $$\|z_i\|^2 \leqslant C(u_0, f) \, (1 - h)^{-1} \left(1 + \frac{h}{1 - h}\right)^{i-1}$$

for $i = 1, 2, \ldots n$.

There exists a C such that $\left(1 + \frac{h}{1 - h}\right)^{i-1} \leqslant C$

for each $h \leqslant h_0$ and $i = 1, 2, \ldots n$. Thus, from (15) we obtain Assertion 1 ii). From ii), (4) and (11) we easily obtain Assertion 1 i).

Assertion 2. There exist $C(u_0, f)$ and $h_0 > 0$ such that $\left\| \frac{z_1 - z_0}{h} \right\| \leqslant C(u_0, f)$

for each $h \leqslant h_0$.

Proof. From (10) for $j = 1$, $v = z_1 - z_0$ we obtain

(16) $$[A(1) z_1, z_1 - z_0] - [A(1) z_0, z_1 - z_0] + h^{-1} \|z_1 - z_0\|^2 =$$
$$- (f(1), z_1 - z_0) + ([A(0) z_0, z_1 - z_0] - [A(1) z_0, z_1 - z_0]) -$$
$$- [A(0) z_0, z_1 - z_0].$$

Using Lagrange's theorem we have

$$[A(0) z_0, z_1 - z_0] - [A(1) z_0, z_1 - z_0] = [A'(0 + \vartheta t_1) z_0, z_1 - z_0] \, . \, h$$

for suitable $0 \leqslant \vartheta \leqslant 1$. Hence, due to (6) we have

$$(17) \qquad |[A(0)\, z_0\,,\, z_1 - z_0] - [A(1)\, z_0\,,\, z_1 - z_0]| \leqslant h\, \|z_1 - z_0\|_V (C_1 + C_2\, r(\|z_0\|_V) \leqslant C_1\, h \|z_1\|_V + h\, C_2(u_0)\,.$$

Since $A(0)\, z_0 \equiv A(0)\, u_0 \in H$, the estimate

$$(18) \qquad |[A(0)\, z_0\,,\, z_1 - z_0]| \leqslant \|A(0)\, z_0\|\, \|z_1 - z_0\|$$

holds. From (3), (16), (17) and (18) we deduce

$$\left\| \frac{z_1 - z_0}{h} \right\|^2 \leqslant \|f(1)\| \left\| \frac{z_1 - z_0}{h} \right\| + \|A(0)\, u_0\| \left\| \frac{z_1 - z_0}{h} \right\| + C_1\, \|z_1\|_V + C_2(u_0)$$

and hence applying (13) we obtain

$$(19) \qquad \left\| \frac{z_1 - z_0}{h} \right\|^2 \leqslant C_1\,(u_0, f) + C_2\, \|z_1\|_V \leqslant C_2(u_0, f) + C_2\, \|z_1\|_V\, r(\|z_1|_V)\,.$$

From (10) for $j = 1$ and $v = z_1$ we have

$$[A(1)\, z_1\,,\, z_1] = -\left(\frac{z_1 - z_0}{h}\,,\, z_1 \right) + (f(1), z_1)\,.$$

Thus, due to (3), (13), (19) and Assertion 1 ii) we have

$$\|z_1\|_V\, r(\|z_1\|_V) \leqslant \left\| \frac{z_1 - z_0}{h} \right\| \|z_1\| + \|f(1)\|\, \|z_1\| \leqslant$$

$$\leqslant 2^{-1}\, \varepsilon^2\, C_2(u_0, f) + 2^{-1}\, \varepsilon^2\, C_2 \|z_1\|_V\, r(\|z_1\|_V) +$$

$$+ 2^{-1}\, \varepsilon^{-2}\, \|z_1\|^2 + \|f(1)\|\, \|z_1\| \leqslant C_3(u_0, f, \varepsilon) +$$

$$+ 2^{-1}\, \varepsilon^2\, C_2\, \|z_1\|_V\, r(\|z_1\|_V)\,.$$

Let us put $\varepsilon = \dfrac{1}{\sqrt{C_2}}$. Then, the estimate

$$(20) \qquad \|z_1\|_V\, r(\|z_1\|_V) \leqslant C_4(u_0, f)$$

is valid and hence, due to (19), the proof of Assertion 2 follows.

Estimating $\left\| \dfrac{z_j - z_{j-1}}{\mathrm{h}} \right\|$ we use a variational method. The idea of such an estimation is due to P. P. Mosolov [3]. Analogously as in [3] (Lemma 1 and Lemma 6) we prove Assertions 3 and 4.

Assertion 3. The inequality

$$\Phi(t_i, z_i) \leqslant \Phi(t_i, z) + h^{-1}(z - z_i, z_i - z_{i-1}) - (f(i), z - z_i) \tag{21}$$

is valid for all $z \in V \cap H$.

For completness we sketch the proof of this assertion. $\Phi(t, u)$ is convex in u, since $A(t)$ is a monotone (see (3) and (5)). From the minimality property of z_i for $\Psi(t_i, z, z_{i-1})$ (see (9)) we have $\Psi(t_i, z_i, z_{i-1}) \leqslant \Psi(t_i, r z_i + s z, z_{i-1})$ for all $0 \leqslant r, s \leqslant 1$ with $r + s = 1$ and $z \in V \cap H$. Thus, from the identity

$$(r u + s v - w, r u + s v - w) = r(u - w, u - w) + \\ + s(v - w, v - w) - r s(u - v, u - v),$$

where $u, v, w \in H$, $0 \leqslant r, s \leqslant 1$ with $r + s = 1$ and the convexity of $\Phi(t, u)$ we obtain

$$\Phi(t_i, z_i) + (2h)^{-1} \|z_i - z_{i-1}\|^2 - (f(i), z_i) \leqslant \\ \leqslant r\, \Phi(t_i, z_i) + s\, \Phi(t_i, z) + (2h)^{-1} r\|z_i - z_{i-1}\|^2 + \\ + (2h)^{-1} s\|z - z_{i-1}\|^2 - (2h)^{-1} r s \|z_i - z\|^2 - \\ - r(f(i), z_i) - s(f(i), z)$$

and hence

$$\Phi(t_i, z_i) \leqslant \Phi(t_i, z) - (2h)^{-1} \|z_i - z_{i-1}\|^2 + \\ + (2h)^{-1} \|z - z_{i-1}\|^2 - (2h)^{-1} r\|z_i - z\|^2 + (f(i), z_i - z).$$

From this inequality and from the identity

$$- \|z_i - z_{i-1}\|^2 + \|z - z_{i-1}\|^2 - r\|z_i - z\|^2 = \\ = 2(z - z_i, z_i - z_{i-1}) + s\|z - z_i\|^2$$

we deduce

$$\Phi(t_i, z_i) \leqslant \Phi(t_i, z) + h^{-1}(z - z_i, z_i - z_{i-1}) + \\ + (f(i), z_i - z) + s \|z - z_i\|^2.$$

Thus, by limiting process $s \to 0$ we obtain (21).

Assertion 4. There exist $C(u_0, f)$, C and $h_0 > 0$ such that $\left\| \frac{z_j - z_{j-1}}{h} \right\|^2 \leqslant C(u_0, f) + C \max_{1<p\leq j} \|z_p\|_V\, r(\|z_p\|_V)$ holds for each $h \leqslant h_0$ and $j = 1, 2, \ldots n$.

Proof. Consider (21) with $i = j$, $z = z_{j-1}$ and with $i = j - 1$, $z = z_j$. Summing up these inequalities we obtain

$$(22) \qquad \Phi(t_j, z_j) - \Phi(t_j, z_{j-1}) + \Phi(t_{j-1}, z_{j-1}) - \Phi(t_{j-1}, z_j) + $$
$$+ h^{-1}\|z_j - z_{j-1}\|^2 \leqslant h^{-1}(z_j - z_{j-1}, z_{j-1} - z_{j-2}) +$$
$$+ (f(j) - f(j-1), z_j - z_{j-1}).$$

Let us denote

$$\Phi_j = \Phi(t_j, z_j) + \Phi(t_{j-1}, z_{j-1}) - \Phi(t_{j-1}, z_j) - \Phi(t_j, z_{j-1}).$$

From (22) and (13) we obtain

$$(23) \qquad \left\| \frac{z_j - z_{j-1}}{h} \right\|^2 \leqslant \left\| \frac{z_j - z_{j-1}}{h} \right\| \|f(j) - f(j-1)\| +$$
$$+ 2^{-1}\left\| \frac{z_j - z_{j-1}}{h} \right\|^2 + 2^{-1} \left\| \frac{z_{j-1} - z_{j-2}}{h} \right\|^2 - \frac{\Phi_j}{h}$$

Due to (7) and (13) we have

$$\left\| \frac{z_j - z_{j-1}}{h} \right\| \|f(j) - f(j-1)\| \leqslant \left\| \frac{z_j - z_{j-1}}{h} \right\| L\,h \leqslant$$
$$\leqslant \left\| \frac{z_j - z_{j-1}}{h} \right\|^2 2^{-1}L\,h + 2^{-1}L\,h$$

and hence from (23) we obtain

$$(24) \qquad \left\| \frac{z_j - z_{j-1}}{h} \right\|^2 (1 - L\,h) \leqslant \left\| \frac{z_{j-1} - z_{j-2}}{h} \right\|^2 + L\,h - \frac{2\Phi_j}{h}$$

Let us aussme that $h_0 < L^{-1}$. Thus, from (24) we obtain successively

$$(25) \qquad \left\| \frac{z_j - z_{j-1}}{h} \right\|^2 (1 - L\,h)^{j-1} \leqslant \left\| \frac{z_1 - z_0}{h} \right\|^2 + L\,h \sum_{i=2}^{j} (1 - Lh)^{i-2} -$$
$$- \sum_{i=2}^{j} \frac{2\Phi_i}{h} (1 - L\,h)^{i-2}.$$

The inequality $1 \geqslant (1 - L\,h)^i \geqslant \exp(-L\,T)$ holds and $(1 - L\,h)^i$ is decreasing in i. Thus, using
Abel's summation formula we estimate

$$\left|\sum_{i=2}^{j} \frac{2\Phi_i}{h} (1 - L\,h)^{i-2}\right| \leqslant \max_{1\le j\le p} \left|\sum_{i=2}^{j} \frac{2\Phi_i}{h}\right|$$

and hence, owing to Assertion 2, from (25) we obtain

$$(26) \qquad \left\|\frac{z_j - z_{j-1}}{h}\right\|^2 \leqslant C(u_0, f) + C \max_{1\le p\le j} \left|\sum_{i=2}^{p} \frac{2\Phi_i}{h}\right|$$

since $L\,h \sum_{i=2}^{j} (1 - L\,h)^{i-2} \leqslant L\,h.\,(j-2) < LT$.

The strength of the variational method used consists in the following estimate

$$(27) \qquad \left|\sum_{i=2}^{p} \frac{\Phi_i}{h}\right| \leqslant C(u_0, f) + C\|z_p\|_V\, r(\|z_p\|_V).$$

Indeed, the sum in (27) can be rewritten into the form

$$(28) \qquad \sum_{i=2}^{p} \frac{\Phi_i}{h} = h^{-1}(\Phi(t_p, z_p) - \Phi(t_{p-1}, z_p)) -$$

$$- h^{-1}(\Phi(t_2, z_1) - \Phi(t_1, z_1)) - \sum_{i=3}^{p} h^{-1}(\Phi(t_i, z_{i-1}) -$$

$$- \Phi(t_{i-1}, z_{i-1})) - h^{-1}(\Phi(t_{i-1}, z_{i-1}) - \Phi(t_{i-2}, z_{i-1})).$$

The formula $\Phi(t, u) = \int_0^1 [A(t)\,\tau u, u]\,d\tau$ is true

and thus, using Lagrange's formula and the assumption (6), the expression in the last sum in (28) can be estimated by

$$|h^{-1} \int_0^1 [A(t_i)\,\tau z_{i-1} - 2\,A(t_{i-1})\,\tau z_{i-1} + A(t_{i-2})\,\tau z_{i-1},$$

$$, z_{i-1}]\,d\tau| \leqslant h\,\|z_{i-1}\|_V \int_0^1 (C_1 + C_2\, r(\tau\,|z_{i-1}\|_V)\,d\tau \leqslant$$

$$\leqslant C_1\,h\,\|z_{i-1}\|_V + C_2 h\,\|z_{i-1}\|_V\, r(\|z_{i-1}\|_V) \leqslant$$

$$\leqslant h\,C_3\,\|z_{i-1}\|_V\, r(\|z_{i-1}\|_V) + h\,C_4,$$

since $r(s)$ is nondecreasing for $s \geqslant s_0$ and bounded in $\langle 0, s_0\rangle$. Analogously, from (6) we deduce

$$|h^{-1}(\Phi(t_p, z_p) - \Phi(t_{p-1}, z_p))| \leqslant C_1 + C_2 \|z_p\|_V \, r(\|z_p\|_V)$$

and

$$|h^{-1}(\Phi(t_2, z_1) - \Phi(t_1, z_1))| \leqslant C_1 + C_2 \|z_1\|_V \, r(\|z_1\|_V) \leqslant C(u_0, f),$$

where the estimate (20) has been used. From these estimates, Assertion 1, (28), (27) and (26) the proof follows.

Assertion 5. There exist $C(u_0, f)$ and $h_0 > 0$ such that

$$\text{i)} \quad \left\| \frac{z_j - z_{j-1}}{h} \right\| \leqslant C(u_0, f), \qquad \text{ii)} \quad \|z_j\|_V \leqslant C(u_0, f)$$

holds for each $h \leqslant h_0$ and $j = 1, 2, \ldots n$.

Proof. Suppose that

$$\max_{1 \le p \le n} \|z_p\|_V \, r(\|z_p\|_V) = \|z_{p_0}\|_V \, r(\|z_{p_0}\|_V).$$

Then, owing to Assertion 4 we obtain

$$\left\| \frac{z_{p_0} - z_{p_0-1}}{h} \right\|^2 \leqslant C(u_0, f) + C \|z_{p_0}\|_V \, r(\|z_{p_0}\|_V),$$

where $C(u_0, f)$ and C are from Assertion 4. Using (13) and Assertion 1 we estimate

$$(29) \qquad \left| \left(\frac{z_{p_0} - z_{p_0-1}}{h}, z_{p_0} \right) \right| \leqslant (2\varepsilon^2)^{-1} \|z_{p_0}\|^2 + $$
$$+ \varepsilon^2 2^{-1} \left\| \frac{z_{p_0} - z_{p_0-1}}{h} \right\|^2 \leqslant C(u_0, f, \varepsilon) +$$
$$+ 2^{-1} \varepsilon^2 C \|z_{p_0}\|_V \, r(\|z_{p_0}\|_V).$$

Let us choose $\varepsilon > 0$ so that $\varepsilon^2 C = 2^{-1}$. From (10) for $j = p_0$, $v = z_{p_0}$ and with respect to (29) and Assertion 1 we obtain

$$[A(p_0) z_{p_0}, z_{p_0}] \leqslant C(u_0, f) + 2^{-1} \|z_{p_0}\|_V \, r(\|z_{p_0}\|_V).$$

Hence, due to (4) we deduce

$$\|z_{p_0}\|_V \, r(\|z_{p_0}\|_V) \leqslant C(u_0, f)$$

from which Assertion ii) follows. From ii) and Assertion 4 we deduce Assertion i) and the proof of Assertion 5 is complete.

Remark 4. Assertion 5 holds true if (4), (6) are substituted by (4′), (6′).

Indeed, we work with the expression $[A(t)\, u, u]$ instead of $\|u\|_V\, r(\|u\|_V)$. Assertion 2 can be proved on the base of (6′) ii). In estimating (20) in Assertion 4 we use Lagrange's formula and the inequality

$$|\Phi(t, u)| \leqslant C(1 + |\Phi(t', u)|),$$

which we obtain from (6′) i) with C independent of either t, t' or u. Then, using (6′) i) and iii) we infer

$$\left\| \frac{z_j - z_{j-1}}{h} \right\|^2 \leqslant C(u_0, f) + C \max_{1 \leq p < j} [A(t_p)\, z_p, z_p]$$

from which we obtain Assertion 5.

Let us define the step function f^n by

$$f^n(t) = f(j) \quad \text{for} \quad t_{j-1} < t \leqslant t_j, \quad j = 1, 2, \ldots n$$

and

$$f^n(0) = f(0).$$

Similarly we define the operator $A^n(t)$ by

$$A^n(t) = A(t_j) = A(j) \quad \text{for} \quad t_{j-1} < t \leqslant t_j, \quad j = 1, 2, \ldots n$$

and

$$A^n(0) = A(0).$$

Rothe's function $z^n(t)$ (see (*)) is differentiable from the left and

$$\frac{\mathrm{d}^- z^n(t)}{\mathrm{d}\, t} = \frac{z_j - z_{j-1}}{h} \quad \text{for} \quad t \in (t_{j-1}, t_j\rangle,$$

$$j = 1, 2, \ldots n,$$

where $\dfrac{\mathrm{d}^-}{\mathrm{d}\, t}$ is the derivative from the left.

With respect to this notation relation (10) can be rewritten in the form

$$\left(\frac{\mathrm{d}^- z^n(t)}{\mathrm{d}\, t}, v \right) + [A^n(t)\, x^n(t), v] = (f^n(t), v) \tag{30}$$

for all $v \in V \cap H$ and $t \in \langle 0, T\rangle$.

Before we carry out the limiting process in (30) we prove some assertions.

Assertion 6 There exists $C(u_0, f)$ such that

$$\|A^n(t)\, x^n(t)\| \leqslant C(u_0, f) \quad \text{for all } n \text{ and } \quad t \in \langle 0, T\rangle.$$

Proof. Due to Assertion 5 from (30) we conclude

$$|[A^n(t)\, x^n(t), v]| \leqslant C(u_0, f)\, \|v\|$$

for all n, $t \in \langle 0, T\rangle$ and $v \in V \cap H$. Since $V \cap H$ is dense in H we have

$$A^n(t)\, x^n(t) \in H \quad \text{and} \quad \|A^n(t)\, x^n(t)\| \leqslant C(u_0, f).$$

Assertion 7. There exists $C(u_0, f)$ such that

$$|[A^n(t)\, x^n(t), v - v']| \leqslant C(u_0, f)\, \|v - v'\|$$

holds for all v, $v' \in V \cap H$ and $t \in \langle 0, T\rangle$.

Proof. From (30) we deduce

$$[A^n(t)\, x^n(t), v - v'] = -\left(\frac{\mathrm{d}^- z^n(t)}{\mathrm{d}\, t}, v - v'\right) + (f(t), v - v').$$

On the base of Assertion 5 i) we have

$$\left\|\frac{\mathrm{d}^- z^n(t)}{\mathrm{d}\, t}\right\| \leqslant C(u_0, f) \quad \text{for all } n \text{ and } \quad t \in \langle 0, T \quad ,$$

from which we obtain the required result.

From the definition of $z^n(t)$, $x^n(t)$ (see (*) and (**)) and Assertion 5 i) we immediately obtain

$$\|z^n(t) - x^n(t)\| \leqslant C(u_0, f)\, n^{-1}. \tag{31}$$

From (7) we deduce

$$\|f^n(t) - f(t)\| \leqslant TL\, n^{-1}. \tag{32}$$

Assertion 8. There exists $u(t) : \langle 0, T\rangle \to H$ such that $z^n(t) \to u(t)$, $x^n(t) \to$ $\to u(t)$ for $n \to \infty$ in H uniformly on $\langle 0, T\rangle$.

Proof.

$$\frac{\mathrm{d}^-}{\mathrm{d}\, t} \|z^m - z^n\|^2 = 2\left(\frac{\mathrm{d}^- z^m(t)}{\mathrm{d}\, t} - \frac{\mathrm{d}^- z^n(t)}{\mathrm{d}\, t}, z^m(t) - z^n(t)\right) - \tag{33}$$

$$= 2(f^m(t) - f^n(t), z^m(t) - z^n(t)) -$$

$$- 2\,[A^m(t)\, x^m(t) - A^n(t)\, x^n(t), z^m(t) - z^n(t)]$$

Now, we estimate

$$(34)\qquad [A^m(t)\, x^m(t) - A^n(t)\, x^n(t), z^m(t) - z^n(t)] =$$
$$[A^m(t)\, x^m(t) - A^n(t)\, x^n(t), z^m(t) - z^n(t) - x^m(t) + x^n(t)] +$$
$$+ [A^m(t)\, x^m(t) - A^n(t)\, x^n(t), x^m(t) - x^n(t)].$$

From (31) and Assertion 7 we conclude

$$(35)\qquad |[A^m(t)\, x^m(t) - A^n(t)\, x^n(t), z^m(t) - x^m(t) - z^n(t) + x^n(t)]| \leqslant$$
$$\leqslant C(u_0, f)\, (m^{-1} + n^{-1}).$$

From (3) we deduce

$$(36)\qquad [A^m(t)\, x^m(t) - A^n(t)\, x^n(t), x^m(t) - x^n(t)] =$$
$$= [A^m(t)\, x^m(t) - A^m(t)\, x^n(t), x^m(t) - x^n(t)] +$$
$$+ [A^m(t)\, x^n(t) - A^n(t)\, x^n(t), x^m(t) - x^n(t)] \geqslant$$
$$\geqslant [A^m(t)\, x^n(t) - A^n(t)\, x^n(t), x^m(t) - x^n(t)].$$

Using Lagrange's theorem and (6) we have

$$[A(t')\, v - A(t'')\, v, z] = (t - t')\, [A'(t'' + \tau(t' - t'')\, v, z]$$

for a suitable $0 \leqslant \tau \leqslant 1$ and thus

$$(36a)\qquad |[A(t')\, v - A(t'')\, v, z]| \leqslant |t - t'|\, \|z\|_V (C_1 + C_2\, r(\|v\|_V)).$$

On the base of these estimates, Assertion 5 ii)

$$(\|x^n(t)\|_V + \|x^m(t) - x^n(t)\|_V \leqslant C(u_0, f))$$

and the definitions of $A^n(t)$, $x^n(t)$ we conclude

$$(37)\qquad |[A^m(t)\, x^n(t) - A^n(t)\, x^n(t), x^m(t) - x^n(t)]| \leqslant (m^{-1} + n^{-1})\, C(u_0, f).$$

Hence, from (33)—(37) we conclude

$$\frac{d^-}{dt} \|z^m(t) - z^n(t)\|^2 \leqslant 2\|f^m(t) - f^n(t)\|\, \|z^m(t) - z^n(t)\| + C(u_0, f)\, (m^{-1} + n^{-1})$$

and hence

$$(38)\qquad \|z^m(t) - z^n(t)\|^2 \leqslant 2 \int_0^T \|f^m(t) - f^n(t)\|\, \|z^m(t) - z^n(t)\|\, dt +$$
$$+ TC(u_0, f)\, (m^{-1} + n^{-1}) \leqslant C(u_0, f)\, (m^{-1} + n^{-1})$$

since

$$\|z^m(t) - z^n(t)\| \leqslant C(u_0, f)$$

and

$$\|f^m(t) - f^n(t)\| \leqslant L(m^{-1} + n^{-1}) \quad \text{for all} \quad t \in \langle 0, T\rangle .$$

From this fact it follows that there exists $u(t) \in H$ for $t \in \langle 0, T\rangle$ such that $z^n(t) \to u(t)$ in H for $n \to \infty$ uniformly with respect to $t \in \langle 0, T\rangle$. Thus, from (31) it follows $x^n(t) \to u(t)$ in H uniformly with respect to $t \in \langle 0, T\rangle$ and the proof of Assertion 8 is complete.

Assertion 9. Let $u(t)$ be the function from Assertion 8. Then,

i) $u(t)$ is Lipschitz continuous from $\langle 0, T\rangle$ into H

ii) $u(t) \in V \cap H$ for each $t \in \langle 0, T\rangle$

iii) $u(t)$ is weakly continuous in $V \cap H$ with respect to $t \in$ 0, $T\rangle$

iv) $u(t) \in L_\infty(\langle 0, T\rangle, V \cap H)$.

Proof.

i) Using triangle inequality and Assertion 5 i) we obtain easily

$$\|z^n(t) - z^n(t')\| \leqslant C(u_0, f)\, |t - t'| \tag{39}$$

and hence, owing to Assertion 8, we obtain i).

ii) Due to Assertion 1 ii) and Assertion 5 ii) we have

$$\|x^n(t)\|_V + \|x^n(t)\| \leqslant C(u_0, f)$$

and hence owing to the reflexivity of $V \cap H$ there exists a subsequence $\{x^{n_k}(t)\}$ and $w_t \in V \cap H$, so that $x^{n_k}(t) \rightharpoonup w_t$ in $V \cap H$, where t is a fixed point from $\langle 0, T\rangle$. Thus,

$$\|w_t\|_V + \|w_t\| \leqslant C(u_0, f).$$

On the other hand $x^n(t) \to u(t)$ in H for $n \to \infty$ and thus $u(t) \equiv w_t$. From this fact it follows $x^n(t) \rightharpoonup u(t)$ in $V \cap H$ for each $t \in \langle 0, T\rangle$ and

$$\|u(t)\|_V + \|u(t)\| \leqslant C(u_0, f) \quad \text{for each} \quad t \in \langle 0, T\rangle. \tag{40}$$

Thus, Assertion ii) is proved.

iii) Suppose that $t_n \to t_0$ for $n \to \infty$, t_n, $t_0 \in \langle 0, T\rangle$. From (40) it follows that there exists a subsequence $\{u(t_{n_k})\}$ from $\{u(t_n)\}$ and $v \in V \cap H$ such that $u(t_{n_k}) \rightharpoonup v$ in $V \cap H$ for $k \to \infty$. On the other hand from Assertion 9 i) it follows $u(t_{n_k}) \to u(t_0)$ in H for $k \to \infty$ and thus $u(t_0) \equiv v$. From this fact it follows $u(t_n) \rightharpoonup u(t_0)$ in $V \cap H$ for $n \to \infty$ and iii) is proved.

iv) Since $u(t) \in V \cap H$ for each $t \in \langle 0, T\rangle$ and (40) holds, it suffices to prove that $u(t)$ is measurable. For this purpose it suffices to prove (see [9] Theorem of Pettis) that the set $\{u(t);$ for each $t \in \langle 0, T\rangle\}$ is separable in $V \cap H$ and that $u(t)$ is weakly measurable, i.e., $x^*(u(t))$ is a measurable function in $t \in \langle 0, T\rangle$ for each $x^* \in (V \cap H)'$ (dual space), where $x^*(x)$ is the value of

$x^* \in (V \cap H)'$ at the point $x \in V \cap H$. Since $u(t)$ is weakly continuous in $V \cap H$ with respect to $t \in \langle 0, T \rangle$, it is weakly measurable. Let us consider the countable set $M = \{u(r)$, for each rational number $r \in \langle 0, T \rangle\}$.

Let $L(M)$ be the smallest closed linear subspace of $V \cap H$ containing M. Then, $L(M)$ is a separable space. We prove that $u(t) \in L(M)$ for each $t \in \in \langle 0, T \rangle$.

Let $t \in \langle 0, T \rangle$ be a fixed point. There exist r_n, $n = 1, 2, \ldots$ ($r_n \in \langle 0, T \rangle$ rational points) such that $r_n \to t$ for $n \to \infty$. From iii) we have $u(r_n) \rightharpoonup u(t)$ in $V \cap H$ for $n \to \infty$. Since $u(r_n) \in L(M)$ and $L(M)$ is weakly closed, $u(t) \in \in L(M)$ and the proof of iv) is complete.

Assertion 10.

$$A^n(t)\, x^n(t) \rightharpoonup A(t)\, u(t) \quad \text{in} \quad H \quad \text{for} \quad n \to \infty,$$

for all $t \in \langle 0, T \rangle$.

Proof. From Assertion 6 it follows that there exists a subsequence $\{x^{n_k}(t)\}$ of $\{x^n(t)\}$ and $g_t \in H$ ($t \in \langle 0, T \rangle$ is a fixed point) such that

$$A^{n_k}(t)\, x^{n_k}(t) \rightharpoonup g_t \quad \text{in } H \quad (\text{also in } (V \cap H)').$$

From the inequality

$$|[A^{n_k}(t)\, x^{n_k}(t), x^{n_k}(t)] - [g_t, u(t)]| \leqslant$$
$$\leqslant |[A^{n_k}(t)\, x^{n_k}(t) - g_t, u(t)]| + |[A^{n_k}(t)\, x^{n_k}(t), x^{n_k}(t) - u(t)]|$$

and owing to the assertions 7 and 8 we conclude that

$$[A^{n_k}(t)\, x^{n_k}(t), x^{n_k}(t)] \to [g_t, u(t)]. \tag{41}$$

From (3) we have

$$[A^{n_k}(t)\, v - A^{n_k}(t)\, x^{n_k}(t), v - x^{n_k}(t)] \geqslant 0 \tag{42}$$

for all $v \in V \cap H$.

From (36a) it follows $A^{n_k}(t)\, v \to A(t)\, v$ in $(V \cap H)'$ for $k \to \infty$. Since $x^{n_k}(t) \rightharpoonup u(t)$ in $V \cap H$ for $k \to \infty$ (see the proof of Assertion 9 ii)), we have

$$[A^{n_k}(t)\, v, v - x^{n_k}(t)] \to [A(t)\, v, v - u(t)]$$

and hence from (41) and (42) we conclude

$$[A(t)\, v - g_t, v - u(t)] \geqslant 0 \quad \text{for all} \quad v \in V \cap H.$$

We put $v = u(t) + \lambda w$, where $w \in V \cap H$, $\lambda > 0$. By the limiting process $\lambda \to 0$ we obtain

$$[A(t)\, u(t) - g_t, w] = 0 \quad \text{for all} \quad w \in V \cap H$$

and hence $A(t)\, u(t) \equiv g_t$. From this fact follows Assertion 10.

Assertion 11. $A(t)\,u(t)$ is weakly continuous in H with respect to $t \in \langle 0, T\rangle$.

Proof. Consider $[A(t_k)\,u(t_k), v]$, where $v \in V \cap H$ and $t_k \to t_0 \in \langle 0, T\rangle$. Owing to Assertion 10 and 5 i) by the limiting process in (30) we deduce that there exists $w_{t_k} \in H$ such that

$$[A(t_k)\,u(t_k), v] = -(w_{t_k}, v) + (f(t_k), v) \tag{43}$$

for all $v \in V \cap H$, where $\|w_{t_k}\| \leqslant C(u_0, f)$. From (43) we deduce

$$\|A(t_k)\,u(t_k)\| \leqslant C(u_0, f) \quad \text{for each } k.$$

and hence, there exist $g_{t_0} \in H$ and a subsequence

$$A(t_{k_n})\,u(t_{k_n})$$

such that

$$A(t_{k_n})\,u(t_{k_n}) \rightharpoonup g_{t_0} \quad \text{in } H \text{ for } \quad n \to \infty. \tag{44}$$

From (3) we have

$$[A(t_{k_n})\,v - A(t_{k_n})\,u(t_{k_n}), v - u(t_{k_n})] \geqslant 0$$

for all $v \in V \cap H$. Hence, from (44), (43) and the fact $A(t_{k_n})\,v \to A(t_0)\,v$ in $(V \cap H)'$ for $n \to \infty$ (because of (36a)) we conclude that $A(t_0)\,u(t_0) \equiv g_{t_0}$ by the same argument as in Assertion 10 equality $A(t)\,u(t) = g_t$ has been proved. From this fact there follows the required result.

Proof of the theorem.

Integrating (30) over $\langle 0, t\rangle$ we obtain

$$\int_0^t [A^n(s)\,x^n(s), v]\,\mathrm{d}s + (z^n(t), v) = \int_0^t (f^n(s), v)\,\mathrm{d}s + (u_0, v) \tag{45}$$

From Assertion 6 and 10 we have

$$[A^n(t)\,x^n(t), v] \to [A(t)\,u(t), v] \quad \text{for} \quad n \to \infty$$

and each $t \in \langle 0, T\rangle$, where $v \in V \cap H$ is fixed.

The estimate

$$|[A^n(t)\,x^n(t), v]| \leqslant C(u_0, f)\,\|v\| \quad \text{for all} \quad t \in \langle 0, T\rangle \tag{46}$$

holds because of Assertion 6. Hence from Assertion 8, (32) and Lebesgue's theorem by limiting process in (45) we conclude

$$\int_0^t [A(s)\,u(s), v]\,\mathrm{d}s + (u(t), v) = \int_0^t (f(s), v)\,\mathrm{d}s + (u_0, v) \tag{47}$$

from which we deduce that $u(t) \in C_w^1(\langle 0, T\rangle, H)$ because of Assertion 11 and (7). Thus, differentiating (47) with respect to $t \in \langle 0, T\rangle$ we conclude that $u(t)$ is a solution of the problem (1), since $u(0) = u_0$. Now, we prove the properties a)—h).

a)—c) are proved in Assertions 9, 10 and 11.

d) From (47) we deduce that

$$\left(\frac{\mathrm{d}\, u(t)}{\mathrm{d}\, t}, v\right) \in C(\langle 0, T\rangle) \quad \text{for each} \quad v \in V \cap H$$

because of c) and (7). Thus, $\frac{\mathrm{d}\, u(t)}{\mathrm{d}\, t}$ is weakly continuous in H with respect to $t \in \langle 0, T\rangle$ and hence $\frac{\mathrm{d}\, u(t)}{\mathrm{d}\, t}$ is weakly measurable. Analogously as in Assertion 9 iv) we prove that the set $\left\{\frac{\mathrm{d}\, u(t)}{\mathrm{d}\, t}; t \in \langle 0, T\rangle\right\}$ is separable in H and hence $\frac{\mathrm{d}u(t)}{\mathrm{d}\, t}$ is measurable. Due to Assertions 6 and 8 we estimate

$$\left(\frac{\mathrm{d}\, u(t)}{\mathrm{d}\, t}, v\right) = -\, [A(t)\, u(t), v] + (f(t), v) \leqslant C(u_0, f)\, \|v\|$$

for all $t \in \langle 0, T\rangle$ and $v \in V \cap H$.

$$\text{Hence,} \quad \left\|\frac{\mathrm{d}u(t)}{\mathrm{d}\, t}\right\| \leqslant C(u_0, f) \quad \text{for all} \quad t \in \langle 0, T\rangle$$

and the proof of d) is complete.

e) Due to Assertion 8 by a limiting process in (38) we obtain the reguired result.

f) This assertion is proved in (31).

g) From the proof of Assertion 9 ii) it follows $x^n(t) \rightharpoonup u(t)$ in $V \cap H$ for $n \to \infty$ and each $t \in \langle 0, T\rangle$. Analogously, with respect to the estimate $\|z^n(t)\|_V + \|z^n(t)\| \leqslant C(u_0, f)$ for each n and $t \in \langle 0, T\rangle$ (because of Assertion 1 ii) and Assertion 5 ii)), and (31) we prove $z^n(t) \rightharpoonup u(t)$ in $V \cap H$ for $n \to \infty$ and $t \in \langle 0, T\rangle$.

h) Owing to (3) we have

$$\frac{\mathrm{d}}{\mathrm{d}\, t}\, \|u_1(t) - u_2(t)\|^2 = 2\left(\frac{\mathrm{d}\, u_1(t)}{\mathrm{d}\, t} - \frac{\mathrm{d}\, u_2(t)}{\mathrm{d}\, t}, u_1(t) - u_2(t)\right) =$$

$$= 2(f_1(t) - f_2(t), u_1(t) - u_2(t)) -$$

$$- 2[A(t)\, u_1(t) - A(t)\, u_2(t), u_1(t) - u_2(t)] \leqslant$$
$$\leqslant 2\|f_1(t) - f_2(t)\|\, \|u_1(t) - u_2(t)\|.$$

Integrating this inequality over $\langle 0, t\rangle$ we deduce

$$\|u_1(t) - u_2(t)\|^2 \leqslant \|u_1(0) - u_2(0)\|^2 +$$
$$+ 2 \max_{<0, T>} \|u_1(s) - u_2(s)\| \cdot \int_0^T \|f_1(s) - f_2(s)\|\, \mathrm{d}s.$$

From this inequality we obtain

$$\max_{<0, T>} \|u_1(t) - u_2(t)\| \leqslant \|u_{01} - u_{02}\| + 2 \int_0^T \|f_1(s) - f_2(s)\|\, \mathrm{d}s$$

since $u_1(0) = u_{01}$ and $u_2(0) = u_{02}$.

From Assertion h) the uniqueness for the solution of (1) follows. Thus, the proof of Theorem is complete.

Remark 5. Let $u(t)$ be a solution of the problem (1).

Let be $t_0 \in \langle 0, T\rangle$ a fixed point. Consider the problem

$$(1') \qquad \frac{\mathrm{d}\, u_1(t)}{\mathrm{d}\, t} + A(t)\, u_1(t) = f(t) \quad \text{for} \quad t \in \langle 0, T\rangle,\ u_1(t_0) = u(t_0).$$

Since $u(t_0) \in V \cap H$ and $A(t_0)\, u(t_0) \in H$, from Theorem we conclude that there exists a unique solution $u_1(t)$ of (1′). But, $u(t)$ is also a solution of (1′) and thus $u(t) = u_1(t)$ for $t \in \langle t_0, T\rangle$. On the base of this fact transition operators $U_{t_0}(t) : U_{t_0}(t)\, u(t_0) = u(t) \quad t \geq t_0$
are defined and the identities

$$U_{t_0}(t + s) \equiv U_s\,(t + s)\, U_{t_0}(s) \equiv U_t\,(t + s)\, U_{t_0}\,(t), \quad U_{t_0}\,(t_0) \equiv I$$

(I is identity mapping and t, $s \geq t_0$) are valid.

If $f_1(t) = f_2(t) = 0$, then from (48) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \|u_1(t) - u_2(t)\|^2 \leqslant 0.$$

It means that $U_{t_0}(t)$ is a nonexpansive operator on its definition set $D(U_{t_0}) = \{u \in H \cap V;\ A(t_0)\, u \in H\}$

Remark 6. If $A(t) \equiv A$, the Theorem holds true without the assumption (5). Indeed, in this case we deduce easily from (3) the estimate

$$\left\| \frac{z_j - z_{j-1}}{h} \right\| \leqslant C(u_0, f)$$

— see [1] (part I). The more general result in this case ($A(t) \equiv A$) is proved by J. Nečas in [2].

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Received November 14, 1973

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