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ZEROS OF THE POLYNOMIAL SOLUTIONS OF THE DIFFERENTIAL EQUATION $xy'' + (\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma - n\beta_2 x)y = 0$.

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At the conference on differential equations at Dundee in March 1972 (the proceedings of which are shortly to be published by Springer) Arscott showed that polynomial solutions of the above equation have valuable bi-orthogonal properties. The purpose of this note is to investigate the zeros of these polynomials. The main result of this paper is Theorem 3, which is an analogy with Stieltjes' theorem on the zeros of Lamé polynomials [2].

Consider the differential equation

$$xy'' + (\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma - n\beta_2 x)y = 0, \tag{1}$$

where n is a positive integer and $\beta_0 > 0$, β_1, $\beta_2 < 0$, γ are real.

Theorem 1. *For the numbers β_0, β_1, β_2, n restricted as above, there exist $n + 1$ real numbers $\gamma_0 < \gamma_1 < \ldots < \gamma_n$ so that the differential equation*

$$xy'' + (\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_i - n\beta_2 x)y = 0, \tag{2}$$

has a polynomial solution of the degree n, for $i = 0, 1, \ldots, n$.

Proof. Let $y(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial solution of (1). Then $y'(x) = \sum_{i=1}^{n} i a_i x^{i-1}$, $y''(x) = \sum_{i=2}^{n} i(i-1)a_i x^{i-2}$. On substituting $y(x)$, $y'(x)$, $y''(x)$ into equation (1), we obtain the following system of equations for the coefficients a_i, $i = 0, 1, \ldots, n$.

$$\begin{aligned}
\gamma a_0 + \beta_0 a_1 &= 0,\\
\beta_2 n a_0 + (\gamma + \beta_1)a_1 + 2(\beta_0 + 1)a_2 &= 0,\\
-\beta_2(n-1)a_1 + (\gamma + 2\beta_1)a_2 + 3(\beta_0 + 2)a_3 &= 0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&
\end{aligned}$$

(3)

$$-\beta_2(n-k+2)a_{k-2} + [\gamma + (k-1)\beta_1]a_{k-1} + k(\beta_0 + k - 1)a_k = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\beta_2 2a_{n-2} + [\gamma + (n-1)\beta_1]a_{n-1} + n(\beta_0 + n - 1)a_n = 0,$$

$$-\beta_2 a_{n-1} + \quad (\gamma + n\beta_1)a_n = 0.$$

In order that the system (3) may have a nontrivial solution it is necessary that

$$(4) \quad D_{n+1}(\gamma) = \begin{vmatrix} \gamma & 1\beta_0 & & & & \\ -\beta_2 n & \gamma + \beta_1 & 2(\beta_0 + 1) & & & \\ & -\beta_2(n-1) & \gamma + 2\beta_1 & 3(\beta_0 + 2) & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & -\beta_2(n-k+2) & \gamma + (k-1)\beta_1 & k(\beta_0 + k - 1) & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & -\beta_2 2 & \gamma + (n-1)\beta_1 & n(\beta_0 + n - 1) & \\ & & & -\beta_2 & \gamma + n\beta_1 & \end{vmatrix} = 0.$$

Denote by $D_{k.n+1}$ the k-th order determinant obtained from the first rows and the first columns of $D_{n+1}(\gamma)$. We can easily show that

$$(5) \qquad D_{k+1,n+1} = (\gamma + k\beta_1)D_{k,n+1} + \beta_2(n-k+1)k(\beta_0 + k - 1)D_{k\ 1.n+1}$$

for $k = 2, 3, \ldots, n$.

The following properties of $D_{k,n+1}$ were proved by Arscott [1]. (i) the zeros of D_{n+1} and $D_{n,n+1}$ interlace, (ii) at consecutive zeros of D_{n+1} the values of $D_{n,}$ $_{n+1}$ are alternately positive and negative, (iii) all the zeros of D_{n+1} are real and distinct.

If a_n or a_0 is equal to zero, it follows from (3) ($\beta_0 \neq 0$, $\beta_2 \neq 0$) that $a_i = 0$ for all i, i. e. $y(x) \equiv 0$. From this remark and (iii) it follows that for every positive integer n there exist $n + 1$ numbers $\gamma_i^{(n)}$, $i = 0, 1, \ldots, n$ so that for each $\gamma_i^{(n)}$ the system (3) has a nontrivial solution $(a_0^{(i)}, a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)})$, where $a_n^{(i)} \neq 0$, $a_0^{(i)} \neq 0$. This means that the differential equation (2) has $n + 1$ polynomial solutions of the degree n associated with the different numbers $\gamma_0^{(n)} < \gamma_1^{(n)} < \ldots < \gamma_n^{(n)}$. The polynomial solution of equation (2) associated with $\gamma_i^{(n)}$ will be denoted by $y_n^{(i)}(x)$. ($y_n^{(i)}(x)$ doees not mean the i-th order derivative of $y_n(x)$.)

Remark. Every solution $y_n^{(i)}(x)$ depends on the coefficient a_0. Put $a_0 = 1$. Then the solutions $y_n^{(i)}(x)$ satisfy the condition $y_n^{(i)}(0) = 1$. From now on, we shall assume that every polynomial solution $y_n^{(i)}(x)$ has that property.

Lemma 1. *Let $y_n^{(i)}(x)$ be a polynomial solution of equation* (2) *with the property that $y_n^{(i)}(0) = 1$, i. e. $y_n^{(i)}(x) = 1 + a_1^{(i)}x + a_2^{(i)}x^2 + \ldots + a_n^{(i)}x^n$. Then the sign of $a_n^{(i)}$ is $(-1)^i$.*

Proof. The solution $y_n^{(i)}(x)$ is associated with the root $\gamma_i^{(n)}$ of the equation $D_{n+1}(\gamma) = 0$. From (ii) it follows that the sign of $D_{n,n+1}(\gamma_i^{(n)})$ is $(-1)^{n+i}$. Let us carry over the first columm of (3) to the right-hand side and leave out the last row, we thus obtain a nohomogeneous system of equatios (for $a_0 \neq 0$) with the unknows $a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)}$ (associated with $\gamma_i^{(n)}$). By Cramer's rule we obtain

$$a_n^{(i)} = \frac{1}{n!\prod_{i=0}^{n-1}(\beta_0 + i)}(-1)^n D_n(\gamma_i^{(n)}),$$

thus the sign $a_n^{(i)}$ is equal to $(-1)^n$. sign $D_n(\gamma_i^{(n)}) = (-1)^n \cdot (-1)^{n+i} = (-1)^i$. The lemma is thus proved.

Theorem 2. *Let $u(x) = y_n^{(i)}(x)$, $v(x) = y_m^{(j)}(x)$ be two polynomial solutions of* (1). *Let $m \geqq n$ and $\gamma_j^{(m)} \geqq \gamma_i^{(n)}$. Then if the number a is the first zero of $u(x)$ in $(0, \infty)$, $v(x)$ vanishes at some point of $(0, a\rangle$.*

Proof. The differential equation (1) can be transformed in $(-\infty, 0) \cup \cup (0, \infty)$ by the substitution

$$w(x) = y(x)\sqrt{|x|^{\beta_0}e^{\beta_1 x + \frac{1}{2}\beta_2 x^2}}, \tag{6}$$

to the form

$$w''(x) + \left[-n\beta_2 + \frac{\gamma}{x} + f(x)\right] w = 0, \tag{7}$$

where

$$f(x) = -\frac{\beta_2}{4}x^2 - \frac{\beta_1\beta_2}{2}x - \frac{1}{4}(\beta_1^2 + 2\beta_2 + 2\beta_0\beta_2) - \frac{1}{2x}\beta_0\beta_1 + \frac{1}{4x^2}(2\beta_0 - \beta_0^2).$$

From the relation (6) it follows that the zeros of a solution of equation (1) are the same as the zeros of the corresponding solution of equation (7) except possibly for the point $x = 0$.

Let $u(x)$, $v(x)$ be a polynomial solution of (1). Then the function

$$z(x) = u(x)\sqrt{x^{\beta_0}e^{\beta_1 x + \frac{1}{2}\beta_2 x^2}}$$

satisfies the equation

(8) $$z''(x) + \left[-n\beta_2 + \frac{\gamma_i}{x} + f(x)\right] z(x) = 0$$

and the function

$$w(x) = v(x) \sqrt{x^{\beta_0} e^{\beta_1 x + \frac{1}{2}\beta_2 x^2}}$$

satisfies the equation

(9) $$w''(x) + \left[-m\beta_2 + \frac{\gamma_j}{x} + f(x)\right] w(x) = 0.$$

Now suppose that Theorem 2 is false, i. e. that $v(x) > 0$ in $(0, a\rangle$ $(u(0) = v(0) = 1)$. Then $z(x) > 0$ and $w(x) > 0$ in $(0, a)$. Multiplication of (8) by $w(x)$ and (9) by $z(x)$, subtraction of the resulting equations and integration over $(0, a)$ yields

$$\int_0^a [z'(x)w(x) - w'(x)z(x)]'\mathrm{d}x = \int_0^a \left[-\beta_2(m - n) + \frac{1}{x}(\gamma_j - \gamma_i)\right] w(x)z(x)\mathrm{d}x.$$

At the moment, we do not know that these integrals exist (since the integrands are not continuous functions at $x = 0$). But after making substitution (6) we obtain

$$\int_0^a [x^{\beta_0} e^{\beta_1 x + \frac{1}{2}\beta_2 x^2} \; u'(x)v(x) - v'(x)u(x)]'\mathrm{d}x =$$

$$= \int_0^a \left[-\beta_2(m - n) + (\gamma_j - \gamma_i)\frac{1}{x}\right] u(x)v(x)x^{\beta_0}e^{\beta_1 x + \frac{1}{2}\beta_2 x^2}\mathrm{d}x.$$

Since $\beta_0 > 0$, both integrals exist and the integrand on the right-hand side is nonnegative by hypothesis. Therefore the integral on the left-hand side is nonnegative as well, i. e.

(10) $$\{x^{\beta_0} e^{\beta_1 x + \frac{1}{2}\beta_2 x^2} \; [u'(x)v(x) - u(x)v'(x)]\}_{x=0}^{x=a} \geqq 0.$$

But since $u'(a) < 0$, $v(a) > 0$ for $a > 0$, the left-hand side of (10) is negative; we thus have a contradiction and so the theorem is proved.

Similary, we can prove the following theorem.

Theorem 2'. *Let $u(x) = y_n^{(i)}(x)$, $v(x) = y_m^{(j)}(x)$ be two polynomial solutions of (1). Let $m \geqq n$ and $\gamma_j^{(m)} \leqq \gamma_i^{(n)}$. Then if the number $a < 0$ is the last zero of $u(x)$ in $(-\infty, 0)$ (that is to say, the value of a is the greatest of all zeros in $(-\infty, 0)$), $v(x)$ vanishes at some point of $\langle a, 0)$.*

Theorem 3. *Let $y_n^{(i)}(x)$, $n = 1, 2, \ldots, n$ be a polynomial solution of equation*

(2) *corresponding to* $\gamma_i^{(n)}$. *Then every such solution has n zeros in* $(-\infty, \infty.)$; *i zeros lying in* $(0, \infty)$*and* $n-1$ *in* $(-\infty, 0)$.

Proof. From Theorem 1 it follows that every solution $y_n^{(i)}(x)$ is of the degree n, and so has at most n zeros in $(-\infty, \infty)$. Consequently it is sufficient to prove that every such solution has at least n zeros situated as stated in the Theorem. We recall that every solution $y_n^{(i)}(x)$ has the following properties

1. $y_n^{(i)}(0) = 1$.
2. $y_n^{(i)}(x) = 1 + a_1^{(i)}x + \ldots + (-1)^i \mid a_n^{(i)} \mid x^n$.

The proof is by induction on the nubmer n.

Let $n = 1$. The equation $D_2(\gamma) = 0$ has two roots $\gamma_0^{(1)} < 0$ and $\gamma_1^{(1)} > 0$. From (3) it follows that the solution $y_1^{(0)}(x)$ has a zero in $(-\infty, 0)$ and $y_1^{(1)}(x)$ has a zero in $(0, \infty)$. Consequently Theorem 3 is valid for $n = 1$.

Assume that Theorem 3 is valid for the number $n = k$, $i = 0, 1, \ldots, k$, i. e. the solution $y_k^{(i)}(x)$ has i zeros in $(0, \infty)$ and $k - i$ zeros in $(-\infty, 0)$. We shall deduce that the theorem is valid for $n = k + 1$.

At the begining we propose the solution $y_{k+1}^{(i)}(x)$, where $i \neq 0$, $i \neq k + 1$. We divide the proof into two parts a) and b). In part a) we shall show that $y_{k+1}^{(i)}(x)$ has $k + 1 - i$ zeros in $(-\infty, 0)$, and in part b) we shall show that function $y_{k+1}^{(i)}(x)$ has i zeros in $(0, \infty)$.

a). Consider the solution $y_k^{(i)}(x)$ of (1) associated with the number $\gamma_i^{(k)}$. From Theorem 2' it follows that the last zero of $y_{k+1}^{(i)}(x)$ in $(-\infty, 0)$ is greater than the last zero of $y_k^{(i)}(x)$. By the inductive hypothesis, the solution $y_k^{(i)}(x)$ has $k - i$ zeros in $(-\infty, 0)$. Using the Sturm comparison theorem we obtain that $y_{k+1}^{(i)}(x)$ has at least $k - i$ zeros in $(-\infty, 0)$.

Since

$$\lim_{x\to-\infty} y_k^{(i)}(x) = \lim_{x\to-\infty} (-1)^i x^k \neq \lim_{x\to-\infty} (-1)^i x^{k+1} = \lim_{x\to-\infty} y_{k+1}^{(i)}(x),$$

$y_{k+1}^{(i)}(x)$ must have another zero in $(-\infty, 0)$. Consequently $y_{k+1}^{(i)}(x)$ has at least $k + 1 - i$ zeros in $(-\infty, 0)$.

b). In this part we consider the solution $y_k^{(i-1)}(x)$. This solution has $i - 1$ zeros in $(0, \infty)$, by hypothesis.

If $i - 1 = 0$, i. e. $y_k^{(i-1)}(x)$ has no zeros in $(0, \infty)$, then $y_{k+1}^{(i)}(x)$ has at least one zero in $(0, \infty)$, because $y_{k+1}^{(i)}(0) = 1$ and

$$\lim_{x\to\infty} y_{k+1}^{(i)}(x) = \lim_{x\to\infty} (-1)^1 x^{k+1} = -\infty.$$

Let $i > 1$. Then from Theorem 2 it follows that the solution $y_{k+1}^{(i)}(x)$ has its first zero in $(0, \infty)$ before the first zero of $y_k^{(i-1)}(x)$. From Sturm's comparison theorem it follows that $y_{k+1}^{(i)}(x)$ has $i - 2$ other zeros in $(0, \infty)$, and so $y_{k+1}^{(i)}(x)$ has at least $i - 1$ zeros in $(0, \infty)$.

Because

$$\lim_{x\to\infty} y_k^{(i-1)}(x) = \lim_{x\to\infty} (-1)^{i-1}x^k \neq \lim_{x\to\infty} (-1)^i x^{k+1} = \lim_{x\to\infty} y_{k+1}^{(i)}(x),$$

the solution $y_{k+1}^{(i)}(x)$ has another zero in $(0, \infty)$. Consequently $y_{k+1}^{(i)}(x)$ has at least i zeros in $(0, \infty)$.

From a) and b) it follows that $y_{k+1}^{(i)}(x)$ has precisely $k + 1 - i$ zeros in $(-\infty, 0)$ and i zeros in $(0, \infty)$, where $i \neq 0$, $i \neq k + 1$.

We can see that part a) is valid for $i = 0$ as well, i. e. the solution $y_{k+1}^{(0)}(x)$ has at least $k + 1$ zeros in $(-\infty, 0)$, i. e. precisely $k + 1$ zeros in $(-\infty, 0)$ and no zeros in $(0, \infty)$. Similary, part b) is valid for $i = k + 1$, i. e. the solution $y_{k+1}^{(k+1)}(x)$ has precisely $k + 1$ zeros in $(0, \infty)$ and no zeros in $(-\infty, 0)$. Theorem 3 is thus proved completely.

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