D. Opris; I. D. Albu
Geometrical aspects of the covariant dynamics of higher order

Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 3, 395–412

Persistent URL: http://dml.cz/dmlcz/127427

Czechoslovak Mathematical Journal, 48 (123) (1998), 395–412

GEOMETRICAL ASPECTS OF THE COVARIANT DYNAMICS OF HIGHER ORDER

D. Opris and I. D. Albu, Timişoara

(Received October 5, 1995)

Abstract. We present some geometrical aspects of a higher-order jet bundle which is considered a suitable framework for the study of higher-order dynamics in continuous media. We generalize some results obtained by A. Vondra, [7]. These results lead to a description of the geometrical dynamics of higher order generated by regular equations.

MSC 2000: 53C05, 70H35, 58F05

Introduction

The present study is an attempt to emphasize some geometrical aspects of a possible mathematical model for the higher-order dynamics in continuous media as well as for the higher-order field theories.

The mathematicians agree (see [1], [2], [4], etc) that the most suitable framework for this application is a higher-order jet bundle associated to a fibered manifold. A physical field is a section of this "configuration manifold". The partial differential equations describing some higher-order dynamics are the kernels of some operators which appear as sections in a vector bundle of forms over that jet bundle, [1].

A. Vondra initiated such a study for a fibered manifold having the base of dimension 1, [5], [6], [7].

We consider a fibered manifold (E, π_0, B), where B is an orientable manifold of dimension $n \geqslant 1$ ("parameter space" containing $n-1$ "spatial variables" and a "time variable"), E is a manifold of dimension $n+m$ and π_0 is a submersion of E on B.

In [4] one argues the importance of a covariant approach that is the time variable and the other parameters on the whole.

To start the study it is necessary to define some associated structures and geometrical objects as $f(3,-1)$-structures, contact forms, connection of order r, dynamical connections.

Our approach means, in a more general context, to consider the $f(3,-1)$-structure on a jet bundle introduced by Vondra in the case $n=1$, [6].

The results of § 4 ($r=1$) generalize those obtained by Vondra in [7]. These results lead to a description of the geometrical dynamics of higher order generated by regular equations.

We shall use the standard multi-index notation. A multi-index is denoted by $I=(i_1,\dots,i_n)\in\mathbb{N}^n$. The length of I is $|I|=i_1+\dots+i_n$ and its power is $w(I)=|I|!/I!$ where $I!=i_1!\dots i_n!$. $0=(0,\dots,0)$ is the null multi-index and $1_i=(0,\dots,1,\dots 0)$ with 1 at the i-th place. For $I=(i_1,\dots,i_n)$, $J=(j_1,\dots,j_n)$ we define the sum $I+J=(i_1+j_1,\dots,i_n+j_n)$. In particular, $Ij=jI=I+1_j=(i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_n)$. For a family of objects $A=\{a_{i,J},|I|=m,|J|=1\}$ with m,l fixed, we may define a new family $\sigma(A)=\{\sigma_L(A),|L|=m+1\}$ by

$$\sigma_L(A)=\frac{1}{w(L)}\sum_{I+J=L}w(I)w(J)a_{I,J}$$

(the sum is made for all multi-indexes I,J with $I+J=L$). The family of objects $A=\{a_{I,j},|I|=m\}$ is identified with the family $A=\{a_{I,1_j},|I|=m\}$ for which $\sigma(A)=\{\sigma_L(A),|L|=m+1\}$, where

$$\sigma_L(A)=\frac{1}{w(L)}\sum_{I+J=L}w(I)a_{I,J},\ \ |I|=m,\ \ |J|=1.$$

All manifolds and mappings are supposed to be smooth and the summation convention is used as far as possible.

1. Geometric structures on J^pE

Let (E,π_0,B) be a fibered manifold with dim $B=n$, dim $E=n+m$, (U,x^i) a local chart on B and $(U_0=\pi_0^{-1}(U),x^i,u^\alpha)$ the local fibered chart on E adapted to (U,x^i). If $(\overline{U}_0,\bar{x}^i,\bar{u}^\alpha)$ is another chart local fibered charts on E adapted to $(\overline{U},\bar{x}^i)$ and $U\cap\overline{U}\neq\emptyset$ then the coordinate transformations are

$$\text{(1.1)}\qquad \bar{x}^i=\bar{x}^i(x),\ \det\|\overline{B}^i_j\|\neq 0,\ \overline{B}^i_j=\frac{\partial\bar{x}^i}{\partial x^j};$$

$$\bar{u}^\alpha=\bar{u}^\alpha(x,u),\ \det\|\overline{A}^\alpha_\beta\|\neq 0,\ \overline{A}^\alpha_\beta=\frac{\partial\bar{u}^\alpha}{\partial u^\beta}.$$

Let $\Gamma(\pi_0)$ be the set of the sections of π_0 and for a local section on $U \subset B$, $s \in \Gamma_U(\pi_0)$, let us denote

$$u_I^\alpha(x) =: u_{i_1\dots i_n}^\alpha(x) := \frac{\partial^{|I|} s^\alpha(x)}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}, \tag{1.2}$$

where $I = (i_1, \dots, i_n)$ is a multi-index with $|I| \leqslant r$. The equivalence relation in $\Gamma_U(\pi_0)$ is introduced as follows: $s_1 \sim s_2$ iff $\underset{1}{u_I^\alpha}(x) = \underset{2}{u_I^\alpha}(x)$, $0 \leqslant |I| \leqslant r$, $x \in U$ and determines the r-jets of sections of π_0 in x, denoted by $j_x^r s$. Finally, the set of all such r-jets of sections of π_0 is a differentiable manifold denoted by $J^r E$;

$$\begin{aligned} &(J^r E, \pi_r, B), \text{ where} \\ &\pi_r \colon J^r E \to B, \ \pi_r(j_x^r s) = x, \end{aligned} \tag{1.3}$$

is a fibered manifold; for each pair (p, r) such that $0 \leqslant p \leqslant r-1$, $(J^r E, \pi_{pr}, J^p E)$, where

$$\pi_{pr} \colon J^r E \to J^p E, \ \pi_{pr}(j_x^r s) = j_x^p s, \tag{1.4}$$

is a fiber bundle. In particular, $J^r E$ is an affine bundle over $J^{r-1}E$ and $J^0 E = E$. The local fibered chart on $J^r E$ induced by (U, x^i) is $(U_r = \pi_r^{-1}(U), x^i, u_I^\alpha)$, $0 < |I| < r$.

For $f \in \mathscr{F}(J^r E)$ the partial derivative of f in direction x^i is defined by

$$(j^{r+1}s)^*(d_i f) = \partial_i (f \circ j^r s), \ \forall \ s \in \Gamma(\pi_0). \tag{1.5}$$

In the local chart (U_r, x^i, u_I^α) we have

$$d_i^r f = \partial_i f + \sum_{0 \leqslant |I| \leqslant r} u_{iI}^\alpha \partial_\alpha^I f, \tag{1.6}$$

where $0 \leqslant |I| \leqslant r$, $\partial_\alpha^I =: \frac{\partial}{\partial u_I^\alpha}$ and i is identified with 1_i, [3], [1].

For two local fibered charts on $J^r E$, $(\overline{U}_r, \bar{x}^i, \bar{u}_I^\alpha)$, (U_r, x^i, u_J^β) with $\overline{U} \cap U \neq \emptyset$, the coordinate transformations are

$$\begin{aligned} \bar{x}^i &= \bar{x}^i(x), \\ \bar{u}^\alpha &= \bar{u}^\alpha(x, u), \\ &\dots\dots\dots\dots\dots\dots \\ \bar{u}_L^\alpha &= \sigma_L(d_i(\bar{u}_I^\alpha)), \end{aligned} \tag{1.7}$$

where $|L| = 1 + |I|$ and $0 \leqslant |I| \leqslant r-1$. The natural local basis on J^rE is $\{\partial_i, \partial_\alpha^I\}$ and the local co-basis is $\{dx^i, du_I^\alpha\}$, where $0 \leqslant |I| \leqslant r$.

The canonical projection (1.4), $\pi_{pr}\colon\ (x^i, u_I^\alpha) \in J^rE \mapsto (x^i, u_J^\alpha) \in J^pE$, with $0 \leqslant |I| \leqslant r$, $0 \leqslant |J| \leqslant p$, leads to the vector subbundles $V_{pr} = \mathrm{Ker}(\pi_{pr})_*$, $0 \leqslant p \leqslant r-1$, of the tangent bundle $T(J^rE)$. The local fiberes of V_{pr} determine regular differential systems

(1.8) $$V_{pr}\colon\ z \in J^rE \mapsto V_{pr}(z) \subset T_z(J^rE)$$

on J^rE having the property

(1.9) $$V_{r-1r}(z) \subset V_{r-2r}(z) \subset \ldots \subset V_{or}(z).$$

These differential systems are generated by the vector fields $\{\partial_\alpha^I\}, 0 \leqslant |I| \leqslant r$.

We call the contact form $\overset{p}{\theta}$, $1 \leqslant p \leqslant r-1$, the $V(J^pE)$-valued 1-form on J^rE such that

(1.10) $$\overset{p}{\theta}\,((j^rs)_*\nu) = 0,\ \ \forall s \in \Gamma_U(\pi_0),\ \ \forall \nu \in TB,$$
$$\overset{p}{\theta}\,(\xi) = (\pi_{pr})_*\xi,\ \ \forall \xi \in V(J^rE), [3].$$

By using the canonical local basis and co-basis we obtain

(1.11) $$\overset{p}{\theta} = \sum_{|I|=p-1} \overset{p}{\theta}{}_I^\alpha \otimes \partial_\alpha^I,$$

where

(1.12) $$\overset{p}{\theta}{}_I^\alpha = du_I^\alpha - u_{I,i}^\alpha dx^i, \quad |I| = p-1.$$

We can define a $V(J^rE)$-valued contact form θ_2 on J^rE by

(1.13) $$\theta_2 = \sum_{p=1}^{r-1} \overset{p}{\theta} = \sum_{p=1}^{r-1} \sum_{|I|=p} \theta_I^\alpha \otimes \partial_\alpha^I.$$

Finally, we consider a contact map on J^rE which is a $\pi_r^*(T^*B) \otimes T(J^{r-1}E)$-valued 1-form θ_1 locally given by

(1.14) $$\theta_1 = dx^i \otimes d_i^r, \quad \text{where } d_i^r = \partial_i + \sum_{0 \leqslant |I| \leqslant r} u_{iI}^\alpha \partial_\alpha^I.$$

We can also introduce some 1-forms $\overset{p}{J}$, $1 \leqslant p \leqslant r-1$, on $J^r E$, which are $T(J^p E) \otimes T(J^{p+1}E)$-valued and defined by

$$J^p = \sum_{|I|=p-1} \overset{p}{\theta}{}^\alpha_I \otimes \partial^{Ii}_\alpha \otimes d^{p+1}_i, \tag{1.15}$$

where

$$d^{p+1}_i = \partial_i + \sum_{0 \leqslant |I| \leqslant p+1} u^\alpha_{iI} \partial^I_\alpha. \tag{1.16}$$

For each $i \in \{1, \dots, n\}$, let us define a $T(J^{p+1}E)$-valued 1-form on $J^r E$ by

$$\overset{p}{J}{}^i = \sum_{|I|=p-1} \overset{p}{\theta}{}^\alpha_I \otimes \partial^{Ii}_\alpha. \tag{1.17}$$

It follows from (1.17) that

$$\overset{p}{J}{}^i \circ \overset{p}{J}{}^j = 0; \ [\overset{p}{J}{}^i, \overset{p}{J}{}^j]_{FN} = 0, \tag{1.18}$$

where $[\ ,\ ]_{FN}$ is the Frölicher-Nijenhuis bracket defined for the vector valued forms. Consequently, the 1-form $\overset{p}{J}{}^i$ is an almost tangent structure called the almost tangent structure in direction x^i.

2. Connection of order r. Dynamical connection of order r

A connection of order r on (E, π_0, B) is a section $\Lambda\colon J^{r-1}E \to J^r E$ of the bundle $(J^r E, \pi_{r-1r}, J^{r-1}E)$. Any such connection is locally given by

$$\Lambda\colon (x^i, u^\alpha_I) \in J^{r-1}E \mapsto (x^i, u^\alpha_I, \Lambda^\alpha_J) \in J^r E, \ 0 \leqslant |I| \leqslant r-1, \ |J| = r,$$

where

$$\Lambda^\alpha_J = \Lambda^\alpha_J(x^i, u^\alpha_I).$$

The horizontal form h^r of Λ and the vertical form v^r are given by

$$\begin{aligned} h^r &= \theta_1 \circ \Lambda = dx^i \otimes \left(\partial_i + \sum_{0 \leqslant |I| \leqslant r-2} u^\alpha_{iI} \partial^I_\alpha + \sum_{|J|=r-1} \Lambda^\alpha_{iJ} \partial^J_\alpha \right), \\ v^r &= \theta_2 \circ \Lambda = \sum_{0 \leqslant |I| \leqslant r-1} \theta^\alpha_I \otimes \partial^I_\alpha + \sum_{|J|=r-1} (du^\alpha_J - \Lambda^\alpha_{iJ} dx^i) \partial^J_\alpha. \end{aligned} \tag{2.1}$$

The π_{r-1r}-horizontal distribution $\operatorname{Im} h^r$ is called the *semispray distribution* $\Delta_r^{r-1}(\Lambda)$ and it is locally generated on $J^{r-1}E$ by the vector fields

$$\Gamma_i = \partial_i + \sum_{0\leqslant |I|\leqslant r-2} u_{iI}^{\alpha}\partial_{\alpha}^{I} + \sum_{|J|=r-1} \Lambda_{iJ}^{\alpha}\partial_{\alpha}^{J}. \tag{2.2}$$

The forms associated to $\Delta_r^{r-1}(\Lambda)$ are given by

$$\theta_I^{\alpha} = du_I^{\alpha} - u_{iI}^{\alpha}dx^i,\ \ \Psi_J^{\alpha} = du_J^{\alpha} - \Lambda_{iJ}^{\alpha}dx^i, \tag{2.3}$$

$0\leqslant |I|\leqslant r-2$, $|J|=r-2$. The connection Λ of order r determines the direct sum decomposition

$$TJ^{r-1}E = \Delta_r^{r-1}(\Lambda)\oplus V(J^{r-1}E). \tag{2.4}$$

A section $s\in\Gamma_U(\pi_0)$ is called an integral section of Λ if

$$j^r s = \Lambda\circ j^{r-1}s$$

on U. The condition of integrability is locally given by the relations

$$s_J^{\alpha}(x, s_I^{\beta}(x)) = \Lambda_J^{\alpha}(x, s_I^{\beta}(x)),\ \ |J|=r,\ \ 0\leqslant |I|\leqslant r-1. \tag{2.5}$$

From (2.2) and (2.5) it results that s is an integral section if and only if $j^{r-1}s$ is an integral map of $\Delta_r^{r-1}(\Lambda)$.

Let $\widetilde{\pi}_{1,r-1}\colon J^1(J^{r-1}E)\to J^{r-1}E$ be the 1-jet bundle of sections of the bundle $\pi_{r-2r-1}\colon J^{r-1}E\to J^{r-2}E$. If $(U_{r-1}, x^i, u_I^{\alpha}), 0\leqslant |I|\leqslant r-2$, is a local chart on $J^{r-2}E$ and $s(x^i, u_I^{\alpha}) = (x^i, u_I^{\alpha}, s_J^{\alpha}(x, u_I^{\alpha}))$, $0\leqslant |I|\leqslant r-2$, $|J|=r-1$, is a section of π_{r-2r-1}, then

$$j^1_{(x,u)}s = (x^i, u_I^{\alpha}, s_J^{\alpha}, s_{Ji}^{\alpha}, s_{J\beta}^{\alpha I}),\ \text{where} \qquad s_{Ji}^{\alpha} = \frac{\partial s_J^{\alpha}}{\partial x^i},\ \ s_{J\beta}^{\alpha I} = \frac{\partial s_J^{\alpha}}{\partial u_I^{\beta}}. \tag{2.6}$$

A canonical chart on $J^1(J^{r-1}E)$ is given by $(\widetilde{U}_{1r-1} = \widetilde{\pi}_{1r-1}(U_{r-1}),\ x^i, u_I^{\alpha}, u_J^{\alpha}, u_L^{\alpha}, u_{J\beta}^{\alpha I})$, $0\leqslant |I|\leqslant r-2, |J|=r-1, |L|=r$. The contact map on $J^1(J^{r-1}E)$ is

$$\widetilde{\theta}_1 = dx^i\otimes\left(\partial_i + \sum_{0\leqslant |I|\leqslant r-1} u_{iI}^{\alpha}\partial_{\alpha}^{I}\right) + \sum_{0\leqslant |I|\leqslant r-2} du_I^{\alpha}\otimes\left(\partial_{\alpha}^{I} + \sum_{|J|=r-1} u_{J\alpha}^{\beta I}\partial_{\beta}^{J}\right) \tag{2.7}$$

and the contact form is

$$\widetilde{\theta}_2 = \sum_{|J|=r-1} \left(du^\alpha_J - du^\alpha_{iJ} dx^i - \sum_{0\leqslant |I| \leqslant r-2} u^{\alpha I}_{J\beta} du^\beta_I \right) \otimes \partial^J_\alpha. \tag{2.8}$$

A dynamical connection on $J^{r-1}E$ is a section $F_d\colon\ J^{r-1}E \to J^1(J^{r-1}E)$ of $\widetilde{\pi}_{1,r-1}$. Locally, such a connection is given by

$$F_d\colon\ (x^i, u^\alpha_I) \in J^{r-1}E \mapsto (x^i, u^\alpha_I, u^\alpha_J, F^\alpha_L, F^{\alpha I}_{J\beta}) \in J^1(J^{r-1}E),$$

where

$$F^\alpha_L = F^\alpha_L(x^i, u^\beta_I), F^{\alpha I}_{J\beta} = F^{\alpha I}_{J\beta}(x^i, u^\gamma_I),\ 0 \leqslant |I| \leqslant r-2, |J| = r-1, |L| = r.$$

The horizontal form h_{F_d} of F_d and the vertical form v_{F_d} are given by

$$\begin{aligned} h_{F_d} &= \widetilde{\theta}_1 \circ F_d = dx^i \otimes \left(\partial_i + \sum_{0\leqslant |I| \leqslant r-2} u^\alpha_{iI} \partial^I_\alpha + \sum_{|J|=r-1} F^\alpha_{iJ} \partial^J_\alpha \right) \\ &\quad + \sum_{0\leqslant |I| \leqslant r-2} du^\alpha_I \otimes (\partial^I_\alpha + \sum_{|J|=r-1} F^{\beta I}_{J\alpha} \partial^J_\beta), \\ v_{F_d} &= \widetilde{\theta}_2 \circ F_d = \sum_{|J|=r-1} \left(du^\alpha_J - F^\alpha_{iJ} dx^i - \sum_{0\leqslant |I| \leqslant r-2} F^{\alpha I}_{J\beta} du^\beta_I \right) \otimes \partial^I_\alpha. \end{aligned} \tag{2.9}$$

The horizontal distribution $\operatorname{Im} F_d$ on $J^{r-1}E$ is locally generated by the vector fields

$$\begin{aligned} \widetilde{\Gamma}_i &= \partial_i + \sum_{0\leqslant |I| \leqslant r-2} u^\alpha_{iI} \partial^I_\alpha + \sum_{|J|=r-1} F^\alpha_{iJ} \partial^J_\alpha, \\ \widetilde{H}^I_\alpha &= \partial^I_\alpha + \sum_{|J|=r-1} F^{\beta I}_{J\alpha} \partial^J_\beta, \quad 0 \leqslant |I| \leqslant r-2, \end{aligned} \tag{2.10}$$

or equivalently by the forms

$$\widetilde{\Psi}^\alpha_J = du^\alpha_J - \left(F^\alpha_{iJ} - \sum_{0\leqslant I \leqslant r-2} u^\beta_{iI} F^{\alpha I}_{J\beta} \right) dx^i - \sum_{0\leqslant |I| \leqslant r-2} F^{\alpha I}_{J\beta} du^\beta_I, \quad |J| = r-1. \tag{2.11}$$

3. $f(3,-1)$-structure on $J^{r-1}E$

Theorem 3.1. *A tensor field H of type $(1,1)$ on $J^{r-1}E$ which satisfies the relations*

$$\theta_1 \circ H = 0, \quad \theta_2 \circ H = \theta_2, \quad H|_{V(J^{r-1}E)} = -1_{V(J^{r-1}E)} \tag{3.1}$$

is a $f(3,-1)$-structure on $J^{r-1}E$.

Proof. The endomorphism $H\colon T(J^{r-1}E) \to T(J^{r-1}E)$ in the local chart $(U_{r-1}, x^i, u^\alpha_I)$, $0 \leqslant |I| \leqslant r-1$, has the expression

$$\begin{aligned} H = {} & \Big(H^i_j dx^j + \sum_{0\leqslant|I|\leqslant r-2} H^{i,I}_\alpha \theta^\alpha_I + \sum_{|J|=r-1} H^{i,J} du^\alpha_J \Big) \otimes \partial_i \\ & + \sum_{0\leqslant|I|\leqslant r-2} \Big(H^\beta_{I,j} dx^j + \sum_{0\leqslant|L|\leqslant r-2} H^{\beta L}_{I\alpha} \theta^\alpha_L + \sum_{|J|=r-1} H^{\beta J}_{I\alpha} du^\alpha_J \Big) \partial^I_\beta \\ & + \sum_{|J|=r-1} \Big(H^\beta_{J,j} dx^j + \sum_{0\leqslant|I|\leqslant r-2} H^{\beta I}_{J\alpha} \theta^\alpha_I + \sum_{|K|=r-1} H^{\beta K}_{J\alpha} du^\alpha_K \Big) \otimes \partial^J_\beta. \end{aligned} \tag{3.2}$$

The condition $\theta_1 \circ H = 0$ yields $H^i_j = H^{i,I}_j = H^{i,J}_\alpha = 0$; $\theta_2 \circ H = \theta_2$ implies $H^\beta_{I,j} = H^{\beta J}_{I\alpha} = 0$, $H^{\beta L}_{I\alpha} = \delta^\beta_\alpha \delta^L_I$, $0 \leqslant |I| \leqslant r-2$, $0 \leqslant |L| \leqslant r-2$, $|J| = r-1$, where

$$\delta^L_I = \delta^{1_1}_{i_1} \ldots \delta^{1_n}_{i_n}, \quad \text{for } I = (i_1, \ldots, i_n),\ L = (l_1, \ldots, l_n).$$

From the third condition (3.1) we obtain $H^{\beta K}_{J\alpha} = -\delta^\beta_\alpha \delta^K_J$, $|K| = |J| = r-1$, and consequently,

$$H = \sum_{0\leqslant|I|\leqslant r-2} \theta^\alpha_I \otimes \partial^I_\alpha + \sum_{|J|=r-1} \Big(H^\beta_{J,i} dx^i + \sum_{0\leqslant|I|\leqslant r-2} H^{\beta I}_{J\alpha} \theta^\alpha_I - du^\beta_J \Big) \otimes \partial^J_\beta. \tag{3.3}$$

In particular, we have

$$\begin{aligned} H(\partial_i) &= - \sum_{0\leqslant|I|\leqslant r-2} u^\alpha_{iI} \partial^I_\alpha + \sum_{|J|=r-1} H^\beta_{J,i} \partial^J_\beta; \\ H(\partial^I_\alpha) &= \partial^I_\alpha + \sum_{|J|=r-1} H^{\beta I}_{J\alpha} \partial^J_\beta, \quad 0 \leqslant |I| \leqslant r-2; \\ H(\partial^J_\alpha) &= -\partial^J_\alpha; \quad |J| = r-1. \end{aligned} \tag{3.4}$$

From (3.4) we obtain

$$H^2(\partial_i) = -\sum_{0\leqslant |I|\leqslant r-2} u^\alpha_{iI}\partial^I_\alpha - \sum_{|J|=r-1}\Bigg(\sum_{0\leqslant |I|\leqslant r-2} u^\alpha_{iI} H^{\beta I}_{J\alpha} + H^\beta_{J,i}\Bigg)\partial^J_\beta;$$
$$H^2(\partial^I_\alpha) = \partial^I_\alpha,\ 0\leqslant |I|\leqslant r-2;$$
$$H^2(\partial^J_\alpha) = \partial^J_\alpha,\ |J| = r-1.$$

Thus $H^3(\partial_i) = \partial_i, H^3(\partial^I_\alpha) = \partial^I_\alpha, H^3(\partial^J_\alpha) = \partial^J_\alpha$ and H defines a $f(3,-1)$-structure on $J^{r-1}E$. □

Corollary 3.2. *The eigenspace of H corresponding to the eigenvalue 1 is $\mathrm{Im}(H^2 - H) = V_{\widetilde{\pi}_{1,r-1}}(J^{r-1}E)$. The eigenspace of H corresponding to the eigenvalue 0 is $\mathrm{Im}(H^2 - I)$. The eigenspace of H corresponding to the eigenvalue (-1) is $\mathrm{Im}(H^2 + H)$.*

The subbundle

$$H'(J^{r-1}E) = \mathrm{Im}(H^2 + H) \oplus \mathrm{Im}(H^2 - I) \tag{3.5}$$

is called the weak horizontal subbundle associated to H. His generators and the vector fields

$$\overline{\Gamma}_i = \partial_i + \sum_{0\leqslant |I|\leqslant r-2} u^\alpha_{iI}\partial^I_\alpha + \sum_{|J|=r-1}\Bigg(H^\beta_{J,i} + \frac{1}{2}\sum_{0\leqslant |I|\leqslant r-2} u^\alpha_{iJ} H^{\beta I}_{J\alpha}\Bigg)\partial^J_\beta, \tag{3.6}$$
$$\overline{H}^I_\alpha = \partial^I_\alpha + \frac{1}{2}\sum_{|J|=r-1} H^{\beta I}_{J\alpha}\partial^J_\beta, \quad 0\leqslant |I|\leqslant r-2.$$

Also we have

$$T(J^{r-1}E) = H'(J^{r-1}E) \oplus V(J^{r-1}E). \tag{3.7}$$

Theorem 3.3. *Each $f(3,-1)$-structure H on $J^{r-1}E$ defined in Theorem 3.1 induces a canonical dynamical connection F_d on $J^{r-1}E$ by*

$$\mathrm{Im}\, h_{F_d} = H'(J^{r-1}E). \tag{3.8}$$

Locally, F_d is given by

$$F^\alpha_L = \sigma_L(H^\beta_{J,i}) + \frac{1}{2}\sum_{0\leqslant |I|\leqslant r-2} \sigma_L(U^\alpha_{iI} H^{\beta I}_{J\alpha}), \quad |L| = 1 + |J|, \tag{3.9}$$
$$F^{\beta I}_{J\alpha} = \frac{1}{2} H^{\beta I}_{J\alpha}; \quad 0\leqslant |I|\leqslant r-2,\ |J| = r-1.$$

P r o o f. The relation (3.9) follows from (3.6) and (2.10). □

An $f(3,-1)$-structure H on $J^{r-1}E$ defined by (3.1) is called *symmetric* if

$$\sigma_L(H^\beta_{J,i}) = H^\beta_{Ji}, \quad \forall L \text{ with } |L| = r,\ |J| = r-1.$$

Theorem 3.4. *The set of the dynamical connections on $J^{r-1}E$ and the set of the symmetric $f(3,-1)$-structures defined by* (3.1) *have the same cardinality.*

Proof. A bijection is given by

$$\text{(3.10)} \qquad F^\alpha_L = H^\beta_L + \frac{1}{2}\sum_{0\leqslant|I|\leqslant r-2} \sigma_L(u^\alpha_{iI} H^{\beta I}_{J\alpha}), \quad |L| = 1+|J|,\ |J| = r-1,$$

$$F^{\beta I}_{J\alpha} = \frac{1}{2} H^{\beta I}_{J\alpha}, \quad 0 \leqslant |I| \leqslant r-2,\ |J| = r-1,$$

or

$$\text{(3.11)} \qquad H^\beta_L = F^\beta_L - \sum_{0\leqslant|I|\leqslant r-2} \sigma_L(u^\alpha_{iI} F^{\beta I}_{J\alpha}), \quad |L| = 1+|J|,\ |J| = r-1,$$

$$H^{\beta I}_{J\alpha} = 2F^{\beta I}_{J\alpha}, \quad 0 \leqslant |I| \leqslant r-2,\ |J| = r-1.$$

□

Theorem 3.5. *Each connection of order r defines a symmetric $f(3,-1)$-structure.*

Proof. Let $h = \theta_1 \circ \Lambda = dx^i \otimes \Gamma_i$, where Γ_i is given by (2.2), be the horizontal 1-form of a connection Λ of order r. Consider the tensor field

$$\text{(3.12)} \qquad A = \sum_{p=1}^{r-1} [h, \overset{p}{J}{}^i]_{FN} \otimes d_i^{p+1},$$

where $\overset{p}{J}{}^i$ is given by (1.17) and d_i^{p+1} is given by (1.16). Using the definition of the bracket $[\ ,\ ]_{FN}$ we deduce

$$A = \sum_{p=1}^{r-1} dx^k \wedge \mathscr{L}_{\Gamma_k} \overset{p}{J}{}^i \otimes d_i^{p+1}.$$

For the Lie derivation $\mathscr{L}_{\Gamma_k}$ we have

$$\mathscr{L}_{\Gamma_k} \overset{p}{J}{}^i = \sum_{|I|=p-1} (\mathscr{L}_{\Gamma_k}\theta^\alpha_I \otimes \partial^{Ii}_\alpha + \theta^\alpha_I \otimes \mathscr{L}_{\Gamma_k}\partial^{Ii}_\alpha),\ 1 \leqslant p \leqslant r-1;$$

$$\mathscr{L}_{\Gamma_k}\theta^\alpha_I = \theta^\alpha_{Ik},\ 0 \leqslant |I| \leqslant r-2;\ \mathscr{L}_{\Gamma_k}\theta^\alpha_I = du^\alpha_{Ik} - \Lambda^\alpha_{Ikh} dx^h,\ |I| = r-2;$$

$$\mathscr{L}_{\Gamma_k}\partial^{Ii}_\alpha = [\Gamma_k, \partial^{Ii}_\alpha] = -\delta^i_k \partial^I_\alpha - \sum_{|J|=r-1} \partial^{Ii}_\alpha(\Lambda^\beta_{kJ})\partial^J_\beta,\ 0 \leqslant |I| \leqslant r-2.$$

Then we can write

$$
\begin{aligned}
A = & \sum_{p=1}^{r-1} \sum_{|I|=p-1} dx^k \wedge (\mathscr{L}_{\Gamma_k} \theta_I^\alpha \otimes \partial_\alpha^{Ii} + \theta_I^\alpha \otimes \mathscr{L}_{\Gamma_k} \partial_\alpha^{Ii}) \otimes d_i^{p+1} \\
= & \sum_{0 \leqslant |I| < r-2} dx^k \wedge (\theta_{Ik}^\alpha \otimes \partial_\alpha^{Ii} - \delta_k^i \theta_I^\alpha \otimes \partial_\alpha^I) \otimes d_i^r \\
& + \sum_{|I|=r-2} [(du_{Ik}^\alpha - \Lambda_{Ikh}^\alpha dx^h) \otimes \partial_\alpha^{Ii} - \delta_k^i \theta_I^\alpha \otimes \partial_\alpha^I)] \otimes d_i^r \\
& - \sum_{0 \leqslant |I| \leqslant r-2} \sum_{|J|=r-1} \partial_\alpha^{Ii} (\Lambda_{kJ}^\beta) dx^k \wedge \theta_I^\alpha \otimes \partial_\beta^J \otimes d_i^r .
\end{aligned}
$$

Let $\operatorname{tr} A = \sum\limits_{p=1}^{r-1} \mathscr{L}_{\Gamma_k} \overset{p}{J}{}^i dx^k (d_i^{p+1}) = \sum\limits_{p=1}^{r-1} \mathscr{L}_{\Gamma_k} \overset{p}{J}{}^k$. Then

$$
\begin{aligned}
\operatorname{tr} A = & \sum_{0 \leqslant |I| < r-2} (\theta_{Ik}^\alpha \otimes \partial^{Ik} - n \theta_I^\alpha \otimes \partial_\alpha^I) \\
& + \sum_{|I|=r-2} [(du_{Ik}^\alpha - \Lambda_{Ikh}^\alpha dx^h) \otimes \partial_\alpha^{Ik} - n \theta_I^\alpha \otimes \partial_\alpha^I] \\
& - \sum_{0 \leqslant |I| \leqslant r-2} \sum_{|J|=r-1} \partial_\alpha^{Ii} (\Lambda_{iJ}^\beta) \theta_I^\alpha \otimes \partial_\beta^J \\
= & \sum_{0 \leqslant |I| \leqslant r-2} \theta_I^\alpha \otimes \partial_\alpha^I - n \sum_{0 \leqslant |I| \leqslant r-2} \theta_I^\alpha \otimes \partial_\alpha^I \\
& + \sum_{|J|=r-1} du_J^\alpha \otimes \partial_\alpha^J - \sum_{|J|=r-1} \Lambda_{Jh} dx^h \otimes \partial_\alpha^J - \sum_{|J|=r-1} \sum_{0 \leqslant |I| \leqslant r-2} \partial_\beta^{Ii} (\Lambda_{iJ}^\alpha) \theta_I^\beta \otimes \partial_\alpha^J \\
= & (1-n)\theta_2 + \sum_{|J|=r-1} \left(du_J^\alpha - \Lambda_{Jh}^\alpha dx^h - \sum_{0 \leqslant |I| \leqslant r-2} \partial_\beta^{Ii} (\Lambda_{iJ}^\alpha) \theta_I^\beta \right) \otimes \partial_\alpha^J .
\end{aligned}
$$

Now we put

(3.13) $$H = -(n-2)\theta_2 - \operatorname{tr} A, \text{ i.e.}$$

$$H = \theta_2 + \sum_{|J|=r-1} \left(\Lambda_{Ji}^\alpha dx^i + \sum_{0 \leqslant |I| \leqslant r-2} \partial_\beta^{Ii} (\Lambda_{iJ}^\alpha) \theta_I^\beta - du_J^\alpha \right) \otimes \partial_\alpha^I .$$

H is a symmetric $f(3,-1)$-structure on $J^{r-1}E$, satisfying the condition (3.1). □

It is easy to establish the following theorems.

Theorem 3.6. *Each connection of order r defines a dynamical connection. Conversely, each dynamical connection determines a connection of order r.*

If Λ is a connection of order r then the associated dynamical connection F_d is given by

$$(3.14)\quad F^\alpha_L = \Lambda^\alpha_L + \frac{1}{2}\sum_{0\leqslant|I|\leqslant r-2} \sigma_L(u^\beta_{iI}\partial^{Ik}_\beta(\Lambda^\alpha_{kJ})), \quad |L| = 1+|J|,\ |J| = r-1;$$

$$F^{\beta I}_{J\alpha} = \frac{1}{2}\partial^{Ii}_\alpha(\Lambda^\beta_{iJ}), \quad 0\leqslant |I| \leqslant r-2,\ |J| = r-1.$$

A dynamical connection F_d determines a connection of order r given by

$$(3.15)\qquad \Lambda^\alpha_L = F^\alpha_L - \sum_{0\leqslant|I|\leqslant r-2}\sigma_L(u^\beta_{iI}F^{\alpha I}_{J\beta}),\ |L| = 1+|J|,\ |J| = r-1.$$

Theorem 3.7. *Let $\omega\colon J^1(J^{r-1}E)\to J^rE$ be the bundle morphism*

$$\omega\colon (x^i, u^\alpha_I, u^\alpha_J, u^\alpha_L, u^{\alpha I}_{J\beta}) \mapsto (x^i, u^\alpha_I, \widetilde{u}^\alpha_L),\ 0\leqslant |I|\leqslant r-2,\ |J| = r-2,\ |L| = r,$$

where

$$\widetilde{u}^\alpha_L = u^\alpha_L - \sum_{0\leqslant|I|\leqslant r-2}\sigma_L(u^\beta_{iI}u^{\alpha I}_{J\beta}),\ |L| = 1+|J|,$$

and F_d is a dynamical connection on $J^{r-1}E$. The associated connection of order r is given by

$$(3.16)\qquad \Lambda = \omega\circ F_d.$$

4. A geometric study of systems of partial differential equations of second order

A dynamical connection F_d on J^1E is locally characterized by the vector fields $\{\Gamma_i, H_\alpha, V^i_\alpha\}$, where

$$(4.1)\qquad \Gamma_i = \partial_i + u^\alpha_i\partial_\alpha + F^\alpha_{ij}V^j_\alpha,\ H_\alpha = \partial_\alpha + F^\beta_{i\alpha}V^i_\beta,\ V^i_\alpha = \partial^i_\alpha,$$

with $F^\alpha_{ij} = F^\alpha_{ji}$. The 1-forms associated with (4.1) are $\{dx^i, \theta^\alpha, \Psi^\alpha_i\}$, where

$$(4.2)\qquad \theta^\alpha = du^\alpha - u^\alpha_i dx^i;$$

$$\Psi^\alpha_i = du^\alpha_i - F^\alpha_{i\beta}du^\beta - (F^\alpha_{ij} + u^\beta_i F^\alpha_{j\beta})dx^j = du^\alpha_i - F^\alpha_{i\beta}\theta^\beta - F^\beta_{i\beta}dx^j.$$

For the vector fields (4.1) the following relations are satisfied:

$$
\begin{aligned}
(4.3)\qquad [\Gamma_i,\Gamma_j] &= T^\alpha_{ijk}V^k_\alpha,\ T^\alpha_{ijk}=\Gamma_i(F^\alpha_{jk})-\Gamma_j(F^\alpha_{ik}),\\
[\Gamma_i,H_\alpha] &= -F^\beta_{i\alpha}H_\beta+T^\gamma_{ik\alpha}V^k_\gamma,\ T^\gamma_{ik\alpha}=\Gamma_i(F^\gamma_{k\alpha})+F^\beta_{i\alpha}F^\gamma_{k\beta}-H_\alpha(F^\gamma_{ik}),\\
[\Gamma_i,V^j_\alpha] &= -\delta^j_iH_\alpha+T^{j\gamma}_{ik\alpha}V^k_\gamma,\ T^{j\gamma}_{ik\alpha}=\delta^j_iF^\gamma_{k\alpha}-\partial^j_\alpha(F^\gamma_{ik}),\\
[H_\alpha,H_\beta] &= T^\gamma_{\alpha\beta k}V^k_\gamma,\ T^\gamma_{\alpha\beta k}=H_\alpha(F^\gamma_{k\beta})-H_\beta(F^\gamma_{k\alpha}),\\
[V^i_\alpha,V^j_\beta] &= 0.
\end{aligned}
$$

For the forms (4.2) we have

$$
\begin{aligned}
d\theta^\alpha &= -\Psi^\alpha_i\wedge dx^i-F^\alpha_{i\beta}\theta^\beta\wedge dx^i,\\
d\Psi^\alpha_i &= \frac{1}{2}T^\alpha_{jki}dx^j\wedge dx^k+T^\alpha_{ki\beta}\theta^\beta\wedge dx^k-\frac{1}{2}T^\alpha_{\beta\gamma i}\theta^\beta\wedge\theta^\gamma\\
&\quad -\partial^k_\gamma(F^\alpha_{i\beta})\Psi^\gamma_k\wedge\theta^\beta+T^{k\alpha}_{ji\beta}\Psi^\beta_k\wedge dx^j.
\end{aligned}
$$

The tensor field of type (1,1) associated with F_d (see 3.11) is given by

$$
(4.5)\qquad H=\theta^\alpha\otimes\partial_\alpha+(H^\alpha_{ij}dx^j+H^\alpha_{i\beta}du^\beta-du^\alpha_i)\otimes V^i_\alpha,
$$

where

$$
(4.6)\qquad H^\alpha_{ij}=F^\alpha_{ij}-(u^\beta_iF^\alpha_{j\beta}+u^\beta_jF^\alpha_{i\beta}),\ H^\alpha_{i\beta}=2F^\alpha_{i\beta}.
$$

With respect to the basis $\{\Gamma_i,H_\alpha,V^i_\alpha\}$ and the co-basis $\{dx^i,\theta^\alpha,\Psi^\alpha_i\}$ the tensor field H has the form

$$
(4.7)\qquad H=\theta^\alpha\otimes\partial_\alpha+[(F^\alpha_{i\beta}u^\beta_j-F^\alpha_{j\beta}u^\beta_i)dx^j+F^\alpha_{i\beta}\theta^\beta-\Psi^\alpha_i]\otimes V^i_\alpha.
$$

From (4.7) we obtain

$$
(4.8)\qquad H(\Gamma_i)=(F^\alpha_{i\beta}u^\beta_j-F^\alpha_{j\beta}u^\beta_i)V^i_\alpha, H(H_\alpha)=H_\alpha+F^\beta_{i\alpha}V^i_\beta, H(V^i_\alpha)=-V^i_\alpha
$$

and

$$
\begin{aligned}
(4.9)\qquad {}^tH(dx^i) &= dx^i(H)=0,\ t_H(\theta^\alpha)=\theta^\alpha(H)=\theta^\alpha,\\
{}^tH(\Psi^\alpha_i) &= -\Psi^\alpha_i+(F^\alpha_{i\beta}u^\beta_j-F^\alpha_{j\beta}u^\beta_i)dx^j+F^\alpha_{i\beta}\theta^\beta.
\end{aligned}
$$

Let now $\omega=f(x)dx^1\wedge\ldots\wedge dx^n$ be a volume form on B and $\omega_i=\iota_{\partial_i}\omega$ (the interior product with respect to ∂_i). Then

$$
d\omega_i=f^{-1}(\partial_if)\omega,\ dx^j\wedge\omega_i=\delta^j_i\omega.
$$

Consider $\widetilde{J}\colon\ J^1E \to T^*(J^1E) \wedge \Lambda^{n-1}(B) \otimes VT(J^1E)$ defined by

$$\widetilde{J} = \theta^\alpha \wedge \omega_i \otimes V^i_\alpha; \tag{4.10}$$

then

$$\operatorname{Im} \widetilde{J} = \Lambda^{n-1}(B) \otimes VT(J^1E),\ \ \widetilde{J} \circ \widetilde{J} = 0.$$

We call *the Poincaré-Cartan form* of a function $L \in \mathscr{F}(J^1E)$ the n-form θ_L defined by

$$\theta_L = \widetilde{J}(L) + L\omega, \tag{4.11}$$

where $\widetilde{J}(L) +^t \widetilde{J}(dL) = dL(\widetilde{J})$. In a local fibered chart we have

$$\theta_L = \partial^i_\alpha(L)\theta^\alpha \wedge \omega_i + L\omega. \tag{4.12}$$

Now we consider the $(n+1)$-form

$$\Omega_L = d\theta_L. \tag{4.13}$$

Using a dynamical connection F_d on J^1E, the relations (4.4) and the fact that

$$df = \Gamma_i(f)dx^i + H_\alpha(f)\theta^\alpha + \partial^i_\alpha(f)\Psi^\alpha_i,\ \forall\ f \in \mathscr{F}(J^1E)$$

we obtain

$$\begin{aligned}\Omega_L = {} & \partial^j_\beta(\partial^i_\alpha L)\Psi^\beta_j \wedge \theta^\alpha \wedge \omega_i - \frac{1}{2}[H_\beta(\partial^i_\alpha L) \\ & - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i - [\Gamma_k(\partial^k_\alpha L) - \partial_\alpha L - f^{-1}(\partial_i f)\delta^i_\alpha L]\theta^\alpha \wedge \omega.\end{aligned} \tag{4.14}$$

Denoting $A^{ij}_{\alpha\beta} = \partial^i_\alpha(\partial^j_\beta L)$ we have the relations

$$A^{ij}_{\alpha\beta} = A^{ji}_{\alpha\beta} = A^{ij}_{\beta\alpha}. \tag{4.15}$$

We now make a general remark.

Remark. Let T be a tensor field of type (1,1) on a differential manifold M and let Ω be a 3-form on M. We can define in terms of T the following 3-forms on M:

$$\begin{aligned}(T^{(1)}\Omega)(X,Y,Z) &= \Omega(TX,Y,Z) + \Omega(X,TY,Z) + \Omega(X,YTZ), \\ (T^{(2)}\Omega)(X,Y,Z) &= \Omega(TX,TY,Z) + \Omega(TX,Y,TZ) + \Omega(X,TY,TZ).\end{aligned} \tag{4.16}$$

On the other hand, we can associate with T an antiderivation δ_T of degree zero on the algebra of forms on M. δ_T is uniquely determined by the conditions

$$\delta_T f = 0,\ \forall f \in \mathscr{F}(M); \quad \delta_T \theta = {}^t T\theta,\ \forall\ \theta \in \Lambda^1(M).$$

For a k-form $\omega \in \Lambda^k(M)$ we have

$$(\delta_T\omega)(X_1,\dots,X_k) = (T^{(1)}\omega)(X_1,\dots,X_k). \tag{4.17}$$

If we consider the operator d_T given by

$$d_T = \delta_T \circ d - d \circ \delta_T \tag{4.18}$$

then we have

$$\begin{gathered} d\circ d_T = -d_T \circ d,\ d_T^2 \circ d = d \circ d_T^2, \\ \iota_X \circ d_T + d_T \circ \iota_X = \mathscr{L}_{TX} + [\delta_T, \mathscr{L}_X]. \end{gathered} \tag{4.19}$$

Theorem 4.1. *The $(n+1)$-form Ω_L from (4.13) has the decomposition*

$$\Omega_L = \Omega_L^c + H^{(2)}\Omega_L - H^{(1)}\Omega_L, \tag{4.20}$$

where

$$\Omega_L^c = A^{ij}_{\alpha\beta}\Psi^\alpha_i \wedge \theta^\beta \wedge \omega_j, \tag{4.21}$$

$$\begin{aligned} H^{(1)}\Omega_L = &- A^{ij}_{\alpha\beta}F^\alpha_{i\gamma}\theta^\gamma \wedge \theta^\beta \wedge \omega_j - [H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i \\ &- [\Gamma_k(\partial^k_\alpha L) - \partial_\alpha L - f^{-1}(\partial_k f)\partial^k_\alpha L]\theta^\alpha \wedge \omega, \end{aligned} \tag{4.22}$$

$$H^{(2)}\Omega_L = - A^{ij}_{\alpha\beta}F^\alpha_{i\gamma}\theta^\gamma \wedge \theta^\beta \wedge \omega_j - \frac{1}{2}[H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i. \tag{4.23}$$

P r o o f. By using the above remark and (4.9) we have

$$\begin{aligned} H^{(1)}\Omega_L = {}& A^{ij}_{\alpha\beta}[{}^tH(\Psi^\alpha_i) \wedge \theta^\beta \wedge \omega_j + \Psi^\alpha_i \wedge^t H(\theta^\beta) \wedge \omega_j + \Psi^\alpha_i \wedge \theta^\beta \wedge H^{(1)}(\omega_j)] \\ &- \frac{1}{2}[H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)][{}^tH(\theta^\alpha) \wedge \theta^\beta \wedge \omega_i + \theta^\alpha \wedge^t H(\theta^\beta) \wedge \omega_i \\ &+ \theta^\alpha \wedge \theta^\beta \wedge H^{(1)}(\omega_i)] - [\Gamma_k(\partial^k_\alpha L) - \partial_\alpha L - f^{-1}(\partial_k f)\partial^k_\alpha][{}^tH(\theta^\alpha) \wedge \omega \\ &+ \theta^\alpha \wedge H^{(1)}(\omega)] \\ = {}& A^{ij}_{\alpha\beta}[-\Psi^\alpha_i + (F^\alpha_{i\gamma}u^\gamma_k - F^\alpha_{k\gamma}u^\gamma_k)dx^k + F^\alpha_{i\gamma}\theta^\gamma + \Psi^\alpha_i] \wedge \theta^\beta \wedge \omega_j \\ &- \frac{1}{2}[H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)](\theta^\alpha \wedge \theta^\beta \wedge \omega_i + \theta^\alpha \wedge \theta^\beta \wedge \omega_i) \\ &- [\Gamma_k(\partial^k_\alpha L) - \partial_\alpha L - f^{-1}(\partial_k f)\partial^k_\alpha] \wedge \theta^\alpha \wedge \omega \\ = {}& -(A^{ij}_{\alpha\beta}F^\alpha_{i\gamma}u^\gamma_j - A^{ij}_{\alpha\beta}F^\alpha_{i\gamma}u^\gamma_j)\theta^\beta \wedge \omega_j - A^{ij}_{\alpha\beta}F^\alpha_{i\gamma}\theta^\gamma \wedge \theta^\beta \wedge \omega_j \\ &- [H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i - [\Gamma_k(\partial^k_\alpha L) - \partial_\alpha L \\ &- f^{-1}(\partial_k f)\partial^k_\alpha L]\theta^\alpha \wedge \omega. \end{aligned}$$

Similarly,

$$H^{(2)}\Omega_L = A^{ij}_{\alpha\beta}\,{}^tH(\Psi^\alpha_i) \wedge {}^t H(\theta^\beta) \wedge \omega_j - \frac{1}{2}[H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i$$
$$= -A^{ij}_{\alpha\beta}F^\alpha_{i\gamma}\theta^\gamma \wedge \theta^\beta \wedge \omega_j - \frac{1}{2}[H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i.$$

Then

$$H^{(2)}\Omega_L - H^{(1)}\Omega_L = \frac{1}{2}[H_\beta(\partial^i_\alpha L) - H_\alpha(\partial^i_\beta L)]\theta^\alpha \wedge \theta^\beta \wedge \omega_i$$
$$+ [\Gamma_K(\partial^k_\alpha L) - \partial_\alpha L - f^{-1}(\partial_k f)\partial^k_\alpha L]\theta^\alpha \wedge \omega.$$

If Ω^c_L is given by (4.21) then (4.20) is verified. □

The above theorem suggests the following definition:
A dynamical connection F_d is said to be *compatible with* L if $H^{(1)}\Omega_L = H^{(2)}\Omega_L$.

Theorem 4.2. *A dynamical connection F_d is compatible with L iff the following conditions are satisfied:*

(4.24) $$A^{ij}_{\alpha\beta}F^\alpha_{ij} + B_\beta = 0,\ A^{ij}_{\alpha\beta}F^\alpha_{j\gamma} = \frac{1}{2}\partial^i_\beta B_\gamma + R^i_{\beta\gamma},$$

where

(4.25) $$B_\alpha = \partial_k\partial^k_\alpha L + u^\beta_k\partial_\beta u^k_\alpha L - \partial_\alpha L + f^{-1}(\partial_k f)\partial^k_\alpha L$$

and

$$R^i_{\beta\gamma} = R^i_{\gamma\beta}.$$

P r o o f. The definition yields

(4.26) $$\partial_k(\partial^k_\alpha L) + u^\beta_k\partial_\beta\partial^k_\alpha L - \partial_\alpha L + f^{-1}(\partial_k f)\partial^k_\alpha L + A^{ij}_{\alpha\beta}F^\beta_{ij} = 0,$$
$$\partial_\beta\partial^i_\alpha L - \partial_\alpha\partial^i_\beta sL + F^\gamma_{k\beta}A^{ki}_{\gamma\alpha} - F^\gamma_{k\alpha}A^{ki}_{\gamma\beta} = 0;$$

by (4.25) we obtain

$$A^{ij}_{\alpha\beta}F^\alpha_{ij} + B_\beta = 0$$

and

(4.27) $$\partial^i_\beta B_\alpha = \partial_k A^{ik}_{\beta\alpha} + \partial_\beta\partial^i_\alpha L - \partial_\alpha\partial^i_\beta L + u^\gamma_k\partial_\gamma A^{ik}_{\beta\alpha} + f^{-1}(\partial_k f)A^{ik}_{\beta\alpha}.$$

From (4.27) we obtain

$$\partial_\beta\partial^i_\alpha L - \partial_\alpha\partial^i_\beta L = \partial^i_\beta B_\alpha - \partial_k A^{ik}_{\alpha\beta} - u^\gamma_k\partial_\gamma A^{ik}_{\alpha\beta} + f^{-1}(\partial_k f)A^{ik}_{\alpha\beta},$$
$$\partial_\alpha\partial^i_\beta L - \partial_\beta\partial^i_\alpha L = \partial^i_\alpha B_\beta - \partial_k A^{ik}_{\alpha\beta} - u^\gamma_k\partial_\gamma A^{ik}_{\alpha\beta} + f^{-1}(\partial_k f)A^{ik}_{\alpha\beta}$$

and

$$\partial_\beta\partial^i_\alpha L - \partial_\alpha\partial^i_\beta L = \frac{1}{2}(\partial^i_\beta B_\alpha - \partial^i_\alpha B_\beta). \tag{4.28}$$

(4.26) and (4.28) imply

$$\frac{1}{2}(\partial^i_\beta B_\alpha - \partial^i_\alpha B_\beta) + A^{ki}_{\gamma\alpha}F^\gamma_{k\beta} - A^{ki}_{\gamma\beta}F^\gamma_{k\alpha} = 0$$

or

$$\Big(\frac{1}{2}\partial^i_\beta B_\alpha - A^{ki}_{\gamma\beta}F^\gamma_{k\alpha}\Big) - \Big(\frac{1}{2}\partial^i_\beta B_\alpha - A^{ki}_{\gamma\alpha}F^\gamma_{k\beta}\Big) = 0.$$

Therefore

$$A^{ij}_{\alpha\beta}F^\alpha_{j\gamma} = \frac{1}{2}\partial^i_\beta B_\alpha + R^i_{\beta\gamma},\ R^i_{\beta\gamma} = R^i_{\gamma\beta}.$$

A function $L \in \mathscr{F}(J^1E)$ is called *regular* is $\det\|A^{ij}_{\alpha\beta}\| \neq 0$. Let us note that $\|\widetilde{A}^{\alpha\beta}_{ij}\| = \|A^{ij}_{\alpha\beta}\|^{-1}$. □

Theorem 4.3. *If L is regular then the connections F_d compatible with L are given by*

$$F^\alpha_{ij} = \widetilde{A}^{\alpha\beta}_{ih}\Big(P^h_{\beta j} - \frac{1}{n}\delta^h_j B_\beta\Big), \tag{4.29}$$
$$F^\alpha_{i\beta} = \widetilde{A}^{\alpha\gamma}_{ij}\Big(R^j_{\gamma\beta} + \frac{1}{2}\partial^j_\gamma B_\beta\Big),$$

where $P(P^h_{\beta j})$ is a tensor field of type $(1,2)$ with Trace $P_\alpha = 0$ and $(\delta^h_k\delta^j_i - \delta^h_i\delta^j_k)P^l_{\alpha j} = 0$; $R = (R^i_{\alpha\beta})$ is a symmetric tensor field of type $(1,2)$.

P r o o f. We consider the system of linear equations

$$A^{ij}_{\alpha\beta}F^\alpha_{ik} + \frac{1}{n}\delta^j_k B_\beta = P^j_{\beta k}. \tag{4.30}$$

Setting $j = k$ and summing one obtains the first relation (4.24) if Trace $P_\alpha = 0$. From (4.30) we deduce the first relation (4.29). The symmetry of F_{ij} implies

$$\widetilde{A}^{\alpha\beta}_{ij}(P^j_{\beta k} - \frac{1}{n}\delta^j_k B_\beta) = \widetilde{A}^{\alpha\beta}_{kj}\Big(P^j_{\beta i} - \frac{1}{n}\delta^j_i B_\beta\Big),$$

which leads to $(\delta^h_k\delta^j_i - \delta^h_i\delta^j_k)P^l_{\alpha j} = 0$. The second relation (4.29) results from (4.24). □

References

[1] *I. M. Anderson*: Aspects of the inverse problem to the calculus of variations. Arch. Math. (Brno) *24* (1988), 181–202.

[2] *M. Ferraris, M. Francaviglia*: On the Global Structure of Lagrangian and Hamiltonian Formalisms in Higher Order Calculus of Variations, Proceedings of the Meeting "Geometry and Physics". Florence, October 12–15 (1982), 43–70.

[3] *H. Goldschmidt, S. Stemberg*: The Hamilton-Cartan Formalism in the Calculus of Variations. Ann. Inst. Fourier, Grenoble *23* (1973), 203–267.

[4] *M. J. Gotay*: A multisymplectic Framework for Classical Field Theory aud the Calculus of Variations. I. Covariant Hamiltonian Formalism, Mechanics, Analysis and Geometry 200 Years after Lagrange, Amsterdam, 1990.

[5] *A. Vondra*: Semisprays, connections and regular equations in higher order mechanics. Proc. Conf. Diff. Geom. and Its. Appl., World Scientific, Singapore (1990), 276–287.

[6] *A. Vondra*: Some connections related to the geometry of regular higher-order dynamics. Sbornik VA Brno, Řada "B" *2* (1992), 7–18.

[7] *A. Vondra*: Natural Dynamical Connections. Czechoslovak Math. J., Praha *41(116)* (1991), 724–730.

Authors' address: Universitatea de Vest din Timişoara, Facultatea de Matematică, B-dul V. Părvan 4, 1900 Timişoara, Romania.