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THE ASSOCIATED TENSOR NORM TO (q,p)-ABSOLUTELY SUMMING OPERATORS ON $C(K)$-SPACES

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Abstract. We give an explicit description of a tensor norm equivalent on $\mathscr{C}(K)\otimes F$ to the associated tensor norm ν_{qp} to the ideal of (q,p)-absolutely summing operators. As a consequence, we describe a tensor norm on the class of Banach spaces which is equivalent to the left projective tensor norm associated to ν_{qp}.

As far as we know there is no explicit description for the tensor norm ν_{qp} associated to the ideal $\mathcal{P}_{(q,p)}$ of (q,p)-absolutely summing operators. The purpose of this note is to define explicitly a norm equivalent to this one in the case of tensor products of type $\mathscr{C}(K)\otimes F$. As a consequence, we shall be able to give an easy and direct definition of a tensor norm equivalent to the left projective tensor norm $\backslash\nu_{qp}$. The key of our results is the connection on $\mathscr{C}(K)$ spaces of $\mathcal{P}_{(q,p)}$ with the ideal $\mathcal{P}_{p,\sigma}$ of (p,σ)-absolutely continuous operators defined by Matter in [4] and the knowledge of the tensor norm associated to $\mathcal{P}_{p,\sigma}$, which was obtained by the authors in [3].

Throughout this note we use standard Banach space notation. The class of all Banach spaces will be denoted by BAN. If $E\in$ BAN, B_E will be the unit ball of E and J_E will denote the canonical inclusion of E into E''. K will be always a compact Hausdorff topological space and $\mathscr{C}(K)$ the Banach space of all scalar continuous functions on K. If $E\in$ BAN$, B_{E'}$ will be considered as a compact space with the topology $\sigma(E',E)$. We define $I_E\colon E\to\mathscr{C}(B_{E'})$ to be the canonical isometric embedding. We refer the reader to [1] and [7] for all definitions concerning tensor norms and operator ideals respectively. If $1\leqslant p\leqslant\infty$, p' is the extended real number satisfying $\frac{1}{p}+\frac{1}{p'}=1$. $(\mathcal{P}_p,\Pi_p)$ will be the normed ideal of p-absolutely summing

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operators on BAN. For every $E \in \mathrm{BAN}, (x_i) \in E^{\mathbb{N}}, p \in [1,\infty]$ and $\sigma \in [0,1[$ we define (changing Σ by sup when $p=\infty$)

$$\pi_p((x_i)) := \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{1/p}$$

and

$$\delta_{p\sigma}((x_i)) := \sup_{x' \in B_{E'}} \left(\sum_{i=1}^{\infty} (|\langle x_i, x'\rangle|^{1-\sigma}\|x_i\|^{\sigma})^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}.$$

1. Definition. (Matter [4]): Let $0 \leqslant \sigma < 1$ and $E, F \in \mathrm{BAN}$. We say that $T \in \mathscr{L}(E,F)$ is a (p,σ)-absolutely continutous operator if there exist $G \in \mathrm{BAN}$ and an operator $S \in \mathcal{P}_p(E,G)$ such that

$$\|Tx\| \leqslant \|x\|^{\sigma}\|Sx\|^{1-\sigma} \quad \forall x \in E. \tag{1}$$

In such case, we put $\Pi_{p,\sigma}(T) = \inf \Pi_p(S)^{1-\sigma}$, taking the infimum over all G and $S \in \mathcal{P}_p(E,G)$ such that (1) holds. We denote by $(\mathcal{P}_{p,\sigma}, \Pi_{p,\sigma})$ the normed ideal of (p,σ)-absolutely continuous operators in BAN.

We have the following characterization of $\mathcal{P}_{p,\sigma}(\mathscr{C}(K), F)$:

2. Proposition. *Let $F \in \mathrm{BAN}$ and let $T \in \mathscr{L}(\mathscr{C}(K), F)$. Then*

$$T \in \mathcal{P}_{p,\sigma}(\mathscr{C}(K), F)$$

iff there are $C > 0$ and a Radon probability measure λ on K such that

$$\|Tx\| \leqslant C\|x\|^{\sigma} \; \|I_K(x)\|^{1-\sigma} \qquad \forall x \in \mathscr{C}(K), \tag{2}$$

where I_K is the canonical map $I_K\colon \mathscr{C}(K) \to L_p(K,\lambda)$. In addition, $\pi_{p,\sigma}(T)$ is the infimum of numbers C for which (2) holds.

Proof. Let $T \in \mathcal{P}_{p,\sigma}(\mathscr{C}(K), F)$. Then there is $G \in \mathrm{BAN}$ and $S \in \mathcal{P}_p(\mathscr{C}(K)), G)$ such that (1) holds. By Pietsch's factorization theorem (see [7], 17.3.5), there is a probability measure λ on K and $R \in \mathscr{L}(L_p(K,\lambda), G)$ such that $S = RI_K$ and $\Pi_p(S) = \inf \|R\|$ over all R and λ. Then (2) holds and $\inf C \leqslant \inf \|R\|^{1-\sigma} = \Pi_p(S)^{1-\sigma}$. Taking the infimum over all S in (1), we have $\inf C \leqslant \Pi_{p,\sigma}(T)$. Conversely, if (2) holds, the map $S = C^{1/(1-\sigma)} I_K \in \mathcal{P}_p(\mathscr{C}(K), L_p(K,\lambda))$ verifies (1) and hence $T \in \mathcal{P}_{p,\sigma}(\mathscr{C}(K), F)$ and $\Pi_{p,\sigma}(T) \leqslant \Pi_p(S)^{1-\sigma} = C$. Then the conclusion follows. □

The following result is due essentially to Pisier.

3. Proposition. *For all $F \in \mathrm{BAN}, 1 \leqslant p < \infty$ and $0 \leqslant \sigma < 1$ we have*

$$\mathcal{P}_{p,\sigma}(\mathscr{C}(K), F) = \mathcal{P}_{(\frac{p}{1-\sigma}, p)}(\mathscr{C}(K), F).$$

Moreover,

$$\Pi_{(\frac{p}{1-\sigma}, p)}(T) \leqslant \Pi_{p,\sigma}(T) \leqslant \left(\frac{p}{1-\sigma}\right)^{(1-\sigma)/p} \Pi_{(\frac{p}{1-\sigma}, p)}(T).$$

Proof. The inclusion $\mathcal{P}_{p,\sigma} \subset \mathcal{P}_{(\frac{p}{1-\sigma}, p)}$ and the first inequality are immediate from theorem 4.1 of Matter in [4]. On the other hand, by theorem 2.4 of Pisier in [8], every $T \in \mathcal{P}_{(\frac{p}{1-\sigma}, p)}(\mathscr{C}(K), F)$ verifies our proposition 2 and the second inequality. □

It is well known that $\mathcal{P}_{(q,p)}(\mathscr{C}(K), F)$ does not depend on the parameter p (see [6] and [8]). From proposition 3 we get

4. Corollary. *Let $F \in \mathrm{BAN}$, $1 \leqslant p < \infty$, $0 < \sigma < 1$ and $q = \frac{p}{1-\sigma}$. Then for every $1 \leqslant s < \infty$ and $0 < \tau < 1$ such that $\frac{s}{1-\tau} = q$, $\mathcal{P}_{p,\sigma}(\mathscr{C}(K), F) = \mathcal{P}_{s,\tau}(\mathscr{C}(K), F)$. Moreover, if $\tau \geqslant \sigma$ there is a $C \geqslant 1$ such that $\Pi_{s,\tau}(T) \leqslant \Pi_{p,\sigma}(T) \leqslant C\Pi_{s,\tau}(T)$ for every $T \in \mathcal{P}_{p,\sigma}(C(K), F)$.*

Proof. It follows from theorem 4.1 in [4], the fact that $g(\sigma) = a^{1-\sigma}b^{\sigma}$ is an increasing function on $[0,1[$ for $0 \leqslant a \leqslant b < \infty$ and the open mapping theorem. □

We have defined in [3] a family $\alpha_{q,\nu,q,\sigma}$ of tensor norms on BAN which generalizes the known tensor norms α_{pq} of Lapresté (see [2] and [1]). In particular, choosing $\nu = 0$ and $q = 1$ we get the following:

5. Definition. Let $1 \leqslant p \leqslant \infty$ and $0 \leqslant \sigma < 1$. The tensor norm $d_{p,\sigma}$ on BAN is defined by

$$d_{p,\sigma}(z; E \otimes F) := \inf\left\{ \delta_{p'\sigma}((x_i))\pi_{\left(\frac{p'}{1-\sigma}\right)'}((y_i)) \;\Big|\; z = \sum_{i-1}^{n} x_i \otimes y_i \right\} \quad \forall z \in E \otimes F.$$

It is proved in [3] that $d'_{p,\sigma}$ is the associated tensor norm to the deal $\mathcal{P}_{p',\sigma}$, i.e. $(E\hat{\otimes}_{d_{p,\sigma}} F)' = \mathcal{P}_{p',\sigma}(E, F')$. Hence

6. Corollary. *If $F \in BAN$, $1 \leqslant p \leqslant \infty$, $0 \leqslant \sigma < 1$ and $q = \frac{p}{1-\sigma}$, then $(\mathscr{C}(K) \otimes_{d_{p',\sigma}} F)'$ is isomorphic to $\mathcal{P}_{(q,p)}(\mathscr{C}(K), F')$, i.e. on $\mathscr{C}(K) \otimes F$, the associated tensor norm to $\mathcal{P}_{(q,p)}$ is equivalent to $d'_{p',\sigma}$.*

7. Definition. Let $(\mathscr{U}, U)$ be a normed operator ideal in BAN and $E, F \in BAN$. We say that $T \in \mathscr{U}(E, F)$ has the extension property if there is $\bar{T} \in \mathscr{U}(\mathscr{C}(B_{E'}), F'')$ such that $J_F T = \bar{T} I_E$.

Note that this definition is not coincident with the given one by Matter in [4] section 5. We denote by $\mathscr{U}^{\text{ext}}(E,F)$ the set of all operators $T \in \mathscr{U}(E,F)$ with the extension property. It is easy to see that

$$U^{\text{ext}}(T) = \inf\{U(\bar{T}) \mid \bar{T}|_E = T \text{ and } \bar{T} \in \mathscr{U}(\mathscr{C}(B_{E'}), F'')\}$$

is a norm in $\mathscr{U}^{\text{ext}}(E,F)$.

When $\mathscr{U}$ is a maximal operator ideal with associated tensor norm α, we denote by $\backslash\mathscr{U}$ the maximal operator ideal associated to the left projective tensor norm $\backslash\alpha$. The following characterization shows, in particular, that $(\mathscr{U}^{\text{ext}}, U^{\text{ext}})$ is a normed operator ideal in BAN and gives us an easy description of the ideal $\backslash\mathscr{U}$:

8. Proposition. *The following are equivalent:*

1) $T \in \mathscr{U}^{\text{ext}}(E,F)$
2) $T \in \backslash\mathscr{U}(E,F)$
3) *There are a compact space K, a Radon measure μ on K and operators $R \in \mathscr{L}(E, L_\infty(K,\mu))$ and $\bar{T} \in \mathscr{U}(L_\infty(K,\mu), F'')$ such that $J_F T = \bar{T}R$.*

Moreover $U^{\text{ext}}(T) = \backslash U(T) = \inf \|R\| U(\bar{T})$, taking the infimum over all factorizations as in 3).

Proof. 1) $\Rightarrow$ 2). This implication and the inequality $\backslash U(T) \leqslant U^{\text{ext}}(T)$ follow from proposition 20.12 in [1].

2) $\Rightarrow$ 3). Use again proposition 20.12 in [1].

3) $\Rightarrow$ 1). Suppose that $J_F T$ admits a factorization as in 3). Since $L_\infty(K,\mu)$ is isometric to some $\mathscr{C}(W)$ where W is a compact Stonean space (see for instance the section 3.10 of [1]), R has a norm preserving extension $H \in \mathscr{L}(\mathscr{C}(B_{E'}), L^\infty(\mu))$. Thus $U^{\text{ext}}(T) \leqslant U(\bar{T}H) \leqslant \|H\| U(\bar{T}) \leqslant \|R\| U(\bar{T})$ and $U^{\text{ext}}(T) \leqslant \backslash U(T)$. □

9. Corollary. *Let $q \geqslant p$ and $\sigma \in [0,1[$ such that $q = \frac{p}{1-\sigma}$. For all $E, F \in$ BAN, $\mathcal{P}^{\text{ext}}_{p,\sigma}(E,F)$ is isomorphic to $\mathcal{P}^{\text{ext}}_{q,p}(E,F)$.*

When $\mathscr{U} = \mathcal{P}_{p,\sigma}$ we can determine explicitly the tensor norm associated with $\mathscr{U}^{\text{ext}} = \backslash\mathscr{U}$. Given $E, F \in$ BAN, let $\alpha_{p,\sigma}$ be the norm on $E \otimes F$

$$\alpha_{p,\sigma}(z; E \otimes F) = d_{p,\sigma}((I_E \otimes \mathrm{Id}_F))(z); \mathscr{C}(B_{E'}) \otimes F).$$

$\alpha_{p,\sigma}$ is a tensor norm in BAN as consequence of the following theorem:

10. Theorem. *Given $q \geqslant p$, let $\sigma \in [0,1[$ be such that $q = \frac{p}{1-\sigma}$. Then*

$$(E \otimes_{\alpha_{p',\sigma}} F)' = \mathcal{P}^{\text{ext}}_{(q,p)}(E, F')$$

i.e. $\alpha'_{p',\sigma}$ *is equivalent to the tensor norm associated to* $\mathcal{P}^{\text{ext}}_{(q,p)}$.

Proof. $E \otimes_{\alpha_{p',\sigma}} F$ is a topological subspace of $\mathscr{C}(B_{E'}) \otimes_{d_{p',\sigma}} F$. Then

$$(E \otimes_{\alpha_{p',\sigma}} F)' = (\mathscr{C}(B_{E'}) \otimes_{d_{p',\sigma}} F)'/(E \otimes F)^{\perp} = \mathcal{P}_{p,\sigma}(\mathscr{C}(B_{E'}), F')/(E \otimes F)^{\perp}$$

where $(E \otimes F)^{\perp}$ is the orthogonal to $E \otimes F$ in $\mathcal{P}_{p,\sigma}(\mathscr{C}(B_{E'}), F')$. Let $\|.\|_0$ be the norm on $\mathcal{P}_{p,\sigma}(\mathscr{C}(B_{E'}), F')/(E \otimes F)^{\perp}$. It is clear that every element $\widehat{T}$ of this quotient ($\widehat{\ }$ denotes the classes in the quotient) defines an unique operator T in $\mathcal{P}^{\text{ext}}_{p,\sigma}(E, F')$ such that

$$\Pi^{\text{ext}}_{p,\sigma}(T) \leqslant \inf\{\Pi_{p,\sigma}(S) | S \in \widehat{T}\} = \|\widehat{T}\|_0.$$

Conversely, since there is a projection P from F''' onto F' of norm 1, every $T \in \mathcal{P}^{\text{ext}}_{p,\sigma}(E, F')$ has an extension $S \in \mathcal{P}_{p,\sigma}(\mathscr{C}(B_{E'}), F''')$. If $T_0 = PS$, then $\widehat{T_0} \in \mathcal{P}_{p,\sigma}(\mathscr{C}(B_{E'}), F')/(E \otimes F)^{\perp}$ and $\|\widehat{T_0}\|_0 \leqslant \Pi_{p,\sigma}(PS) \leqslant \Pi_{p,\sigma}(S)$. Hence $\|\widehat{T_0}\|_0) \leqslant \Pi^{\text{ext}}_{p,\sigma}(T)$ and $\mathcal{P}_{p,\sigma}(\mathscr{C}(B_{E'}), F')/(E \otimes F)^{\perp}$ is isometric with $\mathcal{P}^{\text{ext}}_{p,\sigma}(E, F')$. Corollary 9 gives the conclusion. □

11. Corollary. *If* $q \geqslant p \in [1, \infty[$, *and* $\sigma \in [0, 1[$ *is such that* $q = \frac{p}{1-\sigma}$, *then* $\alpha'_{p',\sigma}$ *is equivalent to* $\backslash\nu_{qp}$.

References

[1] *A. Defant, K. Floret*: Tensor norms and Operator Ideals. North-Holland Mathematics Studies 176. Amsterdam-London-New York-Tokyo, 1993.

[2] *J. T. Lapresté*: Operateurs sommantes et factorization à travers les espaces L^p. Studia Math. *56* (1976), 47–83.

[3] *J. A. López Molina, E. A. Sánchez Pérez*: Ideales de operadores absolutamente continuos. Rev. Real Acad. Ciencias Exactas, Fisicas y Naturales, Madrid *87* (1993), 349–378.

[4] *U. Matter*: Absolutely continuous operators and super-reflexivity. Math. Nachr. *130* (1987), 193–216.

[5] *U. Matter, H. Jarchow*: Interpolative constructions for operator ideals. Note di Matematica *VIII* (1988), no. 1, 45–56.

[6] *B. Maurey*: Sur certaines propriétés des opérateurs sommants. C. R. Acad. Sci. Paris *A 277* (1973), 1053–1055.

[7] *A. Pietsch*: Operator Ideals. North-Holland Publ. Company, Amsterdam-New York-Oxford, 1980.

[8] *G. Pisier*: Factorization of operators through $L_{p,\infty}$ or $L_{p,1}$ and non-commutative generalizations. Math. Ann. *276* (1986), 105–136.

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