Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 46 (1996), No. 3, 501–512

Persistent URL: `http://dml.cz/dmlcz/127311`

Czechoslovak Mathematical Journal, 46 (121) 1996, Praha

FINITE-VALUED SUBGROUPS OF LATTICE-ORDERED GROUPS

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(Received August 15, 1994)

0. Introduction

A lattice-ordered group, written ℓ-group, is a partially ordered group $(G, \leqslant)$ where the partial order is a lattice (meaning that each pair of elements a, b of G has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$). An ℓ-subgroup A of an ℓ-group G is both a subgroup and a sublattice of G. A is a convex ℓ-subgroup of G, if $a, b \in A$ and $a < g < b$ imply that $g \in A$. A convex ℓ-subgroup P of G is prime if $a \wedge b = 0$ in G implies that either $a \in P$ or $b \in P$. A convex ℓ-subgroup which is maximal with respect to not containing some $g \in G$ is called regular and is a value of g. Element g is special if it has a unique value. Regular subgroups of G form a root system under conclusion, written $\Gamma(G)$ (i.e. $\Gamma(G)$ is a partially ordered set for which $\{\alpha \in \Gamma(G) \mid \alpha \geqslant \gamma\}$ is totally ordered, for any $\gamma \in \Gamma(G)$.) A subset $\Delta \subseteq \Gamma(G)$ is plenary if $\bigcap \Delta = \{0\}$ and Δ is a dual ideal in $\Gamma(G)$; that is, if $\delta \in \Delta$, $\gamma \in \Gamma(G)$ and $\gamma > \delta$, then $\gamma \in \Delta$. If G is an abelian ℓ-group, then G is ℓ-isomorphic to an ℓ-subgroup of $V(\Gamma(G), R)$ such that if γ is a value of $g \in G$, then γ is a maximal component of g after the embedding, where $V(\Gamma(G), R)$ is an abelian ℓ-group of all functions v on $\Gamma(G)$ for which $v(\gamma) \in R$ and the support of each v satisfies ascending chain condition. This is the result of the Conrad-Harvey-Holland embedding theorem for abelian lattice-ordered groups. Actually, for any abelian ℓ-group G, there exists such an embedding of G into $V(\Delta, R)$, where Δ is any plenary subset of $\Gamma(G)$.

$\Sigma(\Delta, R)$ is an ℓ-subgroup of $V(\Delta, R)$ containing all elements $v \in V$ with finite supports. $F(\Delta, R)$ is an ℓ-subgroup of $V(\Delta, R)$ containing all elements $v \in V$ whose supports are contained in a finited number of roots in Δ.

For any $g \in G$, $G(g) = \{h \in G \mid |h| \leqslant n|g|$, for some positive integer $n\}$ the principal convex ℓ-subgroup of G generated by g is the least convex ℓ-subgroup of G that contains g.

An element b of G is basic if the set $\{g \in G \mid 0 < g \leqslant b\}$ is totally-ordered. An ℓ-group G has a basis if G possesses a maximal pairwise disjoint set of elements g_λ, and in addition, each $G(g_\lambda)$ is a totally ordered-group.

An ℓ-group is laterally complete (conditionally laterally complete) if for any subset (bounded subset) $\{g_\alpha \mid \alpha \in A\}$ of disjoint positive elements, $\bigvee_A g_\alpha$ exists.

An ℓ-group G is finite-valued if every element of G has only a finite number of values; this is equivalent to that every element of G can be expressed as a finite sum of disjoint special elements. Each element of G is also called finite-valued. An ℓ-group G is special-valued if G has a plenary subset of special values; this is equivalent to that each positive element of G can be expressed as the join of a set of pairwise disjoint positive special elements. A positive element g of G is special-valued if g can be expressed as the join of disjoint special elements.

An ℓ-group is archimedean if for any elements g and h, $ng \leqslant h$ for all positive integers n implies that $g \leqslant 0$. Two positive elements g and h are a-equivalent if there exists a positive integer n so that $g \leqslant nh$ and $h \leqslant ng$. If G is an ℓ-subgroup of H, and for each $h \in H^+$, there exists $g \in G^+$ so that h and g are a-equivalent, then we say that H is an a-extension of G. H is a-closed if H admits no a-extensions. H is an a-closure of G, if H is an a-closed a-extension of G.

A torsion class is a class of lattice-ordered groups that is closed under convex ℓ-subgroups, ℓ-homomorphic images, and joins of convex ℓ-subgroups. For an ℓ-group G and a torsion class T, $T(G)$ indicates the join of all the convex ℓ-subgroups of G that belong to T. $T(G)$ is then the largest convex ℓ-subgroup of G that belongs to T, called the torsion radical of G. A quasi-torsion class is a class of ℓ-groups which is closed under convex ℓ-subgroups, complete ℓ-homomorphic images, and joins of convex ℓ-subgroups. Finite-valued ℓ-groups form a torsion class F_v, and special-valued ℓ-groups form a quasi-torsion classes S.

1. Maximal finite-valued subgroups

Definition. A finite-valued subgroup of an ℓ-group G is an ℓ-subgroup U such that each $g \in U$ is finite-valued in G.

An ℓ-subgroup of G that is finite-valued as an ℓ-group may not be a finite-valued subgroup of G. For example, if $G = \prod_{i=1}^{\infty} R_i$, then the subgroup $[(1,1,1,\dots)]$ generated by (1,1,1, ...) is an ℓ-subgroup of G that is finite valued as an ℓ-group but is not a finite-valued subgroup of G.

Let U be a finite-valued subgroup of G, then each ℓ-subgroup of U is a finite-valued subgroup of G. Moreover, U is finite-valued as an ℓ-group. For let P be a value of

$0 < u \in U$. Then there exists a value Q of u in G such that $Q \cap U = P$ [8]. Since u has only a finite number of values in G, it has only a finite number of values in U.

If $\ldots \subseteq C_\alpha \subseteq C_\beta \subseteq \ldots$ is a chain of finite-valued subgroups of G, then $\bigcup C_\alpha$ is a finite-valued subgroup of G. So each finite-valued subgroup U is contained in a maximal finite-valued subgroup of G.

If W is an a-extension of a finite-valued subgroup U of G, then W is a finite-valued subgroup of G. Thus each maximal finite-valued subgroup of G is a-closed in G. For if $0 < w \in W$, then there exists $0 < u \in U$, such that $nw > u$ and $nu > w$ for some $n > 0$. In particular, w and u have the same values in G, so w is finite-valued in G. If α is an ℓ-automorphism of G, then $U\alpha$ is a finite-valued subgroup of G, and if U is maximal, then so is $U\alpha$. In fact, $g \in G$ is finite-valued if and only if $g\alpha$ is finite-valued. Thus, of course, $F_v(G)\alpha = F_v(G)$, where $F_v(G)$ is the finite-valued torsion radical for G.

Proposition 1.1. *If U is a finite-valued subgroup of G, then so is $U + F_v(G)$, where $F_v(G)$ is the torsion radical of G for the torsion class of finite-valued ℓ-groups. Thus if U is a maximal finite-valued subgroup of G, then $U \supseteq F_v(G)$.*

Proof. $U + F_v(G)$ is an ℓ-subgroup of G, since U is an ℓ-subgroup and $F_v(G)$ is an ℓ-ideal. Now consider $0 < g = a + b \in U + F_v(G)$, where $a \in U$ and $b \in F_v(G)$. We have $g + F_v(G) = a + F_v(G)$, so without loss of generality, we may assume that $a > 0$.

$$0 < g = a + b = (a_1 \vee a_2 \vee \ldots \vee a_n) + b$$
$$= a_1 \vee a_2 \vee \ldots \vee a_k \vee a_{k+1} \vee a_{k+2} \vee \ldots \vee a_n + b$$

where $0 < a_i$ are disjoint and special. $a_i \notin F_v(G)$, for $i = 1, \ldots, k$, and $a_i \in F_v(G)$, for $i = k+1, \ldots, n$.

Now $a_{k+1} \vee a_{k+2} \vee \ldots \vee a_n + b = b_1 + b_2 + \ldots + b_m$, where b_i are special and $|b_i| \wedge |b_j| = 0$.

Now we use the fact that the sum of two positive finite-valued elements is finite-valued. If each $b_i > 0$, then g is finite-valued. Suppose that $b_1 < 0$, then since g is positive, $|b_i| \ll a_j$, for a unique j, so

$$0 < g = a_1 \vee a_2 \vee \ldots \vee a_j + b_1 \vee \ldots \vee a_k + b_2 + b_3 + \ldots + b_m.$$

Continue this process until the remaining b_i are positive. But then g is the sum of two positive finite-valued elements. □

Theorem 1.2. *$F_v(G)$ is the intersection of all maximal finite-valued subgroups of G.*

Proof. By the last proposition, $F_v(G)$ is contained in the intersection of all maximal finite-valued subgroups of G. We will show that for each $0 < a \in G \setminus F_v(G)$, there exists a maximal finite-valued subgroup that does not contain a. If a has an infinite number of values, then a does not belong to any finite-valued subgroup. Now suppose a is finite-valued, then

$$a = a_1 \vee a_2 \vee \ldots \vee a_n$$

where $a_i > 0$ are disjoint and special.

Without loss of generality, we assume that $a_1 \notin F_v(G)$, so $a_1 \gg b > 0$, where b is infinite-valued; thus the ℓ-subgroup of G generated by $a + b$ is a finite-valued subgroup that contains $a + b$ but not a. Each maximal finite-valued subgroup that contains $a + b$ does not contain a. □

Corollary 1.3. *C is the largest finite-valued subgroup of G if and only if $C = F_v(G) =$ all the finite-valued elements of G.*

Corollary 1.4. *For an ℓ-group G, the following are equivalent.*

(1) *There exists a largest finite-valued subgroup of G.*

(2) *$F_v(G)$ consists of all the finite-valued elements of G.*

(3) *If $0 < a < b$, and b is special, then a is finite-valued.*

(4) *If b is special, then each regular subgroup of $G(b)$ is special.*

(5) *$F_v(G)$ contains all the special elements of G.*

Proof. By Corollary 1.3, (1) $\longleftrightarrow$ (2).

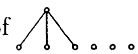
Clearly (2) $\longrightarrow$ (3) $\longrightarrow$ (5) $\longrightarrow$ (2).

By Theorem 2.2 [8], (4) holds if and only if each $G(b)$ with b special is finite-valued, so (4) if and only if (2). □

The set of all convex ℓ-subgroups of G is denoted $\mathscr{C}(G)$. $\mathscr{C}(G)$ forms a distributive lattice where the meet operation is the intersection and the join operation is the join as subgroups of G.

Note that $F_v(G) = \bigcup\{G(b) \mid \text{each regular subgroup of } G(b) \text{ is special}\}$, and so is an invariant of the lattice $\mathscr{C}(G)$. Hence $\mathscr{C}(G)$ determines whether or not G has a largest finite-valued subgroup.

Suppose G is a special-valued ℓ-group, and let Δ be the plenary set of special values of $\Gamma(G)$. We consider the following properties of G.

(a) Δ contains no copies of 

(b) $G(g)$ has a finite basis for each special element $g \in G$.

(c) $F_v(G)$ consists of all the finite-valued elements in G, so $F_v(G) = F(G)$, where F is the torsion class of all ℓ-groups such that $G(g)$ has a finite basis for each $g \in G$.

(d) $F_v(G)$ is the largest finite-valued subgroup of G.

(e) There exists a largest finite-valued subgroup of G.

(f) $F_v(G)$ consists of all the finite-valued elements of G.

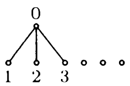
Proposition 1.5. (a) $\longleftrightarrow$ (b) $\longleftrightarrow$ (c) $\longrightarrow$ (d) $\longrightarrow$ (e) $\longleftrightarrow$ (f), *and if G is conditionally laterally complete, then* (e) $\longrightarrow$ (a).

Proof. Clearly (a) $\longleftrightarrow$ (b) and (c) $\longrightarrow$ (d) $\longrightarrow$ (e) $\longleftrightarrow$ (f).

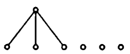
(b) $\longrightarrow$ (c) If $0 < g \in G$ is finite-valued, then $g = g_1 \vee g_2 \vee \ldots \vee g_n$, where g_i are disjoint and special. Each $G(g) \in F$, so $G(g_i) \subseteq F(G)$, and hence $G(g) \subseteq F(G)$.

(c) $\longrightarrow$ (b) If g is special, then $g \in F(G)$. So $G(g) \subseteq F(G)$, and hence $G(g)$ has only a finite number of roots.

Now suppose that G is conditionally laterally complete, and Δ contains a copy of

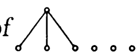


Let $g_i > 0$ be special with value i, and let $g = \bigvee_{i=1}^{\infty} g_i$, then $g_0 > g$, which contradicts (3) of the above corollary. So (e) is false. Therefore we have (e) $\longrightarrow$ (a). $\square$

In general, (e) $\longrightarrow$ (a) is not true. For example, if $G = \Sigma(\Delta, R)$, and Δ contains a copy of 
then G satisfies (f) but not (a).

Now we consider ℓ-groups $\Sigma(\Delta, R)$ and $F(\Delta, R)$. They are both finite-valued subgroups of $V(\Delta, R)$.

Corollary 1.6. *The following are equivalent.*

(1) *Δ contains no copy of* 

(2) *The principal convex ℓ-subgroup $V(v)$ of $V(\Delta, R)$ has a finite basis for each special element $v \in V(\Delta, R)$.*

(3) *$F(V)$ consists of all the finite-valued elements in $V(\Delta, R)$.*

(4) *$F(V)$ is the largest finite-valued subgroup of $V(\Delta, R)$.*

(5) *There exists a largest finite-valued subgroup of $V(\Delta, R)$.*

(6) *$F_v(V)$ consists of all the finite-valued elements in $V(\Delta, R)$.*

Let U be a finite-valued subgroup of G. If $0 < u \in U$, then $u = u_1 + u_2 + \ldots + u_n$, where u_i are disjoint and special in G. We say that U is saturated if each $u_i \in U$.

Theorem 1.7. *Each maximal finite-valued subgroup of an ℓ-group G is saturated.*

Proof. Let A be a finite-valued ℓ-subgroup of G; let $g \in A$ and x be a component of g. Let B be the ℓ-subgroup of G generated by A and x, we will show that B is finite-valued.

Let $h \in B$, then $h = \bigvee_I \wedge_J w_{ij}$, where I and J are finite sets and w_{ij} is in the subgroup of G generated by A and x. Let M be a value of h in G. Then $M + h = M + (\bigvee_I \wedge_J w_{ij}) = \bigvee_I \wedge_J (M + w_{ij})$ and so there exists $(i, j) \in I \times J$ such that $M + h = M + w_{ij}$. Thus M is also a value of w_{ij}. Thus if each w_{ij} can be shown to be finite-valued, then the values of h are in the union of the sets of values of the w_{ij}'s, and this union is necessarily finite.

So let M be a value of w_{ij}. Now w_{ij} can be written in the form $(\varepsilon_1 x) + a_1 + (\varepsilon_2 x) + a_2 + \ldots + (\varepsilon_{n+1} x)$, where ε_i can be $+$ or $-$, and $a_i \in A$. Define u_0 to be 0 and u_i to be equal to $(\varepsilon_1 x) + a_1 + (\varepsilon_2 x) + a_2 + \ldots + (\varepsilon_i x) + a_i$. For $0 \leqslant i \leqslant n$, define $b_{i+1} \in A$ by

$$b_{i+1} = \begin{cases} 0, & \text{if } x \in -u_i + M + u_i; \\ g, & \text{if } x \notin -u_i + M + u_i. \end{cases}$$

Thus if $x \in M$, then $M + (\varepsilon_1 x) = M + 0 = M + (\varepsilon_1 b_1)$, while if $x \notin M$, then $g - x \in M$, and so $M + (\varepsilon_1 x) = M + (\varepsilon_1 g) = M + (\varepsilon_1 b_1)$. So in either cases, $M + (\varepsilon_1 x) + a_1 = M + (\varepsilon_1 b_1) + a_1$. Likewise, the choice of b_2 guarantees that $M + (\varepsilon_1 b_1) + a_1 + (\varepsilon_2 x) = M + (\varepsilon_1 b_1) + a_1 + (\varepsilon_2 b_2)$ and so $M + (\varepsilon_1 x) + a_1 + (\varepsilon_2 x) = M + (\varepsilon_1 b_1) + a_1 + (\varepsilon_2 b_2)$. Continuing, we see that $M + w_{ij} = M + (\varepsilon_1 b_1) + a_1 + \ldots + (\varepsilon_n b_n) + a_n + (\varepsilon_{n+1} b_{n+1})$. Thus M is a value of $(\varepsilon_1 b_1) + a_1 + \ldots + (\varepsilon_n b_n) + a_n + (\varepsilon_{n+1} b_{n+1})$.

Now $(\varepsilon_1 b_1) + a_1 + \ldots + (\varepsilon_n b_n) + a_n + (\varepsilon_{n+1} b_{n+1}) \in A$ and so has only finitely many values. Since this element of A was constructed from w_{ij} by choosing either 0 or g for each occurrence of x in the representation of w_{ij}, the values of w_{ij} are a subset of the values of the at most 2^n possible elements formed from w_{ij} in this fashion. Since all of these have only finitely many values, w_{ij} can have only finitely values as well. □

2. Finite-valued subgroups of an abelian ℓ-group

Proposition 2.1. *For a finite-valued subgroup U of an abelian ℓ-group G, the following are equivalent.*

(1) U is a maximal finite-valued subgroup of G.

(2) U is a-closed in G, and for each special value δ in $\Gamma(G)$, there exists a special element $u \in U$ with value δ.

Proof. ($1 \longrightarrow 2$) Suppose there is no special element in U with value δ. Then since U is saturated, there is no element in U with value δ. Since G is abelian, we can embed G into $V(\Gamma(G), R)$ so that the characteristic function χ_δ on δ belongs to G. We will show that $U \oplus [\chi_\delta]$ is a finite-valued subgroup of G, but this contradicts the maximality of U.

Clearly $U \oplus [\chi_\delta]$ is a subgroup of G. Consider $u + n\chi_\delta$, and we need to show that $(u + n\chi_\delta) \vee 0 \in U \oplus [\chi_\delta]$. This is clear if u has a value greater than δ, or if $|u| \wedge |\chi_\delta| = 0$. Now suppose u has values that are less than δ, we have $u = u_1 + u_2 + \ldots + u_n$, with $|u_i|$ disjoint and special. Suppose that $|u_1 + \ldots + u_m| < |\chi_\delta|$, and $|u_{m+1} + \ldots + u_n| \wedge |\chi_\delta| = 0$.

If $n < 0$, then $(u + n\chi_\delta) \vee 0 = (u_{m+1} + \ldots + u_n) \vee 0$, and

If $n > 0$, then $(u + n\chi_\delta) \vee 0 = u_1 + \ldots + u_m + n\chi_\delta + (u_{m+1} + \ldots + u_n) \vee 0$, Here again we use the fact that U is saturated to get that in both cases, $(u + n\chi_\delta) \vee 0$ belongs to $U \oplus [\chi_\delta]$. Thus $U \oplus [\chi_\delta]$ is an ℓ-subgroup of G, and we have shown that each positive element in $U \oplus [\chi_\delta]$ is finite-valued in G. Since a maximal finite-valued subgroup is a-closed, U is a-closed in G.

($2 \longrightarrow 1$) If $U \subseteq W \subseteq G$, where W is a finite-valued subgroup of G, then W is an a-extension of U, but U is a-closed. Therefore $U = W$. $\square$

Corollary 2.2. *If U is a maximal finite-valued subgroup of an abelian ℓ-group G, then $\Gamma(U) \cong \Delta$, where Δ is the set of special values of $\Gamma(G)$.*

Thus if U and V are maximal finite-valued subgroups of G, then they are a-equivalent in the following sense:

For each $0 < u \in U$, there exists $0 < v \in V$ such that $nu > v$ and $nv > u$ for some $n > 0$, and

For each $0 < v \in V$, there exists $0 < u \in U$ such that $nv > u$ and $nu > v$ for some $n > 0$.

For non-abelian ℓ-groups, example 4.1 shows that the proposition and the corollary do not hold.

For the rest of this section, we will use G_δ to denote the regular subgroup of G, and G^δ the cover of G_δ.

Proposition 2.3. *If U is a maximal finite-valued subgroup of a divisible abelian ℓ-group G, and δ is a special value in $\Gamma(G)$, then $U^\delta/U_\delta \cong G^\delta/G_\delta$. Thus if G is a vector lattice, then $U^\delta/U_\delta \cong R$.*

Proof. Since U is a-closed in the divisible group G, U is divisible. Hence $U + G_\delta$ is divisible as well.

Suppose that s is special in G with value G_δ, and that $s \notin U + G_\delta$. Then, since $U + G_\delta$ is divisible and $s \notin U + G_\delta$, $ns \notin U + G_\delta$ for any integer $n \neq 0$. Since U is a maximal finite-valued ℓ-subgroup of G, $U + [s]$ is not finite-valued. So there exist $u \in U$ and an integer $n \neq 0$ such that $u + ns$ is not finite-valued.

Now $u = u_1 + u_2 + \ldots + u_k$ as a sum of pairwise disjoint special elements of G. If there exists $1 \leqslant i \leqslant k$ such that the value of u_i contains G_δ, then $u_i + ns$ is special with the same value as that of u_i, and so $u + ns$ is finite-valued, which of course contradicts the statement above that $u + ns$ is not finite-valued. If the value of each u_i is incomparale to G_δ, then $u + ns$ is finite-valued. So there exists a subset $S \subseteq \{u_1, u_2, \ldots, u_k\}$ such that the values of $u_i \in S$ are strictly contained in G_δ. Without loss of generality, we can assume there exists $0 < m \leqslant k$ such that if $0 < i \leqslant m$, $u_i \in S$, and if $i > m$, $u_i \notin S$. But then $(u_1 + u_2 + \ldots + u_m) + ns$ is special with value G_δ, and so $ns + u$ is finite-valued.

Thus every special element of G with value G_δ is in $U + G_\delta$. Now let $0 < g \in G$ with value G_δ and let $s > 0$ be a special element of G with value G_δ. There exists an integer k such that $ks + G_\delta > g + G_\delta > G_\delta$, and so $ks \wedge g$ is special in G with value G_δ. Since $(ks \wedge g) + G_\delta = g + G_\delta$ and $ks \wedge g \in U + G_\delta$, $g \in U + G_\delta$. Thus $G^\delta = U^\delta + G_\delta$, and so $\frac{G^\delta}{G_\delta} = \frac{U^\delta + G_\delta}{G_\delta} \cong \frac{U^\delta}{U^\delta \cap G_\delta} = \frac{U^\delta}{U_\delta}$. □

Theorem 2.4. *Suppose G is a special-valued divisible abelian ℓ-group, then without loss of generality,*

$$\Sigma(\Delta, R_\delta) \subseteq G \subseteq V(\Delta, R_\delta)$$

where $G^\delta/G_\delta \cong R_\delta$ a divisible subgroup of R and G is an ℓ-subgroup of $V(\Delta, R_\delta)$.

If U is a maximal finite-valued subgroup of G, then there exists an ℓ-automorphism σ such that

$$\Sigma(\Delta, R_\delta) \subseteq U\sigma \subseteq G\sigma \subseteq V(\Delta, R_\delta)$$

and $U\sigma$ is an a-closure of $\Sigma(\Delta, R_\delta)$ in $G\sigma$.

Proof. We have shown that U is special-valued with plenary set Δ of special values, each $U^\delta/U_\delta \cong R_\delta$, and U is a-closed in G.

Thus there exists an ℓ-isomorphism σ of U into $V(\Delta, R_\delta)$ such that

$$\Sigma(\Delta, R_\delta) \subseteq U\sigma \subseteq V(\Delta, R_\delta)$$

and σ can be extented to G, and hence to an ℓ-automorphism of $V(\Delta, R_\delta)$. Since $U\sigma$ is finite-valued, it is an a-extension of $\Sigma(\Delta, R_\delta)$ in $G\sigma$. □

Corollary 2.5. *Δ satisfies the descending chain condition if and only if that $\Sigma(\Delta, R_\delta) = F(\Delta, R_\delta)$. If this is the case, then each maximal finite-valued subgroup of G is isomorphic to $\Sigma(\Delta, R_\delta)$.*

Proof. The proof of Theorem 4.4 in [8] shows that $F(\Delta, R_\delta)$ is an a-closure of $\Sigma(\Delta, R_\delta)$ in $V(\Delta, R_\delta)$, for any choice of R_δ. □

Corollary 2.6. *If U is a maximal finite-valued subgroup of $V(\Delta, R)$, then there exists an ℓ-automorphism of U such that $\Sigma(\Delta, R) \subseteq U\sigma$, and $U\sigma$ is an a-closure of $\Sigma(\Delta, R)$. Thus the maximal finite-valued subgroups of $V(\Delta, R)$ are the a-closures of $\Sigma(\Delta, R)$.*

Corollary 2.7. *If Δ satisfies the descending chain condition, then $\Sigma(\Delta, R) = F(\Delta, R)$ is a-closed, and $V(\Delta, R) = \Sigma(\Delta, R)^L$, the lateral completion of $\Sigma(\Delta, R)$. Thus each maximal finite-valued subgroup U of $V(\Delta, R)$ is isomorphic to $\Sigma(\Delta, R)$. In fact, there exists an ℓ-automorphism σ of $V(\Delta, R)$ such that $U\sigma = \Sigma(\Delta, R)$. In particular, U is a vector lattice.*

Now we consider all the maximal finite-valued subgroups of $V(\Delta, R)$. Since each maximal finite-valued subgroup is a-closed, by the results of [5], we have the following proposition.

Proposition 2.8. *For $V(\Delta, R)$, the following are equivalent.*

(1) *$\overline{\Delta}$ satisfies the descending chain condition, where $\overline{\Delta}$ contains all the branch points of Δ.*

(2) *$F(\Delta, R)$ is the unique abelian a-closure of $\Sigma(\Delta, R)$.*

(3) *$\Sigma(\Delta, R)$ has a unique abelian a-closure.*

(4) *Each maximal finite-valued subgroup of $V(\Delta, R)$ is isomorphic to $F(\Delta, R)$.*

(5) *All the maximal finite-valued subgroups of $V(\Delta, R)$ are isomorphic.*

(6) *Each finite-valued subgroup of $V(\Delta, R)$ has a unique a-closure.*

Proof. The equivalence of (1), (2), and (3) is given in Theorem 1.2.6 in [5]. The rest follows the fact that each maximal finite-valued subgroup is a-closed. □

3. Structure of $F_v(G)$

Let $C \neq 0$ be a convex ℓ-subgroup of an ℓ-group G, and $\mathscr{I}$ be the collection of prime subgroups of G that do not contain C; $\mathscr{I}$ is an ideal of the set of all prime subgroups of G. Let $\mathscr{S}$ be the collection of proper prime subgroups of C. For each $M \in \mathscr{I}$, we have

$$\sigma \colon M \longrightarrow M\sigma = M \cap C$$

σ is a 1-1 inclusion preserving map of $\mathscr{I}$ onto $\mathscr{S}$. M is regular in G if and only if $M\sigma$ is regular in C and M is special in G if and only if $M\sigma$ is special in C.

Let Δ be the set of regular subgroups of G that do not contain C. Then Δ is an ideal of $\Gamma(G)$ and σ induces an isomorphism of Δ onto $\Gamma(C)$, and $\Delta \cong \Gamma(C)$. $C \in F_v$ if and only if Δ consists of special elements.

We say Λ is a special ideal of $\Gamma(G)$, if it is an ideal of special elements in $\Gamma(G)$.

Proposition 3.1. *Let Λ be a special ideal of $\Gamma(G)$. For each $\lambda \in \Lambda$, pick a special element $0 < c_\lambda \in G$ with value λ.*

1. The principal convex ℓ-subgroup $G(c_\lambda) \in F_v$ and $\Gamma(G(c_\lambda)) \cong$ principal ideals $\langle\lambda\rangle$ of Λ = principal ideals $\langle\lambda\rangle$ of $\Gamma(G)$.

2. $H = \bigvee_\Lambda G(c_\lambda) \in F_v$ and $\Gamma(H) \cong \Lambda$.

3. There is a 1-1 order preserving map between finite-valued convex ℓ-subgroups C of G and special ideals Λ of $\Gamma(G)$. So the finite valued convex ℓ-subgroups are freely generated by the largest special ideals Δ of $\Gamma(G)$.

$$\bigvee_\Lambda G(c_\lambda) \longleftarrow \Lambda$$

$$C \longrightarrow \Gamma(C) \cong \Lambda$$

Proof. 1. Let G_λ be the regular subgroup of G. The regular subgroups of $G(c_\lambda)$ correspond to the regular subgroups of G contained in G_λ and these are all special. Thus each regular subgroup of $G(c_\lambda)$ is special, so $G(c_\lambda) \in F_v$.

2. This follows from the fact that F_v is a torsion class.

3. If C is a finite-valued convex ℓ-subgroup of G, then $\Gamma(C)$ is isomorphic to a special ideal Λ of $\Gamma(G)$, i.e. the $G_\lambda \in \Gamma(G)$ that do not contain C.

Conversely, given a special ideal Λ of $\Gamma(G)$, let H be as in 2, then $\Gamma(H) \cong \Lambda$. □

Corollary 3.2. *If $G \in F_v$, then $\Delta = \Gamma(G)$ is special valued, so $\Gamma(G)$ freely generates $\mathscr{C}(G)$.*

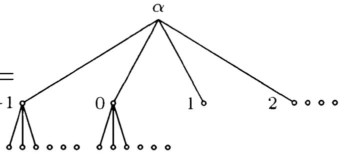
If Λ_1 and Λ_2 are special ideals in $\Gamma(G)$, then $\Lambda_1 \cup \Lambda_2$ is a special ideal in $\Gamma(G)$.

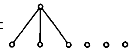
Let Δ be the largest special ideal in $\Gamma(G)$ and $\Lambda = \{\lambda \in \Delta \mid \langle\lambda\rangle$ contains only a finite number of roots$\}$. Then Λ is an ideal of Δ and hence an ideal of $\Gamma(G)$.

Proposition 3.3. *1. $F_v(G) = \bigvee_\Delta G(c_\delta)$, and $G \in F_v$ if and only if $\Delta = \Gamma(G)$.*

2. $F(G) = \bigvee_\Lambda G(c_\lambda)$, and $G \in F$ if and only if $\Lambda = \Gamma(G)$.

3. $F(G) = F_v(G)$ if and only if $\Lambda = \Delta$.

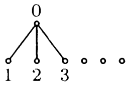
4.1 Let $\Delta =$ 

and let $\Lambda =$ 

Let $H = V(\Lambda, Z)$ and let $G = Z \mathscr{W}\mathrm{r} H$ be the wreath product of Z and H. G is a special-valued ℓ-group, and $G = V(\Delta, Z)$ as a set, but with a different operation. Let $U = \sum_{-\infty}^{\infty} U_i$ where

$$U_n = \begin{cases} \Sigma(\Lambda, Z), & \text{if } n = 0; \\ [\left(\begin{smallmatrix} 1 \\ 1,1,1,1,\dots \end{smallmatrix}\right)], & \text{if } n \neq 0. \end{cases}$$

U is a finite-valued subgroup of G, and ℓ-subgroup of G that contains U and an element with value α is not finite-valued. So proposition 2.1 does not hold for non-abelian ℓ-groups.

4.2 Let $\Delta =$ 

and $V = V(\Delta, R)$.

Let $a = \left(\begin{smallmatrix} 1 \\ 1,0,1,0,1,0,\dots \end{smallmatrix}\right)$, and $b = \left(\begin{smallmatrix} \pi \\ 0,\pi,0,\pi,0,\pi,\dots \end{smallmatrix}\right)$.

Then $S = Qa + Qb + \sum_{i=1}^{\infty} R_i$ is a finite valued subgroup of V, so it is contained in a maximal finite-valued subgroup U of V. U is not a subspace of V, otherwise

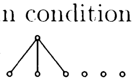
$$\pi a - b = \left(\begin{matrix} 0 \\ \pi, -\pi, \pi, -\pi, \pi, -\pi, \dots \end{matrix}\right) \in U$$

so

$$(\pi a - b) \vee 0 = \left(\begin{matrix} 0 \\ \pi, 0, \pi, 0, \pi, 0, \dots \end{matrix}\right) \in U$$

but is not finite-valued.

Note that we know each maximal finite-valued subgroup of V is isomorphic to $\Sigma(\Delta, R)$, (since Δ satisfies the descending chain condition), so U is a vector lattice.

Let C be the characteristic function on $\Delta =$ 

Then $H = RC + \sum_{i=1}^{\infty} R_i$ is a finite-valued subgroup of $V(\Delta, R)$ that does not contain $\Sigma(\Delta, R)$. We show that H is a maximal finite-valued subgroup of V. For suppose U is finite-valued subgroup of V that properly contains H, and we pick $v = \left(\begin{smallmatrix} v_0 \\ v_1,v_2,v_3,\dots \end{smallmatrix}\right) \in U \setminus H$, then

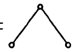
$$\left(\begin{matrix} v_0 \\ v_1, v_2, v_3, \dots \end{matrix}\right) - \left(\begin{matrix} v_0 \\ v_0, v_0, v_0, \dots \end{matrix}\right) = \left(\begin{matrix} 0 \\ v_1 - v_0, v_2 - v_0, v_3 - v_0, \dots \end{matrix}\right) \in U$$

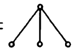
so all but a finite number of the $v_i - v_0$ must be zero, but $v \in H$. contradiction.

4.3 A maximal ℓ-subgroup does not necessarily have property F.

Let $\Lambda =$ 

$\Lambda_1 =$

$\Lambda_2 =$ 

$\Lambda_3 =$ 

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and $V = V(\Lambda, R)$. Then

$$\Sigma(\Lambda_1, R) \subseteq \Sigma(\Lambda_2, R) \subseteq \Sigma(\Lambda_3, R) \subseteq \Sigma(\Lambda_4, R) \subseteq \dots$$

all $\Sigma(\Lambda_i, R)$ have property F, i.e., each $0 < g \in \Sigma(\Lambda_i, R)$ exceeds at most finite number of disjoint elements. But the join of them does not have property F.

References

[1] *M. Anderson, P. Bixler, and P. Conrad*: Vector lattices with no proper a-subspaces. Archiv Math. (Basel) *41* (1983), 427–433.

[2] *M. Anderson, and T. Fei*: Lattice-ordered groups: an introduction. D. Reidel, 1987.

[3] *P. Bixler, and M. Darnel*: Special-valued ℓ-groups. Algebra Universalis *22* (1986). 172–191.

[4] *R. Byrd*: Lattice-ordered groups. Ph.D. Dissertation, Tulane University, 1966.

[5] *Y. Chen*: Torsion classes of lattice-ordered groups and special-valued subgroups. Ph.D. dissertation, University of Kansas, 1992.

[6] *P. Conrad, J. Harvey, and W.C. Holland*: The Hahn embedding theorem for lattice-ordered groups. Trans. Amer. Math. Soc. *108* (1963), 143–149.

[7] *P. Conrad*: Archimedean extensions of lattice-ordered groups. J. Indian Math. Soc. *30* (1966), 199–221.

[8] *P. Conrad*: Torsion radicals of lattice-ordered groups. Symposia Math. *21* (1977), 479–513.

[9] *M. Darnel*: Lattice-ordered groups. Preprint.

[10] *J. Martinez*: Torsion theory for lattice-ordered groups. Czech. Math. J. *25* (1975), 284–299.

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