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COMPLETE GENERATORS AND MAXIMAL COMPLETIONS OF MV-ALGEBRAS

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MV-algebras are also called algebras of the infinite valued Łukasiewicz logic [2]. If $\mathcal{A}$ is an MV-algebra, then the notion of a set of generators of $\mathcal{A}$ has the usual meaning.

By considering complete MV-algebras (cf., e.g., [2], [8]) we can define the notion of the complete homomorphism and of the set of the complete generators. (For definitions, cf. Section 1 below.)

Analogous definitions have been applied for complete Boolean algebras, complete lattice ordered groups and complete vector lattices.

In [5] it was proved that if α is an infinite cardinal, then there exists no free complete Boolean algebra with α free complete generators. A similar result was proved in [10] for complete lattice ordered groups and in [11] for complete vector lattices.

In the present paper we show that an analogous result is true also in the case of complete MV-algebras.

It is well-known that each MV-algebra $\mathcal{A}$ can be constructed by means of an appropriately chosen abelian lattice ordered group G with a strong unit u (cf. [12]).

There exist exactly three nonisomorphic types of lattice ordered groups G with one generator. In each of these cases G is complete. If X is a set of nonzero orthogonal elements of G, then $\operatorname{card} X \leqslant 2$. Analogous results hold for Boolean algebras with one generator.

In view of the above mentioned relation between MV-algebras and abelian lattice ordered groups we can ask whether analogous results are valid for MV-algebras with one generator. The answer is "No".

An MV-algebra with one generator need not be complete; moreover, it need not be archimedean. There exist infinitely many nonisomorphic complete MV-algebras with

one complete generator. If $\mathcal{A}$ is an MV-algebra with one generator (or a complete MV-algebra with one complete generator) and if X is an orthogonal subset of $\mathcal{A}$, then the set X can be infinite.

Further we investigate maximal completions of MV-algebras. The analogous notion for abelian lattice ordered groups was studied in [3] and [6]. It will be shown that each MV-algebra possesses a unique maximal completion.

1. Preliminaries

We apply the terminology and notation from [7]. Thus an MV-algebra is a system $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$, where A is a nonempty set, $\oplus, *$ are binary operations, $\neg$ is a unary operation and $0, 1$ are nulary operations on A such that the identities (m_1)–(m_9) from [7] are satisfied.

We quote the following results which will be needed in the sequel.

1.1. Theorem. (Cf. [4].) *Let $\mathcal{A}$ be an MV-algebra. For each $x, y \in A$ put $x \vee y = (x * \neg y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Then $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge)$ is a distributive lattice with the least element 0 and the greatest element 1.*

1.2. Theorem. (Cf. [12].) *Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For each a and b in A we put*

$$a \oplus b = (a + b) \wedge u,\ \neg a = u - a,\ 1 = u,\ a * b = \neg(\neg a \oplus \neg b).$$

Then $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ is an MV-algebra.

The MV-algebra $\mathcal{A}$ from 1.2 will be denoted by $\mathcal{A}_0(G, u)$.

1.3. Theorem. (Cf. [12].) *Let $\mathcal{A}$ be an MV-algebra. Then there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \mathcal{A}_0(G, u)$.*

Let us remark that if $\mathcal{A}$ and G are as in 1.2, then the partial order on A inherited from G is the same as the partial order on A defined by means of 1.1.

An MV-algebra $\mathcal{A}$ is called complete if the lattice $\mathcal{L}(\mathcal{A})$ is complete.

For $a \in A$ we put $1 \cdot a = a$ and for $n > 1$ we define by induction $n \cdot a = a \oplus (n-1) \cdot a$. The MV-algebra $\mathcal{A}$ is called archimedean, if there exists no $a \in A$ such that $n \cdot a < (n + 1) \cdot a < u$ for each positive integer n. (A formally different but equivalent definition was introduced in [9].)

Let A_1 be a nonempty subset of A which is closed with respect to the operations $\oplus$, $*$, $\neg$, 0, 1. Then $\mathcal{A}_1 = (A_1, \oplus, *, \neg, 0, 1)$ is a *subalgebra* of $\mathcal{A}$. If no misunderstanding can occur, then we do not distinguish between $\mathcal{A}_1$ and A_1.

Suppose that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$. If $\mathcal{L}(\mathcal{B})$ is a closed sublattice of $\mathcal{L}(\mathcal{A})$, then $\mathcal{B}$ is said to be a closed subalgebra of $\mathcal{A}$.

The notion of homomorphism of MV-algebras has the usual meaning. A homomorphism φ of an MV-algebra $\mathcal{A}$ into an MV-algebra $\mathcal{B}$ is said to be complete if it satisfies the following condition (c) and the condition (c') dual to (c).

(c) Whenever $\{a_i\}_{i\in I} \subseteq A, a \in A$ and $a = \bigvee_{i\in I} a_i$ is valid in $\mathcal{L}(\mathcal{A})$, then $\varphi(a) = \bigvee_{i\in I} \varphi(a_i)$ is valid in $\mathcal{L}(\mathcal{B})$.

We apply the following standard definitions:

Let X be a subset of A. If each subalgebra $\mathcal{B}_1$ of $\mathcal{A}$ with $X \subseteq \mathcal{B}_1$ coincides with $\mathcal{A}$, then X is called *a system of generators* of $\mathcal{A}$. If, moreover, for each MV-algebra $\mathcal{C}$, each mapping ψ of X into the underlying set C of $\mathcal{C}$ can be extended to a homomorphism φ of $\mathcal{A}$ into $\mathcal{C}$, then X is a set of *free generators* of $\mathcal{A}$.

For the case of complete MV-algebras we modify the above definition as follows.

Let $\mathcal{A}$ be a complete MV-algebra and let $X \subseteq A$. If for each closed subalgebra $\mathcal{B}_1$ of $\mathcal{A}$ with $X \subseteq \mathcal{B}$, the relation $\mathcal{B}_1 = \mathcal{A}$ is valid, then X is said to be *a system of complete generators* of $\mathcal{A}$.

If, moreover, for each complete MV-algebra $\mathcal{C}$, each mapping $\psi\colon X \to C$ can be extended to a complete homomorphism of $\mathcal{A}$ into $\mathcal{C}$, then X is a system of *free complete generators* of $\mathcal{A}$. In such a case we also say that $\mathcal{A}$ is a free complete MV-algebra with α free complete generators, where $\alpha = \operatorname{card} X$.

We will use analogous notions for complete lattice ordered groups and for complete Boolean algebras.

2. Generators and complete generators

We need the following result (cf. [3]).

2.1. Proposition. *Let α be an infinite cardinal. There exists a complete Boolean algebra B_α which satisfies the following conditions:*

(i) $\operatorname{card} B_\alpha \geqslant \alpha$.

(ii) *There exists a denumerable system X of complete generators of B_α.*

In fact, this is also a consequence of Theorem K in [13], p. 157. (It suffices to consider the Dedekind completion of the Boolean algebra from Theorem K; in [13] it is remarked that the method of constructing this Boolean algebra is due to Hales [5].)

2.2. Proposition. *Let α be an infinite cardinal. There exists a complete MV-algebra $\mathcal{A}$ such that*

(i) $\operatorname{card} A \geqslant \alpha$,

(ii) *there exists a denumerable system X of complete generators of $\mathcal{A}$.*

P r o o f. Let $B = B_\alpha$ be as in 2.1. We consider the vector lattice E of all elementary Carathéodory functions on B and then we construct the lattice ordered group G as in the concluding part of [7]. Put $\mathcal{A} = \mathcal{A}_0(G; u)$. Hence the lattice $\mathcal{L}(\mathcal{A})$ coincides with the Boolean algebra B. Thus $\operatorname{card} A \geqslant \alpha$. Let X be as in 2.1. Then X is a system of complete generators of $\mathcal{A}$. $\square$

2.3. Theorem. *Let β be an infinite cardinal. There exists no free complete MV-algebra with β free complete generators.*

P r o o f. By way of contradiction, assume that $\mathcal{A}_\beta$ is a free complete MV-algebra with a system X_β of free complete generators such that $\operatorname{card} X_\beta = \beta$. Let A_β be the underlying set of $\mathcal{A}_\beta$. There exists a cardinal α with $\alpha > \operatorname{card} A_\beta$. Let $\mathcal{A}$ be as in 2.2.

There exists a denumerable subset X_1 of X_β. Hence there is an injective mapping ψ of X_1 onto X. For each $x_\beta \in X_\beta \backslash X_1$ we put $\psi_1(x_\beta) = 0$; for $x_\beta \in X_1$ we set $\psi_1(x_\beta) = \psi(x_\beta)$. According to the assumption there exists a complete homomorphism φ of $\mathcal{A}_\beta$ into $\mathcal{A}$ such that φ is an extension of ψ_1. Thus $\varphi(X_1) = X$ and hence from 2.2 (ii) we obtain that $\varphi(\mathcal{A}_\beta) = \mathcal{A}$. Therefore $\operatorname{card} A_\beta \geqslant \operatorname{card} A \geqslant \alpha$, which is a contradiction. $\square$

The additive group of all integers with the natural linear order will be denoted by $\mathbb{Z}$. The free lattice ordered group with one free generator is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. (Cf. [1], Chap. XIII, §4.) As a free generator we can take either the element $(1, -1)$ or the element $(-1, 1)$. From this we immediately obtain:

2.4. Lemma. *Let G_1 be a lattice ordered group with one generator. Then G_1 is isomorphic to some of the following lattice ordered groups: $\{0\}$, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$. In each of these three cases G_1 is complete.*

2.5. Example. Consider the lattice ordered group $\mathbb{Z}$ and a positive integer n. Put $u = n$ and let us construct the MV-algebra $\mathcal{A} = \mathcal{A}_n$ as in Theorem 1.2. Then $\mathcal{A}_n$ is a complete MV-algebra; the element 1 of $\mathcal{A}_n$ is a generator of $\mathcal{A}_n$. If $n(1)$ and $n(2)$ are distinct positive integers, then $\mathcal{A}_{n(1)}$ fails to be isomorphic to $\mathcal{A}_{n(2)}$. Hence the situation concerning MV-algebras with one generator essentially differs from that concerning lattice ordered groups with one generator.

2.6. Example. Let $G = \mathbb{Z} \circ \mathbb{Z}$ (where $\circ$ denotes the operation of lexicographic product). Put $u = (1, 0)$. Then u is a strong unit of G and hence we can construct the MV-algebra $\mathcal{A}$ according to 1.2. It is obvious that $\mathcal{A}$ is not complete; moreover,

it is not archimedean. The underlying set A of $\mathcal{A}$ consists of all elements $(m,n) \in G$ such that either (i) $m = 0$, or (ii) $m = 1$ and $n \leqslant 0$.

Put $x = (0,1)$. If $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}$ with the underlying set A_1 and if $x \in A_1$, then clearly $(0,n) \in A_1$ for each $n \geqslant 0$; hence $(1,-n) = u - (0,n)$ also belongs to A_1. Thus $A_1 = A$ and hence x is a generator of $\mathcal{A}$.

2.7. Example. For each positive integer n let $G_n = \mathbb{Z}$ and $G = \prod_{n=1}^{\infty} G_n$. If $x \in G$, then we denote by x_n the component of x in G_n. Let $u \in G$ be such that $u_n = n$ for each positive integer n. The convex ℓ-subgroup of G which is generated by u will be denoted by G'. Hence u is a strong unit of G'. We can construct the MV-algebra $\mathcal{A} = \mathcal{A}_0(G', u)$ with the underlying set A. It is obvious that G' is complete, hence in view of [8], 1.1, $\mathcal{A}$ is complete as well. There exists $x \in A$ such that $x_n = 1$ for each $n \in \mathbb{N}$.

Let $\mathcal{A}_1$ be a closed subalgebra of $\mathcal{A}$. Suppose that A_1 is the underlying set of $\mathcal{A}_1$ and $x \in A_1$. Hence $u - x \in A_1$. Put $(u-x) \oplus (u-x) = y$. Then in view of 1.2,

$$y = ((u-x) + (u-x)) \wedge u = (2u - 2x) \wedge u,$$

whence

$$y_n = (2n-2) \wedge n$$

for each $n \in \mathbb{N}$. Therefore

$$y_n = \begin{cases} 0 & \text{if } n = 1, \\ n & \text{if } n > 1. \end{cases}$$

Put $x^1 = u - y$. We have $y \in A_1$, hence $x^1 \in A_1$. Clearly $x^1_1 = 1$ and $x^1_n = 0$ for $n > 1$.

Further we have

$$(u-x)_1 = 0 = (x - x^1)_1,$$

and

$$(u-x)_n = u_{n-1}, \quad (x - x^1)_n = x_{n-1}$$

for $n > 1$. Thus if the elements u and x in the above calculation are replaced by $u - x$ and $x - x^1$, respectively, then we obtain that there exists $x^2 \in A_1$ such that $x^2_2 = 1$ and $x^2_n = 0$ for each $n \in \mathbb{N} \setminus \{2\}$.

By applying the obvious induction we conclude that for each $m \in \mathbb{N}$ there is x^m in A_1 such that $x^m_m = 1$ and $x^m_n = 0$ whenever $n \neq m$.

Let $m \in \mathbb{N}$. We put $z^{m1} = x^m$. If $1 < k \in \mathbb{N}$, $k \leqslant m$, then we define by induction

$$z^{m,k} = z^{m,k-1} \oplus z^{m1}.$$

Thus $z^{m,k} \in A_1$ for $k = 1, 2, \dots, m$. In view of 1.2 we easily verify that $z_n^{m,k} = 0$ if $n \neq m$, and $z_m^{m,k} = k$.

Let $t \in A$. Hence $t_m \leqslant m$ for each $m \in \mathbb{N}$. Then

$$t = \bigvee_{m \in \mathbb{N}} z^{m,t_m} \tag{1}$$

is valid in $\mathcal{A}$. Since $\mathcal{A}_1$ is a closed subalgebra of $\mathcal{A}$, we obtain that t belongs to A_1. Thus $\mathcal{A} = \mathcal{A}_1$.

Therefore $\mathcal{A}$ is a complete MV-algebra having one complete generator x. The subset $\{x^n\}_{n \in \mathbb{N}}$ of A is orthogonal and infinite. Let us also remark that the cardinality of A equals the power of the continuum.

2.8. Example. Let us apply the same notation as in 2.7. Further let $\mathcal{A}_0$ be the subalgebra of $\mathcal{A}$ generated by the element x and let A_0 be the underlying set of A. Thus $\operatorname{card} A_0 \leqslant \aleph_0$. Moreover, all elements $z^{m,k}$ $(m \in \mathbb{N}, 1 \leqslant k \leqslant m)$ belong to A_0. The MV-algebra $\mathcal{A}_0$ fails to be complete.

This can be verified as follows. By way of contradiction, suppose that $\mathcal{A}_0$ is complete. Since $\operatorname{card} A_0 < \operatorname{card} A$, there exists $t \in A \setminus A_0$. Consider the system $\{z^{m,t_m}\}_{m \in \mathbb{N}} = S$. Thus there exists $t' \in A_0$ such that $t' = \sup S$ is valid in $\mathcal{A}_0$. Then $t'_m \geqslant z_m^{m,t_m} = t_m$ for each $m \in \mathbb{N}$, hence $t' \geqslant t$. Since $t \notin A_0$, we conclude that $t' > t$. Therefore there exists $m \in \mathbb{N}$ with $t'_m \geqslant t_m + 1$.

Further there exists t'' in A_0 such that

$$t'' \oplus x^m = t'.$$

Then $t'' < t'$ and t'' is an upper bound of the system S, which is a contradiction.

3. Maximal completions

For a subset X of a lattice L we denote by X^u and X^ℓ, respectively, the set of all upper bounds and the set of all lower bounds of X in L. Let $d(L)$ be the system of all sets $(X^u)^\ell$, where X runs over the system of all nonempty upper bounded subsets of L. The system $d(L)$ is partially ordered by the set-theoretical inclusion. Then $d(L)$ is a conditionally complete lattice.

The mapping $\varphi\colon L \to d(L)$ defined by

$$\varphi(x) = (\{x\}^u)^\ell \quad \text{for each} \quad x \in L$$

is an isomorphism of L into $d(L)$. When no misunderstanding can occur we will identify x with $\varphi(x)$ for each $x \in L$. Then L turns out to be a sublattice of $d(L)$.

Moreover, if X_1 is a subset of L and if x_1 is the supremum of X_1 in L, then x_1 is also the supremum of X_1 in $d(L)$. The corresponding dual assertion is valid as well. If X is a nonempty upper bounded subset of L, then the relation

$$(X^u)^\ell = \bigvee x_i \quad (x_i \in X) \tag{1}$$

holds in $d(L)$.

Let $a, b \in L$, $a < b$, and let L_1 be the interval $[a, b]$ in L. For $\emptyset \neq X \subseteq L_1$ we denote

$$X^{u(1)} = X^u \cap L_1, \quad X^{\ell(1)} = X^\ell \cap L_1.$$

Hence $d(L_1)$ is the system of all sets $(X^{u(1)})^{\ell(1)}$, where X runs over the system of all nonempty subsets of L_1.

Let L_1^* be the interval with the endpoints a and b in $d(L)$. For $Z \in L_1^*$ and $T \in d(L_1)$ we put

$$\varphi_1(Z) = Z \cap [a, b], \quad \varphi_2(T) = (T^u)^\ell.$$

By applying (1) we obtain

3.1. Lemma. *φ_1 is an isomorphism of L_1^* onto $d(L_1)$ and $\varphi_2 = \varphi_1^{-1}$. Moreover, $\varphi_1(x) = x$ for each $x \in L_1$.*

Now let $\mathcal{A}$ be an MV-algebra and let G be a lattice ordered group with a strong unit u such that $\mathcal{A} = \mathcal{A}_0(G, u)$.

Denote $d(\mathcal{L}(\mathcal{A})) = d(A)$. Let A^* be the interval with the endpoints 0 and u in $d(G)$. For each $P \in A^*$ we put $\varphi_1(P) = P \cap A$. From 3.1 we obtain

3.2. Corollary. *φ_1 is an isomorphism of A^* onto $d(A)$. Moreover, $\varphi_1(x) = x$ for each $x \in A$.*

For $Y_1, Y_2 \in d(G)$ we put

$$Y_1 + Y_2 = (\{y_1 + y_2 \colon\ y_1 \in Y_1 \text{ and } y_2 \in Y_2\}^u)^\ell.$$

In view of (1) we have

$$Y_1 + Y_2 = \sup\{y_1 + y_2\} \quad (y_1 \in Y_1, y_2 \in Y_2), \tag{2}$$

where the supremum is taken with respect to $d(G)$.

The following results 3.3 and 3.4 have been proved in [3]; cf. also [6], 1.1 and 1.2.

3.3. Proposition. *The set $d(G)$ with the operation $+$ is a semigroup. The element 0 is a neutral element of $(d(G);+)$. If $a,b,c \in d(G)$, $a \leqslant b$, then $a+c \leqslant b+c$. Next, G is a subsemigroup of $d(G)$.*

3.4. Theorem. *Let $M(G)$ be the set of all elements of $d(G)$ which have an inverse in the semigroup $(d(G),+)$. Then*

(a) *$(M(G);+,\leqslant)$ is a lattice ordered group;*
(b) *$(M(G);\leqslant)$ is a sublattice of $d(G)$.*

In what follows we will write $M(G)$ instead of $(M(G);+,\leqslant)$. In [3], $M(G)$ is called the Dedekind completion of G; in [6], $M(G)$ was called the maximal completion of G.

We define a binary operation $\oplus$ on $d(A)$ as follows. For $T_1, T_2 \in d(A)$ we put

$$T_1 \oplus T_2 = (\{t_1 \oplus t_2 \colon\ t_1 \in T_1 \quad \text{and} \quad t_2 \in T_2\}^{u(1)})^{\ell(1)}$$

where $u(1)$ and $\ell(1)$ have analogous meanings as in the case of L_1 above.

Then according to (1) we have

$$T_1 \oplus T_2 = \sup\{t_1 \oplus t_2\} \quad (t_1 \in T_1 \text{ and } t_2 \in T_2) \tag{3}$$

where the supremum is taken with respect to $d(A)$.

It is easy to verify that the just defined operation $\oplus$ on $d(A)$ is an extension of the original operation $\oplus$ on A.

From the fact that the operation $\oplus$ on A is commutative and associative we infer (by applying (3))

3.5. Lemma. *The set $d(A)$ with the operation $\oplus$ is an abelian semigroup.*

3.6. Definition. *Let $\mathcal{A}$ be as above and let $\mathcal{B}$ be an MV-algebra such that the following conditions are satisfied:*

(a) *$\mathcal{A}$ is a subalgebra of $\mathcal{B}$.*
(b) *$\mathcal{L}(\mathcal{B})$ is a sublattice of $d(A)$.*
(c) *$(B;\oplus)$ is a subsemigroup of the semigroup $(d(A);\oplus)$.*

Then $\mathcal{B}$ is called a c-extension of $\mathcal{A}$.

3.7. Definition. *Let $\mathcal{B}_1$ be a c-extension of $\mathcal{A}$. If for each c-extension $\mathcal{B}$ of $\mathcal{A}$ the MV-algebra $\mathcal{B}$ is a subalgebra of $\mathcal{B}_1$, then $\mathcal{B}_1$ is called a maximal completion of $\mathcal{A}$.*

The above definition yields that if a maximal completion of $\mathcal{A}$ does exist, then it is uniquely determined.

For structures dealt with in the present section we can apply the following diagram:

$$\begin{tikzcd}
A \arrow[rr] \arrow[d] & & G \arrow[d] \arrow[dr] & \\
d(A) & A^* \arrow[l, "\varphi_1"'] & d(G) \arrow[l] & M(G) \arrow[l]
\end{tikzcd}$$

Here the mapping φ_1 is as in 3.2; all the remaining mappings are embeddings.

We define a binary operation $\oplus$ on A^* as follows. For $Z_1, Z_2 \in A^*$ we put

$$Z_1 \oplus Z_2 = \sup\{(z_1 + z_2) \wedge u\colon\ z_1 \in Z_1 \text{ and } z_2 \in Z_2\}, \tag{4}$$

where the supremum is taken with respect to the complete lattice A^*.

3.8. Lemma. *φ_1 is an isomorphism of $(A^*, \oplus)$ onto $(d(A), \oplus)$.*

P r o o f. Let $Z_1, Z_2 \in A^*$. Put $\varphi_1(Z_i) = T_i$ $(i = 1, 2)$. In view of (3),

$$\begin{aligned} T_1 \oplus T_2 &= \sup\{(t_1 + t_2) \wedge u\colon\ t_1 \in T_1 \text{ and } t_2 \in T_2\} = \\ &= \sup\{(\varphi(z_1) + \varphi(z_2)) \wedge \varphi(u)\colon\ z_1 \in Z_1 \text{ and } z_2 \in Z_2\} = \\ &= \sup\{\varphi((z_1 + z_2) \wedge u)\colon\ z_1 \in Z_1 \text{ and } z_2 \in Z_2\}, \end{aligned}$$

where sup is taken with respect to $d(A)$. Hence in view of (4) and 1.2, $T_1 \oplus T_2 = \varphi(Z_1 \oplus Z_2)$. $\square$

From the construction of $M(G)$ we infer that u is a strong unit of $M(G)$. Let M_0 be the interval of $M(G)$ with the endpoints 0 and u. Hence we can construct (by means of 1.2) the MV-algebra $\mathcal{A}_0(M(G), u)$; we will denote this MV-algebra by the symbol M_0 of its underlying set.

3.9. Lemma. *The MV-algebra $\mathcal{A}$ is a subalgebra of M_0.*

P r o o f. Since G is an ℓ-subgroup of $M(G)$, from the relations

$$\mathcal{A} = \mathcal{A}_0(G, u), \quad M_0 = \mathcal{A}_o(M(G), u)$$

we obtain that $\mathcal{A}$ is a subalgebra of M_0. $\square$

3.10. Lemma. *$\varphi_1(M_0)$ is a sublattice of $d(A)$.*

P r o o f. In view of 3.4 (b), $(M(G); \leqslant)$ is a sublattice of $d(G)$. This yields that M_0 is a sublattice of A^*. Hence according to 3.2, $\varphi_1(M_0)$ is a sublattice of $d(A)$. $\square$

The following result is well-known.

($*$) Let $a, b, c \in G^+$, $c \leqslant a+b$. Then there are $a_1, b_1 \in G$ such that $a_1 \in [0, a]$, $b_1 \in [0, b]$ and $c = a_1 + b_1$.

3.11. Lemma. *Let $a, b \in A$. Then there are $a_1 \in [0, a]$ and $b_1 \in [0, b]$ such that $a \oplus b = a_1 + b_1$.*

Proof. We have $a \oplus b = (a + b) \wedge u$. Put $c = a \oplus b$. Now it suffices to apply ($*$). □

3.12. Corollary. *Let $T_1, T_2 \in d(A)$. Then*

$$T_1 \oplus T_2 = \sup\{t_1 + t_2 \colon\ t_1 \in T_1, t_2 \in T_2 \text{ and } t_1 + t_2 \leqslant u\},$$

where the supremum is taken with respect to $d(A)$.

3.13. Lemma. *$(\varphi_1(M_0); \oplus)$ is a subsemigroup of the semigroup $(d(A), \oplus)$.*

Proof. This is a consequence of 3.10 and 3.12. □

3.14. Lemma. *$\varphi_1(M_0)$ is a c-extension of $\mathcal{A}$.*

Proof. This follows from 3.10, 3.9 and 3.13. □

3.15. Lemma. *Let $\mathcal{B}$ be a c-extension of $\mathcal{A}$. Let $Z \in B$, $Z \neq u$. Then there are $a \in A$ and $Z_1 \in d(A)$ such that $a < u$ and $Z \oplus Z_1 = a$.*

Proof. There exists a lattice ordered group G' such that u is a strong unit in G' and $\mathcal{B} = \mathcal{A}_0(G', u)$. Since $Z \neq u$, the set Z must be upper bounded in $A \setminus \{u\}$. Hence there is $a \in A$ such that $Z \leqslant a < u$. This implies that there is $b \in B$ such that $Z + b = a$ holds in G'. Then $0 \leqslant b < u$ and therefore $a = Z \oplus b$ is valid in $\mathcal{B}$. Now it suffices to put $Z_1 = b$. □

3.16. Lemma. *Let Z be as in 3.15. Then $\varphi_1^{-1}(Z) \in M_0$.*

Proof. We have $Z \oplus Z_1 = a$. Hence in view of 3.8 the relation

$$\varphi_1^{-1}(Z) \oplus \varphi_1^{-1}(Z_1) = a$$

is valid in A^*. Since $a < u$, we get

$$a = \varphi_1^{-1}(Z) + \varphi_1^{-1}(Z_1).$$

From the relation $a \in M(G)$ we infer that there is $-a \in M(G)$; hence

$$0 = \varphi_1^{-1}(Z) + (\varphi_1^{-1}(Z_1) - a).$$

Now according to 3.4, $\varphi_1^{-1}(Z)$ belongs to $M(G)$. Next, $\varphi_1^{-1}(Z) \in A^*$ and thus $\varphi_1^{-1}(Z) \in M_0$. $\square$

3.17. Corollary. *Let $\mathcal{B}$ be a c-extension of $\mathcal{A}$. Then $B \subseteq \varphi_1(M_0)$.*

Therefore we obtain

3.18. Theorem. *Let $\mathcal{A}$ be an MV-algebra. Let φ_1 and M_0 be as above. Then $\varphi_1(M_0)$ is the maximal completion of $\mathcal{A}$.*

3.19. Proposition. *Let $\mathcal{A}$ be an MV-algebra. Then the maximal completion of $\mathcal{A}$ is the set of all $T \in d(A)$ which satisfy the following condition:*

(c) *either $T = u$, or there are $a \in A$ and $T_1 \in d(A)$ such that $a < u$ and $T \oplus T_1 = a$.*

P r o o f. a) Let T belong to the maximal completion of $\mathcal{A}$. Then clearly $T \in d(A)$. According to 3.15, the condition (c) is satisfied.

b) Let $T \in d(A)$ and suppose that the condition (c) is valid. If $T = u$, then in view of 3.18, u belongs to the maximal completion $\varphi_1(M_0)$ of $\mathcal{A}$. Let $T < u$. In view of 3.16, $T \in \varphi_1(M_0)$. $\square$

An MV-algebra will be called m-complete if it coincides with its maximal completion. We conclude by remarking without proof that the class of all m-complete MV-algebras is closed with respect to direct products, but it fails to be closed with respect to homomorphic images.

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