Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 4, 687–700

Persistent URL: `http://dml.cz/dmlcz/127447`

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LOCALIZATION OF GLOBAL EXISTENCE OF HOLOMORPHIC SOLUTIONS OF HOLOMORPHIC DIFFERENTIAL EQUATIONS WITH INFINITE DIMENSIONAL PARAMETER

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(Received November 13, 1995)

1. Introduction

Let S be an infinite dimensional locally convex space with the finite open topology τ_0, let Ω be a pseudoconvex domain in the product space $\mathbb{C} \times S$ and $\mathscr{O}$ be the sheaf of germs of holomorphic functions over Ω. Let m be a positive integer and $a(z,s)$ be an m dimensional square matrix valued holomorphic function on Ω. We introduce the sheaf homomorphism $T\colon \mathscr{O}^m \to \mathscr{O}^m$ by the differential operator

$$T := \frac{\mathrm{d}}{\mathrm{d}z} - a(z,s). \tag{1}$$

By the short exact sequence of sheaves

$$0 \longrightarrow \operatorname{Ker} T \longrightarrow \mathscr{O}^m \xrightarrow{T} \mathscr{O}^m \longrightarrow 0,$$

we have the long exact sequence of cohomology groups

$$\ldots \longrightarrow \mathrm{H}^0(\Omega, \mathscr{O}^m) \xrightarrow{T} \mathrm{H}^0(\Omega, \mathscr{O}^m) \longrightarrow \mathrm{H}^1(\Omega, \operatorname{Ker} T) \longrightarrow \mathrm{H}^1(\Omega, \mathscr{O}^m) \longrightarrow \ldots.$$

Since we have

$$\mathrm{H}^1(\Omega, \mathscr{O}^m) = 0 \tag{2}$$

[1] Supported in part by Grant-in-Aid for General Scientific Research (B) no. 046452012 from the Ministry of Education, Science and Culture of Japan, 1995.
[2] Supported in part by the Basic Research Institute Program of the Ministry of Education of Korea, no. 941-0100-037-2, 1995.
[3] Supported in part by Grant-in-Aid, for JSPS Research Fellow no. 07 2309, from the Ministry of Education, Science and Culture of Japan, 1995.

by Dineen [6], we have the isomorphism

$$\mathrm{H}^1(\Omega, \mathrm{Ker}_{\bigcap}) = \mathrm{H}^0(\Omega, \mathscr{O}^m)/T(\mathrm{H}^0(\Omega, \mathscr{O}^m)). \tag{3}$$

We seek a necessary condition and a sufficient condition that

$$\text{for any } g \in \mathrm{H}^0(\Omega, \mathscr{O}^m), \text{ there exist } g \in \mathrm{H}^0(\Omega, \mathscr{O}^m) \text{ with } Tf = g. \tag{4}$$

By the isomorphism (3), the above condition (4) for global existence is equivalent to the vanishing

$$\mathrm{H}^1(\Omega, \mathrm{Ker}\, T) = 0 \tag{5}$$

of the cohomology group of the first degree with coefficients in the sheaf $\mathrm{Ker}\, T$.

Let $\pi\colon\ \mathbb{C} \times S \to S$ be the canonical projection. For any $(z, s) \in \Omega$, let $\Omega(z, s)$ be the connected component of $\pi^{-1}(s)$ containing (z, s). We consider the set of cuts

$$\tilde{\Omega} := \{\Omega(z, s);\ (z, s) \in \Omega\} \tag{6}$$

and induce in the set $\tilde{\Omega}$ the strongest topology so that the canonical mapping

$$\varphi\colon\ \Omega \to \tilde{\Omega} \tag{7}$$

is continuous. The topological space $\tilde{\Omega}$ is a factor space of Ω and it is not necessarily a Hausdorff space. Let $\tilde{\varphi}\colon\ \tilde{\Omega} \to S$ be the canonical mapping with $\tilde{\varphi} \circ \varphi = \pi$.

For any finite dimensional $\mathbb{C}$-linear subspaces L and M of S with $L \subset M$, any element of $\mathrm{H}^0(\pi^{-1}(L), \mathscr{O}^m)$ is holomorphically continued to an element of $\mathrm{H}^0(\pi^{-1}(M), \mathscr{O}^m)$ by the theorem of Oka[15]-Cartan[1]-Sèrre[18] applied to the analytic subset $\pi^{-1}(L)$ of the Stein manifold $\pi^{-1}(M)$, and by induction, finally to an element of $\mathrm{H}^0(\Omega, \mathscr{O}^m)$ holomorphic on the whole space Ω and the vanishing (5) is valid for the analytic subset $\pi^{-1}(L)$, that is,

$$\mathrm{H}^1(\pi^{-1}(L), \mathrm{Ker}\, T) = 0. \tag{8}$$

Especially, for any cut $\Omega(z, s)$, we have the vanishing

$$\mathrm{H}^1(\Omega(z, s), \mathrm{Ker}\, T) = 0 \tag{9}$$

of the cohomology group. Then, by Kajiwara [10], either

$$\Omega(z, s) \quad \text{is simply connected} \tag{10}$$

or

$$\text{(11)} \qquad \Omega(z,s) \quad \text{is doubly connected and} \quad \mathrm{H}^0(\Omega(z,s), \operatorname{Ker} T) = 0.$$

Under the additional condition (A) besides (5), given at the beginning of Section 5, for the coefficient $a(z,s)$, by the argument in Kajiwara-Mori [12], the space $\tilde{\Omega}$ is a Hausdorff space and, moreover, each cut $\Omega(z,s)$ is simultaneously either a simply connected domain or a doubly connected domain with $\mathrm{H}^0(\Omega(z,s), \operatorname{Ker} T) = 0$.

In the former case, $(\tilde{\Omega}, \tilde{\varphi})$ is a domain of holomorphy. In the latter case, each inhomogeneous solution f of $Tf = g$ is unique for any holomorphic g in any cut $\Omega(z,s)$. Conversely, the above condition implies the validity of the condition (4).

Since the Levi problem has been affirmatively solved by Dineen [6] and Gruman [8], the former condition is characterized by the local condition that each cut is simply connected and $(\tilde{\Omega}, \tilde{\varphi})$ is a pseudoconvex domain over S. In the latter case, the above condition is also local.

Thus, we have characterized the global existence of holomorphic solutions of linear differential equations $Tf = g$ with the infinite dimensional parameter $s \in S$ by the conditions which are local concerning the parameter space S.

2. Connectivity

For the sake of brevity and clearness of explanations, in this section we discuss exclusively the case without any parameter.

Let m be a positive integer and D be a domain in the complex plane $\mathbb{C}$, let $\mathscr{O}$ be the sheaf of germs of holomorphic functions over D and $a(z)$ an m dimensional square matrix valued holomorphic function on D. We define a sheaf homomorphism $T\colon \mathscr{O}^m \to \mathscr{O}^m$ by the differential operator

$$\text{(12)} \qquad T := \frac{\mathrm{d}}{\mathrm{d}z} - a(z).$$

By the Weierstrass theorem, the domain D is a domain of holomorphy and we have

$$\text{(13)} \qquad \mathrm{H}^1(D, \mathscr{O}^m) = 0$$

by Oka [15]. Hence, by the isomorphism (3), the global existence

$$\text{(14)} \qquad \mathrm{H}^0(D, \mathscr{O}^m) = T(\mathrm{H}^0(D, \mathscr{O}^m))$$

of the non homogeneous differential equation (4) is equivalent to the vanishing

$$\mathrm{H}^1(D, \mathrm{Ker} \bigcap) = 0 \tag{15}$$

of cohomology with coefficients in the sheaf $\mathrm{Ker}\, T$ of germs of holomorphic solutions f of the homogeneous equation

$$Tf = 0. \tag{16}$$

Theorem 1. *The necessary and sufficient condition for* (14) *is that either D is simply connected or D is a doubly connected domain with $H^0(D, \mathrm{Ker}\, T) = 0$.*

P r o o f. Assume that D were neither simply connected nor doubly connected. There would exist subdomains D_r and D_ℓ of the domain D satisfying the following conditions: (a) $D = D_r \cup D_\ell$. (b) Each connected component of $D_r \cap D_\ell$ is simply connected. (c) $D_r \cap D_\ell$ has at least three connected components Δ_b, Δ_m and so on.

We denote by $\Delta_t \neq \emptyset$ the complement of the disjoint union $\Delta_b \cup \Delta_m$ with respect to D.

We define an open covering

$$\mathscr{U} := \{D_r, D_\ell\} \tag{17}$$

of the domain D. Let $h(z)$ be a holomorphic solution of the homogeneous equation (16) on the simply connected domain D_r. We define a homogeneous solution $k(z)$ of (16) on the open set $D_r \cap D_\ell = \Delta_b \cup \Delta_m \cup \Delta_t$ by putting $k := h$ on Δ_b and $k := 0$ on the open set $\Delta_m \cup \Delta_t$ and regard $k \in \mathrm{H}^0(D_r \cap D_\ell, \mathrm{Ker}\, T)$ as a cocycle $\in \mathrm{Z}^1(\mathscr{U}, \mathrm{Ker}\, T)$.

Since the canonical mapping

$$\mathrm{H}^1(\mathscr{U}, \mathrm{Ker}\, T) \to \mathrm{H}^1(D, \mathrm{Ker}\, T) \tag{18}$$

is injective, by (3) the assumption (14) implies

$$\mathrm{Z}^1(\mathscr{U}, \mathrm{Ker}\, T) \cong \mathrm{B}^1(\mathscr{U}, \mathrm{Ker}\, T). \tag{19}$$

Hence there exist, respectively, holomorphic solutions h_r and h_ℓ of the homogeneous equation (16) on the simply connected domains D_r and D_ℓ such that we have

$$h_r - h_\ell = k \tag{20}$$

on the intersection $D_r \cap D_\ell = \Delta_b \cup \Delta_m \cup \Delta_t$. Especially, we have

$$h_r - h_\ell = h \tag{21}$$

on Δ_b, and

$$h_r - h_\ell = 0 \tag{22}$$

on $\Delta_m \cup \Delta_t$.

Now, let z_b and z_m be, respectively, points of Δ_b and Δ_m. For each $j \in \{1, 2, \dots, m\}$, let e_j be the m-dimensional column vector whose j-th component is 1 and whose k-th component is 0 except $k = j$. Let f_j be the holomorphic solution in the simply connected domain Δ_b of the homogeneous equation (16) satisfying the initial condition

$$f_j(z_b) = e_j \tag{23}$$

and its analytic continuation to the simply connected neighboring domain D_r. We consider the $m \times m$-matrix valued holomorphic function

$$f(z) := (f_1(z), f_2(z), \dots, f_m(z)) \tag{24}$$

in the simply connected domain D_r.

Let b be an m-dimensional column vector. We define a holomorphic solution $f(z)b$, defined by the rule of matrix multiplication, of the homogeneous equation (16) and adopt it as a homogeneous solution $h \in \mathrm{H}^0(\Delta_b, \operatorname{Ker} T)$ in (21), that is, $h := fb$.

Let γ_b and $\gamma_m\colon [0, 1] \to D$ be closed simple smooth curves in D such that $\gamma_b(0) = \gamma_b(1) = z_b \in \Delta_b, \gamma_b(\frac{1}{2}) = \gamma_m(0) = z_m \in \Delta_m, \gamma_m(\frac{1}{2}) \in \Delta_t$, $\gamma_b(t)$ and $\gamma_m(t) \in D_r$ for $t \in [0, \frac{1}{2}]$, and $\gamma_b(t) \in D_\ell$, $\gamma_m(t) \in D_\ell$ for $t \in [\frac{1}{2}, 1]$. Let $f_j(\gamma_b(t))$ be the analytic continuation of the homogeneous solution f_j of (16) along the closed curve γ_b. Let $c_{j,k}$ be the j-th component of the m-dimensional column vector $f_k(\gamma_b(1))$ for $j, k \in \{1, 2, \dots, m\}$ and let c be the $m \times m$ matrix whose (j, k) entry is $c_{j,k}$. In other words, we put

$$c := \Big(f_1(\gamma_b(1)), f_2(\gamma_b(1)), \dots, f_m(\gamma_b(1)) \Big). \tag{25}$$

Then we have

$$f(\gamma_b(1)) = f(\gamma_b(0))c. \tag{26}$$

Since f is the matrix (24) associated to the fundamental system of holomorphic homogeneous solutions of (16), there exists an m-dimensional column vector a such that we have

$$h_r(z) = f(z)a \tag{27}$$

in the simply connected domain D_r. Let $h_r(\gamma_b(t))$ be the analytic continuation of this h_r along the closed curve γ_b. By (27) and (26), we have

$$h_r(\gamma_b(1)) = f(\gamma_b(1))a = f(\gamma_b(0))ca. \tag{28}$$

The relation (22) asserts that the holomorphic homogeneous solution h_ℓ given in the simply connected domain D_ℓ is just the analytic continuation of the holomorphic homogeneous solution h_r given in the simply connected domain D_r along the closed curve γ_b across the simply connected component Δ_m of $D_r \cap D_\ell$ to the simply connected domain D_ℓ. Hence, by (27), (28) and the unicity of the initial value problem, the relation (21) means

$$f(z)a - f(z)ca = f(z)b \tag{29}$$

in the simply connected component Δ_b of the intersection $D_r \cap D_\ell$. Since the matrix $f(z_b)$ is regular, we have

$$a - ca = b, \tag{30}$$

which means that, for any m dimensional column vector b, there exists an m dimensional column vector a satisfying the above linear equation (30). So, the matrix Identity $- c$ is regular, that is,

$$\det(\text{Identity} - c) \neq 0 \tag{31}$$

and a is determined uniquely by

$$a = (\text{Identity} - c)^{-1}b. \tag{32}$$

Hence $a \neq 0$ implies $a - ac \neq 0$. So,

(33) no non trivial holomorphic homogeneous solution of (16) is single valued along γ_b.

Taking (22) into account, we repeat the same argument of analytic continuation along the closed curve γ_m and arrive at the conclusion

(34) every holomorphic homogeneous solution of (16) is single valued along γ_m.

Since we can exchange the role of the subscripts b and m, the above two conclusions (33) and (34) contradict each other.

Hence, the domain D is either simply or doubly connected.

In the latter case, (33) means that $\mathrm{H}^0(D, \mathrm{Ker}\, T) = 0$. Moreover, let g be any single valued m dimensional square matrix valued holomorphic function on the doubly connected domain D. Let f_r be a holomorphic solution of the inhomogeneous equation $Tf_r = g$ on the simply connected domain D_r and let f_ℓ be the direct holomorphic continuation of f_r to the simply connected domain D_ℓ across Δ_m. The m dimensional square matrix valued function $h := -f_r + f_\ell$ is a holomorphic solution in the simply connected domain Δ_b of the homogeneous equation $Th = 0$. For this $h \in \mathrm{H}^0(D_r, \mathrm{Ker}\, T)$, we take the homogeneous solution $h_r \in \mathrm{H}^0(D_r, \mathrm{Ker}\, T)$ satisfying (21) via the solution a given at (32) and revise the inhomogeneous solution f_r on D_r by $f_r + h$. Then $f_r + h$ is the unique single valued holomorphic solution of the inhomogeneous equation (4).

Thus we have proved that the domain D is either a simply connected domain or a doubly connected domain with $\mathrm{H}^0(D, \mathrm{Ker}\, T) = 0$. □

3. Cohomology vanishing and steinness of domains

Let (D, ψ) be a domain over $\mathbb{C}^n$, that is, let D be a Hausdorff space and ψ be a local homeomorphism of D in $\mathbb{C}^n$. Let L a hyperplane of $\mathbb{C}^n$, $\pi\colon\ \mathbb{C}^n \to L$ be the canonical projection and let f be a holomorphic function on the analytic subset $\psi^{-1}(L) \subset D$. There exists a family $\mathscr{U} := \{U_\lambda;\ \lambda \in I\}$ such that $\mathscr{U}$ covers $\psi^{-1}(L)$ and that, for any $\lambda \in \Lambda$, the holomorphic function f on the analytic subset $\psi^{-1}(L) \subset D$ is locally extended to a holomorphic function F_λ on the open set $U_\lambda \subset D$. We put

$$(35) \qquad U_\infty := D - \psi^{-1}(L), \quad \Lambda := I \cup \{\infty\}, \quad \mathscr{V} := \{U_\mu;\ \mu \in \Lambda\}.$$

Then $\mathscr{V} := \{U_\lambda;\ \lambda \in \Lambda\}$ is an open covering of D. We may assume that

$$(36) \qquad L := \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n;\ z_1 = 0\}.$$

We define a 1-cocycle $\mathscr{C} := \{k_{\lambda_1\lambda_2};\ \lambda_1, \lambda_2 \in \Lambda\} \subset \mathrm{Z}^1(\mathscr{V}, \mathscr{O})$ by putting

$$(37) \qquad k_{\lambda_1\lambda_2} := \frac{F_{\lambda_2} - F_{\lambda_1}}{z_1 \circ \psi} \quad (\lambda_1, \lambda_2 \in I), \quad k_{\lambda_1\infty} := \frac{-F_{\lambda_1}}{z_1 \circ \psi} \quad (\lambda_1 \in I).$$

Proposition 1. *If the cocycle $\mathscr{C} \in Z^1(\mathscr{U}, \mathscr{O})$ is a coboundary $\in B^1(\mathscr{U}, \mathscr{O})$, then f is extended to a holomorphic function F on the ambient domain D.*

Proof. There exists a 0-cochain $\{g_\lambda;\ \lambda \in \Lambda\}$ whose coboundary is the cocycle $\mathscr{C}$. We put

$$F := -g_\infty z_1 \circ \psi \tag{38}$$

on the open set $U_\infty = D - \psi^{-1}(L)$ and

$$F := -g_\lambda z_1 \circ \psi + F_\lambda \tag{39}$$

on $U_\lambda \subset D$ for $\lambda \in I$. Then F is a well defined holomorphic extension of f to D. □

Proposition 2. *A Cousin-I domain (D, ψ) over $\mathbb{C}^n$ is a domain of holomorphy if and only if, for any hyperplane L in $\mathbb{C}^n$, the open set $(\psi^{-1}(L), \psi|_{\psi^{-1}(L)})$ is an open set of holomorphy over L.*

Proof. Let x be an ideal boundary of the domain (D, ψ) over $\mathbb{C}^n$. There exists a hyperplane L of $\mathbb{C}^n$ such that x is also regarded as an ideal boundary of the open set $(\psi^{-1}(L), \psi|_{\psi^{-1}}(L))$ of holmorphy over L of a holomorphic function f on $\psi^{-1}(L)$. As in the proof of the above proposition, we can prove the existence of a holomorphic extension F of f to D. Since every ideal boundary of the domain (D, ψ) is an ideal boundary of the envelope of holomorphy of the domain (D, ψ), (D, ψ) is a domain of holomorphy. □

As a corollary, we have the following theorem:

Cartan-Behnke-Stein's Theorem. *Any Cousin-I domain over $\mathbb{C}^2$ is a domain of holomorphy.*

4. The sum space $\sum \mathbb{C}$

For any positive integers $p < q$, we regard $\mathbb{C}^p$ as a $\mathbb{C}$-linear subspace of $\mathbb{C}^q$ by the canonical inclusion

$$\mathbb{C}^p \ni (z_1, z_2, \dots, z_p) \mapsto \pi_{p,q}(z_1, z_2, \dots, z_p) := (z_1, z_2, \dots, z_p, 0, 0, \dots, 0) \in \mathbb{C}^q. \tag{40}$$

Under the above inclusions, we consider the sum space

$$S := \sum \mathbb{C} := \bigcup_{p=1}^{\infty} \mathbb{C}^p. \tag{41}$$

Let $\pi_p\colon \mathbb{C}^p \to S$ be the canonical injection. In the sum space S we induce the strongest topology between those so that each π_p is continuous. We consider the product space $\mathbb{C} \times S$ and, for any positive integer p, let $\sigma_p\colon \mathbb{C} \times \mathbb{C}^p \to \mathbb{C} \times S$ be the canonical injection.

An open set Ω in the product space $\mathbb{C} \times S$ is said to be *pseudoconvex* if, for any positive integer p, the open set

$$\Omega_p := \sigma_p^{-1}(\Omega) \tag{42}$$

is a pseudoconvex open set in the finite dimensional space $\mathbf{C}^{p+1}$. Similarly, a continuous function f on Ω is said to be *holomorphic* if, for any positive integer p, the continuous function $f \circ \sigma_p$ is a holomorphic function in $\Omega_p \subset \mathbb{C}^{p+1}$.

Theorem 2. *Under the assumption $H^1(\Omega, \operatorname{Ker} T) = 0$, either all cuts $\Omega(z, s)$, $(z, s) \in \Omega$ are simultaneously simply connected domains or all cuts $\Omega(z, s)$, $(z, s) \in \Omega$ are simultaneously doubly connected domains with $H^0(\Omega(z, s), \operatorname{Ker} T) = 0$.*

P r o o f. By Theorem 1, for any $(z, s) \in \Omega$, the cut $\Omega(z, s) \subset \mathbb{C}$ is either simply connected or doubly connected. Assume that there exists a point $(z_0, s_0) \in \Omega$ such that the cut $\Omega(z_0, s_0)$ is a doubly connected domain in the complex plane $\mathbb{C}$. Let γ_0 be a closed curve which forms a homology base of the doubly connected cut $\Omega(z_0, s_0)$. The curve γ_0 is a compact set in the product space $\mathbb{C} \times S$. Since the topology of the product space $\mathbb{C} \times S$ is the strongest one so that each canonical mapping $\sigma_p\colon \mathbb{C} \times \mathbb{C}^p \to \mathbb{C} \times S$ is continuous, there exists a positive integer p_0 such that $\gamma_0 \subset \mathbb{C} \times \mathbb{C}^{p_0}$. Let (z, s) be any point of Ω and let γ be one of the closed curves which forms a homology base of the cut $\Omega(z, s)$. There exists a positive integer $p \geqslant p_0$ such that $\gamma \subset \mathbb{C} \times \mathbb{C}^p$. By (8) applied to $L = \mathbb{C}^p \subset S$ for Ω_p, we have

$$\mathrm{H}^1(\Omega_p, \operatorname{Ker} T) = 0. \tag{43}$$

By the finite dimensional results of Kajiwara-Mori [12], the cut $\Omega(z, s)$ is also a doubly connected domain with $\mathrm{H}^0(\Omega(z, s), \operatorname{Ker} T) = 0$, which was to be proved. □

We continue to use the notation in Introduction and in the preceding section. Moreover, we discuss in the present section the case that all cuts $\Omega(z,s), (z,s) \in \Omega$, are simultaneously simply connected. We also present the following supplementary assumption:

(A) There exist a holomorphic function $b\colon\ S \to \mathbb{C}$ and a superdomain O of Ω such that the coefficient $a(z,s)$ is continued to a holomorphic function on O and that, for any $s \in S$, $(b(s),s) \in O$, $\pi^{-1}(s) \cap O$ is a simply connected domain in $\mathbb{C}$.

For each $j \in \{1,2,\dots,m\}$, let e_j be the m-dimensional column vector whose j-th component is 1 and whose k-th component is 0 except $k=j$. For any $s \in S$, let $h_j(z,s)$ be the holomorphic solution in the simply connected domain $\pi^{-1}(s) \cap O$ of the homogeneous equation (16) satisfying the initial condition

$$h_j(b(s),s) = e_j. \tag{44}$$

Then the m homogeneous solutions $h_1(z,s), h_2(z,s), \dots, h_m(z,s)$ form a fundamental system of homogeneous solutions in the simply connected domain $\pi^{-1}(s) \cap O$ and are holomorphic in the domain O as functions of variables z and s.

Let $\tilde{\Omega}$ be the space of cuts, (z,s) running over Ω, and let $\tilde{\varphi}\colon\ \tilde{\Omega} \to S$ be the mapping defined canonically by

$$\tilde{\Omega} \ni \Omega(z,s) \mapsto s \in S. \tag{45}$$

The space $\tilde{\Omega}$ of cuts is not necessarily a Hausdorff space and is not necessarily a complex manifold. However, we can define holomorphic functions on an open subset of D: A continuous function f in an open subset U of $\tilde{\Omega}$ is said to be *holomorphic* if, for any open subset V of U such that $\tilde{\varphi}$ maps V homeomorphically onto an open subset W of S, the function $f \circ (\tilde{\varphi}\big|_V)^{-1}$ is holomorphic in W. Let $\tilde{\mathscr{O}}$ be the sheaf of germs of holomorphic functions over $\tilde{\Omega}$.

Proposition 3. *Under the assumption that $H^1(\Omega, \operatorname{Ker} T) = 0$ and the assumption* (A), *if all cuts $\Omega(z,s), (z,s) \in \Omega$ are simply connected, then $\tilde{\Omega}$ is a Hausdorff space.*

P r o o f. Let x_1 and x_2 be two different points of $\tilde{\Omega}$. We may assume that $\tilde{\varphi}(x_1) = \tilde{\varphi}(x_2)$ which we denote by s_0. There exist open neighborhoods U_1 and U_2, respectively, of x_1 and x_2 in $\tilde{\Omega}$ and an open neighborhood V of s_0 in S such that $\tilde{\varphi}$ maps U_1 and U_2 homeomorphically onto V. By definition of the sum space S, there exists a positive integer p such that $s_0 \in \mathbb{C}^p \subset S$. For this integer p, we consider the subspace $\tilde{\Omega}_p := \tilde{\varphi}^{-1}(\mathbb{C}^p)$ of $\tilde{\Omega}$. Since we have $\mathrm{H}^1(\Omega_p, \operatorname{Ker} T) = 0$, by

Lemma 9 of Kajiwara-Mori [12], the space $\tilde{\Omega}_p$ is a Hausdorff space. Hence, there exists an open neighborhood $W_{1,p} \subset U_1$ and $W_{2,p} \subset U_2$ in $\tilde{\Omega}_p$, respectively, of x_1 and x_2 such that $\tilde{\varphi}$ maps $W_{1,p}$ and $W_{2,p}$ homeomorphically on an open neighborhood P_p of s_0 in $\mathbb{C}^p$ and that $W_{1,p} \cap W_{2,p} = \emptyset$. We consider the open neighborhood $P := \{(z_1, z_2, \ldots, z_p, z_{p+1}, z_{p+2}, z_{p+3}, \ldots) \in S;\ (z_1, z_2, \ldots, z_p) \in P_p\}$ and open neighborhoods $W_1 := U_1 \cap \tilde{\varphi}^{-1}(P)$ and $W_2 := U_2 \cap \tilde{\varphi}^{-1}(P)$, respectively, of x_1 and x_2 in the space $\tilde{\Omega}$. Then we have $W_1 \cap W_2 = \emptyset$, which was to be proved. □

6. Cohomology vanishing and pseudoconvexity

Proposition 4. *Under the assumption* $H^1(\Omega, \operatorname{Ker} T) = 0$, *for a point* $(z_0, s_0) \in \Omega$, *if a cut* $\Omega(z_0, s_0)$ *is simply connected, then every cut* $\Omega(z, s)$ *is simply connected. Moreover, under the supplementary condition* (A), $(\tilde{\Omega}, \tilde{\varphi})$ *is a pseudoconvex domain over* S.

Proof. Since $\tilde{\Omega}$ is a Hausdorff space by Proposition 3, the pair $(\tilde{\Omega}, \tilde{\varphi})$ is a domain over the locally convex space S with the finite open topology and we can apply the theory of pseudoconvex domains by Noverraz [14].

Let p be a positive integer. We put $\tilde{\Omega}_p := \tilde{\varphi}^{-1}(\mathbb{C}^p)$. Then $(\tilde{\Omega}_p, \tilde{\varphi}|_{\tilde{\Omega}_p})$ is an open set over $\mathbb{C}^p$. Let $\tilde{\mathscr{U}} := \{\tilde{U}_\lambda;\ \lambda \in \Lambda\}$ be a Stein covering of $\tilde{\Omega}_p$ and let $\tilde{\mathscr{F}} := \{\tilde{f}_{\lambda\mu};\ \lambda \in \Lambda\}$ be a cocycle $\in \mathrm{Z}^1(\tilde{\mathscr{U}}, \tilde{\mathscr{O}})$, represented by an m dimensional column vector. We consider the Stein covering $\mathscr{U} := \{U_\lambda;\ \lambda \in \Lambda\}$, where $U_\lambda := \varphi^{-1}(\tilde{U_\lambda}), \lambda \in \Lambda$. We use homogeneous holomorphic solutions defined by (44) and put $h(z,s) := (h_1(z,s), h_2(z,s), \ldots, h_m(z,s))$. Then $\{h(z,s)\tilde{f}_{\lambda\mu}(\Omega(z,s));\ \lambda \in \Lambda\}$ is a 1-cocycle $\in \mathrm{Z}^1(\mathscr{U}, \operatorname{Ker} T)$. Since the canonical mapping

$$\mathrm{Z}^1(\mathscr{U}, \operatorname{Ker} T)/\mathrm{B}^1(\mathscr{U}, \operatorname{Ker} T) = \mathrm{H}^1(\mathscr{U}, \operatorname{Ker} T) \to \mathrm{H}^1(\Omega_p, \operatorname{Ker} T) \tag{46}$$

is injective and since its right hand side vanishes by (8), $\{h(z,s)\tilde{f}_{\lambda\mu}(\Omega(z,s); \lambda \in \Lambda\} \in \mathrm{B}^1(\mathscr{U}, \operatorname{Ker} T)$ and it is the coboundary of a 0 cochain $\{f_\lambda;\ \lambda \in \Lambda\} \in \mathrm{Z}^0(\mathscr{U}, \operatorname{Ker} T)$. Since $h(z,s)$ is a fundamental system of homogeneous solutions, there exists $\{\tilde{f}_\lambda(\Omega(z,s)) \in \mathrm{C}^0(\tilde{\mathscr{U}}, \tilde{\mathscr{O}}^m);\ \lambda \in \Lambda\}$ whose coboundary is the above 1 cocycle of $\mathrm{Z}^1(\tilde{\mathscr{U}}, \tilde{\mathscr{O}}^m)$. Thus we have proved

$$\mathrm{H}^1(\tilde{\Omega}_p, \tilde{\mathscr{O}}) = 0. \tag{47}$$

By (47), each $(\tilde{\Omega}_{p+1}, \tilde{\varphi}|_{\tilde{\Omega}_{p+1}})$ is a Cousin-I domain over $\mathbf{C}^{p+1}$. Hence, by induction with respect to p and Proposition 2, $(\tilde{\Omega}_p, \tilde{\varphi}|_{\tilde{\Omega}_p})$ is a pseudoconvex domain over $\mathbb{C}^p$ for any positive integer p, which was to be proved. □

Theorem 3. *Under the assumption that every cut* $\Omega(z,s)$ *is simply connected and under the assumption* (A), *if* $(\tilde{\Omega}, \tilde{\varphi})$ *is a pseudoconvex domain over* S, *then we have* $H^0(\Omega, \mathscr{O}^m) = TH^0(\Omega, \mathscr{O}^m)$.

P r o o f. Let g be an element of $\mathrm{H}^0(\Omega, \mathscr{O}^m)$. Since the topology τ_0 is finite open, it suffices to prove the following proposition $(Q)_p$ with respect to positive integers p: $(Q)_p$ There exists a sequence $\{f_q;\ q = 1, 2, \ldots, p\}$ of holomorphic homogeneous solutions f_q of $Tf_q = g$ on $\Omega_q = \varphi^{-1}(\mathbb{C}^q)$ such that f_{q+1} is a holomorphic extension of the preceding f_q for $q = 1, 2, \ldots, p-1$ to the higher dimensional Ω_{q+1}.

We assume $(Q)_p$. By the results of Kajiwara-Mori [12], for the finite dimensional $\mathbb{C}^{p+1}$ there exists a holomorphic inhomogeneous solution h_{p+1} of $Th_{q+1} = g$ on Ω_{p+1}. Then $h_{p+1}\big|_{\Omega_p} - h_p \in \mathrm{H}^0(\Omega_p, \operatorname{Ker} T)$. There exist $k_1, k_2, \ldots, k_m \in \mathrm{H}^0(\tilde{\Omega}_p, \tilde{\mathscr{O}})$ such that

$$h_p - h_{p+1}\big|_{\Omega_p} = \sum_{\nu=1}^{m} k_\nu(\Omega(z,s)) h_\nu(z,s). \tag{48}$$

By Oka[15]-Cartan[1]-Sèrre[18]'s theorem, each holomorphic function $k_\nu(\Omega(z,s))$ on the open subset $\tilde{\Omega}_p$ of the Stein manifold $\tilde{\Omega}_{p+1}$ is extended to a holomorphic function $K_\nu(\Omega(z,s))$ on $\tilde{\Omega}_{p+1}$. We revise the inhomogeneous solution h_{p+1} and adopt the inhomogeneous solution

$$f_{p+1} = h_{p+1}\big|_{\Omega_p} + \sum_{\nu=1}^{m} K_\nu(\Omega(z,s)) h_\nu(z,s) \tag{49}$$

instead of h_{p+1}. Then f_{p+1} is an inhomogeneous solution of $Tf_{p+1} = g$ and it is the desired extension of the inhomogeneous solution f_p to Ω_{p+1}, which was to be proved. □

6. Main theorem

Main Theorem. *Let* m *be a positive integer,* S *be the sum space* (41), Ω *be a pseudoconvex domain in the product space* $\mathbb{C} \times S$, *let* $a(z,s)$ *be an* m*-dimensional square matrix valued holomorphic function on* Ω *and* T *the differential operator defined by* (1). *If the condition* (4) *of global existence for the equation* $Tf = g$ *is fulfilled, then all cuts* $\Omega(z,s)$ *are either simultaneously simply connected domains for all points* $(z,s) \in \Omega$ *or simultaneously doubly connected domains with* $H^0(\Omega(z,s), \operatorname{Ker} \bigcap) = 0$ *for all points* $(z,s) \in \Omega$.

Under the supplementary condition (A)*, if a cut is simply connected, the necessary and sufficient condition for the global existence* (4) *is that the cut space* $\tilde{\Omega}$ *defined by* (6) *is a Hausdorff space and that* $(\tilde{\Omega}, \tilde{\varphi})$ *is a pseudoconvex domain over* S.

If a cut is doubly connected, the necessary and sufficient condition for the global existence is that every cut $\Omega(z,s)$ *is a doubly connected domain with* $H^0(\Omega(z,s), \mathrm{Ker}\, T) = 0$ *and that the cut space* $\tilde{\Omega}$ *defined by* (6) *is a Hausdorff space.*

P r o o f. We have already discussed the case that every cut is simply connected. We can treat similarly the case that every cut is doubly connected, making use of arguments in Kajiwara-Mori [12] and Kajiwara-Shon [13] given in the finite dimensional case. The key of the proof is based on (32) and (33). □

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