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TRIANGULAR STOCHASTIC MATRICES GENERATED BY INFINITESIMAL ELEMENTS

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Abstract. We show that each element in the semigroup S_n of all $n\times n$ non-singular upper (or lower) triangular stochastic matrices is generated by the infinitesimal elements of S_n, which form a cone consisting of all $n \times n$ upper (or lower) triangular intensity matrices.

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1. Introduction

Let G be a Lie group, let $L(G)$ be its Lie algebra, and let $\exp\colon L(G) \to G$ denote the exponential mapping. Let $\mathrm{gl}(n,\mathbb{R})$ denote the set of all real $n \times n$ matrices and $\mathrm{GL}(n,\mathbb{R})$ the general linear group of degree n over $\mathbb{R}$. Here $\mathbb{R}$ denotes the set of all real numbers and hereafter we shall use this notation. For $G = \mathrm{GL}(n,\mathbb{R})$ and $L(G) = \mathrm{gl}(n,\mathbb{R})$, it is well known that the exponential map $\exp\colon \mathrm{gl}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is defined by $\exp(tX) = I + tX + \frac{1}{2!}(tX)^2 + \dots$ for $X \in \mathrm{gl}(n,\mathbb{R})$.

Let S_n be a subsemigroup of $\mathrm{GL}(n,\mathbb{R})$ and let $X(t)$ be a differentiable matrix function of the real parameter t in an interval $0 \leqslant t \leqslant t_0$ such that $X(t) \in S_n$ for each t and $X(0) = I$. We call the matrix $(\frac{\mathrm{d}X(t)}{\mathrm{d}t})|_{t=0}$ an *infinitesimal element* of S_n and denote the totality of all infinitesimal elements of S_n by $\mathscr{D}(S_n)$. Let $A(t)$ be a sectionwise continuous function of t $(0 \leqslant t \leqslant t_0)$ such that $A(t) \in \mathscr{D}(S_n)$ for each t. It is standard that the differential equation

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = A(t)X(t); \quad X(0) = I$$

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has a unique continuous solution and $X(t_0) \in S_n$. This $X(t_0)$ in S_n is called *generated by the infinitesimal elements* $A(t)$ $(0 \leqslant t \leqslant t_0)$.

Loewner [3] showed that each element in the semigroup of all $n \times n$ non-singular totally positive matrices is generated by the infinitesimal elements of the semigroup, which form a set of all $n \times n$ Jacobi matrices with non-negative off-diagonal elements. In general, a semigroup is not completely recreated from its infinitesimal elements, even if the semigroup is connected, and it is quite difficult to compute a semigroup generated by its infinitesimal elements.

In Section 2, we show that the infinitesimal elements of the semigroup of all $n \times n$ non-singular upper (or lower) triangular stochastic matrices are $n \times n$ upper (or lower) triangular intensity matrices. Finally, in Section 3, we show that each element in the semigroup S_n of all $n \times n$ non-singular upper (or lower) triangular stochastic matrices is generated by the infinitesimal elements of S_n, which form a cone consisting of all $n \times n$ upper (or lower) triangular intensity matrices.

2. Infinitesimal elements of triangular stochastic matrices

Definition. A matrix $A = \|a_{ij}\|$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ over $\mathbb{R}$ is called a *stochastic matrix* if $a_{ij} \geqslant 0$ and $\sum\limits_{j=1}^{n} a_{ij} = 1$ for $i = 1, 2, \ldots, m$. A matrix $B = \|b_{kl}\|$ $(k = 1, 2, \ldots, m;\ l = 1, 2, \ldots, n)$ over $\mathbb{R}$ such that $b_{kl} \geqslant 0$ for $k \neq l$ and $\sum\limits_{l=1}^{n} b_{kl} = 0$ for $k = 1, 2, \ldots, m$ is called an *intensity matrix*. An intensity matrix C is called an *extreme intensity matrix* if C has only one nonzero off-diagonal element which is equal to 1. An extreme intensity matrix $C = \|c_{kl}\|$ is denoted by E_{pq} $(p \neq q)$ if $c_{pp} = -1$ and $c_{pq} = 1$.

It is easy to see that the set of all non-singular $n \times n$ stochastic matrices forms a subsemigroup of $\mathrm{GL}(n, \mathbb{R})$.

Lemma 2.1. *Let S_n be the semigroup of all real $n \times n$ non-singular matrices with non-negative entries. Then $\mathscr{D}(S_n)$ coincides with the set of all real $n \times n$ matrices which are non-negative off the diagonal.*

P r o o f. Let $A = \|a_{ij}\| \in \mathscr{D}(S_n)$. Then $A = (\frac{\mathrm{d}X(t)}{\mathrm{d}t})|_{t=0}$ with $X(t) \in S_n$ for each t and $X(0) = I$. Since $X(t) \in S_n$, $x_{ij}(t) \geqslant 0$ for $i, j = 1, 2, \ldots n$. From $X(0) = I$, $x_{ij}(0) = 0$ for $i \neq j$. Thus $a_{ij} = (\frac{\mathrm{d}x_{ij}(t)}{\mathrm{d}t})|_{t=0} \geqslant 0$ for $i \neq j$.

Conversely let $E_{ij} (i \neq j)$ be an extreme intensity matrix as denoted in the above definition. Since ${E_{ij}}^2 = -E_{ij}$, $\exp(tE_{ij}) = I + tE_{ij} - \frac{t^2}{2!}E_{ij} + \frac{t^3}{3!}E_{ij} + \ldots = I + (1 - \mathrm{e}^{-t})E_{ij}$, and hence $\exp(tE_{ij}) \in S_n$ for $t \geqslant 0$. Since $E_{ij} = \frac{\mathrm{d}}{\mathrm{d}t}(\exp(tE_{ij}))|_{t=0}$,

$E_{ij} \in \mathscr{D}(S_n)$. Let E_k be the matrix whose elements are 0 except that the k-th diagonal element is equal to 1. Since ${E_k}^2 = E_k$, $\exp(tE_k) = I + tE_k + \frac{t^2}{2!}E_k + \frac{t^3}{3!}E_k + \ldots = I + (\mathrm{e}^t - 1)E_k$, and hence $\exp(tE_k) \in S_n$ for $t \geqslant 0$. Thus $E_k \in \mathscr{D}(S_n)$. Similarly we may show $-E_k \in \mathscr{D}(S_n)$. Since $\mathscr{D}(S_n)$ forms a convex cone in the matrix space $\mathrm{gl}(n, \mathbb{R})$, $\sum\limits_{1 \leqslant i \neq j \leqslant n} \alpha_{ij} E_{ij} + \sum\limits_{k=1}^{n} \beta_k E_k - \sum\limits_{k=1}^{n} \gamma_k E_k \in \mathscr{D}(S_n)$ for all α_{ij}, β_k, $\gamma_k \geqslant 0$. Thus every real $n \times n$ matrix which is non-negative off the diagonal is contained in $\mathscr{D}(S_n)$. □

Lemma 2.2. *Let T_n be the semigroup of all real non-singular $n \times n$ matrices with each row sum equal to* 1. *Then*

$$\mathscr{D}(T_n) = \left\{ \|c_{ij}\| \in \mathrm{gl}(n, \mathbb{R})\colon \sum_{j=1}^{n} c_{ij} = 0 \text{ for } i = 1, 2, \ldots, n \right\}.$$

P r o o f. Let $\Omega = \|\omega_{ij}\| \in \mathscr{D}(T_n)$. Then there exists $U(t) \in T_n$ such that $\Omega = (\frac{\mathrm{d}U(t)}{\mathrm{d}t})|_{t=0}$, $\sum\limits_{j=1}^{n} u_{ij}(t) = 1$ for $i = 1, 2, \ldots, n$, and $U(0) = I$. Hence

$$\begin{aligned} \sum_{j=1}^{n} \omega_{ij} &= \sum_{j=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}(u_{ij}(t))|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{n} u_{ij}(t)\right)\Bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(1)\Big|_{t=0} = 0 \quad \text{for } i = 1, 2, \ldots, n. \end{aligned}$$

□

Conversely suppose that $C = \|c_{ij}\|$ with $\sum\limits_{j=1}^{n} c_{ij} = 0$ for $i = 1, 2, \ldots, n$. Let

$$W = \left\{ \|b_{ij}\| \in \mathrm{gl}(n, \mathbb{R})\colon \sum_{j=1}^{n} b_{ij} = 0 \text{ for } i = 1, 2, \ldots, n \right\}.$$

Then W is a cone in $\mathrm{gl}(n, \mathbb{R})$ and $C \in W$. Also

$$C = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tC}\Big|_{t=0} = \lim_{t \to 0^+} \frac{\mathrm{e}^{tC} - I}{t}.$$

Since $C \in W$ and W is a cone, $\exp(tC) \in I + tW = I + W$ for $t \geqslant 0$. Since $\exp(tC)$ is non-singular, $\exp(tC) \in \mathrm{GL}(n, \mathbb{R}) \cap (I + W) \subset T_n$. Thus $C \in \mathscr{D}(T_n)$.

Lemma 2.3. *Let S_n be the semigroup of all $n \times n$ non-singular stochastic matrices. Then $\Omega = \|\omega_{ij}\|$ is an element of $\mathscr{D}(S_n)$ iff Ω is an $n \times n$ intensity matrix.*

P r o o f. It is clear that if S_n and T_n are subsemigroups of $\mathrm{GL}(n, \mathbb{R})$, then $\mathscr{D}(S_n \cap T_n) = \mathscr{D}(S_n) \cap \mathscr{D}(T_n)$. Thus the lemma is proved from Lemma 2.1 and Lemma 2.2. □

Theorem 2.4. *Let S_n be the semigroup of all $n \times n$ non-singular upper (or lower) triangular stochastic matrices. Then A is an element of $\mathscr{D}(S_n)$ iff A is an $n \times n$ upper (or lower) triangular intensity matrix.*

Proof. It is obvious that if T_n is the semigroup of all real $n \times n$ non-singular upper (or lower) triangular matrices, A is an element of $\mathscr{D}(T_n)$ iff A is a real $n \times n$ upper (or lower) triangular matrix. Hence the theorem is proved from Lemma 2.3. $\square$

3. Infinitesimally generated triangular stochastic matrices

Lemma 3.1. *Let A be an $n \times n$ non-singular upper triangular stochastic matrix of the following form:*

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & a_{pp} & a_{pp+1} & \dots & a_{pn} \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Then A can be represented as $A = \exp(t_{pp+1}E_{pp+1})\exp(t_{pp+2}E_{pp+2})\dots\exp(t_{pn}E_{pn})$, where E_{ij} is an extreme intensity matrix as denoted in the definition of Section 2.

Proof. Since A is stochastic, $a_{pp} + a_{pp+1} + \ldots + a_{pn} = 1$. Since A is upper triangular and non-singular, determinant of $A = a_{pp} > 0$. Let

$$x_{p+i} = \frac{a_{pp} + a_{pp+i+1} + \ldots + a_{pn}}{a_{pp} + a_{pp+i} + \ldots + a_{pn}} \quad \text{for } i = 1, 2, \ldots, n.$$

Then $0 < x_{p+i} \leqslant 1$ for $i = 1, 2, \ldots, n$ since $a_{pp} > 0$. For $i = 1$, $x_{p+1} = a_{pp} + a_{pp+2} + \ldots + a_{pn}$. Thus $a_{pp+1} = 1 - x_{p+1}$. Now,

$$x_{p+2} = \frac{a_{pp} + a_{pp+3} + \ldots + a_{pn}}{a_{pp} + a_{pp+2} + \ldots + a_{pn}} = \frac{a_{pp} + a_{pp+3} + \ldots + a_{pn}}{x_{p+1}}.$$

Hence $a_{pp+2} = x_{p+1} - x_{p+1}x_{p+2} = x_{p+1}(1 - x_{p+2})$. Inductively,

$$x_{p+1}x_{p+2}\dots x_{p+k-1} = a_{pp} + a_{pp+k} + \ldots + a_{pn}$$

for $k=2,\ldots,n-p$ and

$$x_{p+1}x_{p+2}\ldots x_{p+k-1}x_{p+k}=a_{pp}+a_{pp+k+1}+\ldots+a_{pn}.$$

Therefore

$$a_{pp+k}=x_{p+1}\ldots x_{p+k-1}(1-x_{p+k})\quad \text{for } k=2,\ldots,n-p.$$

We have

$$\begin{aligned}1&=a_{pp}+a_{pp+1}+a_{pp+2}+\ldots+a_{pn}\\&=a_{pp}+(1-x_{p+1})+x_{p+1}(1-x_{p+2})+\ldots+x_{p+1}\ldots x_{n-1}(1-x_n)\\&=a_{pp}+1-x_{p+1}\ldots x_n.\end{aligned}$$

Hence $a_{pp}=x_{p+1}x_{p+2}\ldots x_n$. Let $A_{x_{p+j}}$ $(j=1,2,\ldots,n-p)$ be an $n\times n$ upper triangular stochastic matrix of the following form:

$$A_{x_{p+j}}=\begin{pmatrix}1&0&\ldots&0&0&0&\ldots&0&\ldots&0\\0&1&\ldots&0&0&0&\ldots&0&\ldots&0\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ldots&\vdots&\ldots&\vdots\\0&0&\ldots&1&0&0&\ldots&0&\ldots&0\\0&0&\ldots&0&x_{p+j}&0&\ldots&1-x_{p+j}&\ldots&0\\0&0&\ldots&0&0&1&\ldots&0&\ldots&0\\\vdots&\vdots&\ldots&\vdots&\vdots&\vdots&\ddots&\vdots&\ldots&\vdots\\0&0&\ldots&0&0&0&\ldots&1&\ldots&0\\\vdots&\vdots&\ldots&\vdots&\vdots&\vdots&\ldots&\vdots&\ddots&\vdots\\0&0&\ldots&0&0&0&\ldots&0&\ldots&1\end{pmatrix},$$

where x_{p+j} is in the pth low and pth column and $1-x_{p+j}$ is in the pth low and $p+j$th column. Then $A=A_{x_{p+1}}A_{x_{p+2}}\ldots A_{x_n}$. Since $0<x_{p+j}\leqslant 1$, $A_{x_{p+j}}=\exp(t_{pp+j}E_{pp+j})$ for some $t_{pp+j}\geqslant 0$. Thus $A=\exp(t_{pp+1}E_{pp+1})\exp(t_{pp+2}E_{pp+2})\ldots\exp(t_{pn}E_{pn})$. □

Lemma 3.2. *If U is an $n\times n$ non-singular upper triangular stochastic matrix, then it can be represented as $U=C_{n-1}C_{n-2}\ldots C_1$, where $C_p=\exp(t_{pp+1}E_{pp+1})\ldots\exp(t_{pn}E_{pn})$ for $p=1,2,\ldots,n-1$ and $t_{ij}\geqslant 0$.*

Analogously, if L is an $n\times n$ non-singular lower triangular stochastic matrix, then it can be represented as $L=H_2H_3\ldots H_n$, where $H_p=\exp(s_{p1}E_{p1})\exp(s_{p2}E_{p2})\ldots\exp(s_{pp-1}E_{pp-1})$ for $p=2,\ldots,n$ and $s_{ij}\geqslant 0$.

P r o o f. Let $U_1, \dots, U_n$ be the rows of U such that $U = (U_1, \dots, U_n)^t$ and I_j be the jth row of $n \times n$ identity matrix. Then $U = C_{n-1}C_{n-2}\dots C_1$, where C_p is an $n \times n$ matrix such that $C_p = (I_1, I_2, \dots, I_{p-1}, U_p, I_{p+1}, \dots, I_n)^t$ for $p = 1, 2, \dots, n-1$. According to the Lemma 3.1, $C_p = \exp(t_{pp+1}E_{pp+1})\dots\exp(t_{pn}E_{pn})$.

The proof for the lower triangular case is similar to that for the upper triangular case. □

Theorem 3.3. *Each element in the semigroup S_n of all $n \times n$ non-singular upper (or lower) triangular stochastic matrices is generated from the infinitesimal elements of S_n, which form a cone consisting of all $n \times n$ upper (or lower) triangular intensity matrices.*

P r o o f. Immediate from Theorem 2.4 and Lemma 3.2. □

References

[1] *I. Chon*: Lie group and control theory. Ph.D. thesis at Louisiana state university, 1988.
[2] *F. R. Gantmacher*: The Theory of Matrices vol. 1 and vol. 2. Chelsea Publ. Comp., New York, 1960.
[3] *C. Loewner*: On totally positive matrices. Math. Zeitschr. *63* (1955), 338–340.
[4] *C. Loewner*: A theorem on the partial order derived from a certain transformation semigroup. Math. Zeitschr. *72* (1959), 53–60.
[5] *H. Min*: One parameter semigroups in Lie groups. Master's thesis at Seoul women's university, 1995.
[6] *V. S. Varadarajan*: Lie Groups, Lie Algebras, and Their Representations. Springer-Verlag, New York, 1984.

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