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COMPLETE DISTRIBUTIVITY OF LATTICE ORDERED GROUPS AND OF VECTOR LATTICES

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Abstract. In this paper we investigate the possibility of a regular embedding of a lattice ordered group into a completely distributive vector lattice.

Keywords: lattice ordered group, vector lattice, complete distributivity, regular embedding

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1. Introduction

We apply the notion of a vector lattice in the same sense as in Birkhoff [2] and Conrad [3]. In the monograph Luxemburg and Zaanen [11] vector lattices are called Riesz spaces. In Russian literature (cf., e.g., Vulikh [18], Kantorovich, Vulikh and Pinsker [9]) the term K-lineal is used.

Let G be an archimedean lattice ordered group. Lapellere and Valente [10] dealt with the possibility of embedding G into a complete vector lattice.

Pinsker [14] proved that if G is complete, then it can be embedded into a complete vector lattice; by applying the Dedekind completion we get that this result is valid for any archimedean lattice ordered group. A shorter and simpler proof of this fact was given by the author [5].

By applying the quoted theorem on the embedding and by using the well-known result on the representation of complete vector lattices (cf. Vulikh [18], Theorem V.4.2; for related results cf. also Maeda and Ogasavara [12] and Yosida [19]) we obtain a representation of archimedean lattice ordered groups by real functions admitting also the values $+\infty$ and $-\infty$ (this was pointed out already in [5]).

A direct proof concerning the representation of archimedean lattice ordered groups (without applying vector lattices) was given by Bernau [1].

Let α and β be cardinals. The notion of (α, β)-distributivity (and, in particular, of complete distributivity) for lattices, Boolean algebras and lattice ordered groups was investigated by several authors (cf., e.g., Pierce [13], Smith and Tarski [17], Redfield [15]).

Let G be an archimedean lattice ordered group. We denote by $S(G)$ the set of all singular elements of G. In the present paper we prove the following results:

(A) Assume that the set $S(G)$ is finite. Then the following conditions are equivalent:

  (i) G is completely distributive.

  (ii) There exists a complete vector lattice V such that G is regularly embedded into V and V is completely distributive.

(B) Let α and β be infinite cardinals. Assume that the set $S(G)$ is finite and that $\operatorname{card}[0, g] \leqslant \beta$ for each $0 < g \in G$. Then the following conditions are equivalent:

  (i) G is (α, β)-distributive.

  (ii) There exists a complete vector lattice V such that G is regularly embedded into V and V is (α, β)-distributive.

2. Preliminaries

For lattice ordered groups we apply the notation and terminology as in [2] and [3].

Let G be a lattice ordered group and let α, β be nonzero cardinals. G is called (α, β)-distributive if, whenever $(g_{ij})_{i \in I, j \in J}$ is an indexed system of elements of G with $\operatorname{card} I \leqslant \alpha$, $\operatorname{card} J \leqslant \beta$ then the relation

$$\bigwedge_{i \in I} \bigvee_{j \in J} g_{ij} = \bigvee_{f \in J^I} \bigwedge_{i \in I} g_{i\varphi(i)}$$

is valid provided the indicated joins and intersections exist.

G is completely distributive if it is (α, β)-distributive for any nonzero cardinals α and β.

Assume that G is an ℓ-subgroup of a lattice ordered group H such that

(i) whenever $(g_i)_{i \in I}$ is an indexed system of elements of G and $\bigvee\limits_{i \in I} g_i = g$ is valid in G, then g is the supremum of $(g_i)_{i \in I}$ in H as well;

(ii) the condition dual to (i) is satisfied.

Then we say that G is regularly embedded into H.

We remark that the term 'regular embedding' is used in an analogous way for Boolean algebras by Sikorski [16].

An element $0 < s \in G$ is called singular if the interval $[0, s]$ of G is a Boolean algebra (or, equivalently: if $x \wedge (s - x) = 0$ for each $x \in [0, s]$). (Cf. Conrad [3].)

Let $S(G)$ be as in Section 1. If $x, y \in G$, $0 < x \leqslant y$ and if $y \in S(G)$, then $x \in S(G)$. We denote by $A(G)$ the set of all atoms of the lattice G^+. Each element of $A(G)$ belongs to $S(G)$. If $S(G)$ is finite, then for each $0 < s \in S(G)$ there exists $a \in A(G)$ with $a \leqslant s$.

In what follows we assume that G is an archimedean lattice ordered group.

Let us consider expressions of the form x/n, where $x \in G$ and n is a positive integer. For x/n and y/m we put $x/n \leqslant y/m$ if $mx \leqslant ny$; if $mx = ny$, then we set $x/n = y/m$. Let G^d be the set of all such expressions (under the mentioned equality); then $\leqslant$ is a partial order on G^d. We define the operation $+$ in G^d by the usual rule

$$\frac{x}{n} + \frac{y}{m} = \frac{mx + ny}{nm}.$$

Then G^d turns out to be a divisible archimedean lattice ordered group. We identify the element $x/1$ with x. Under this identification, G is regularly embedded into G^d; cf., e.g., [5]. (We correct a mistake in [5]: on p. 268 it should be "integrally closed partially ordered group" instead of "abelian partially ordered group".)

G^d is called the divisible hull of G.

The above mentioned embedding of G into G^d is regular. In fact, if $\bigvee_{i \in I} g_i = g$ is valid in G and if $h \in G$, $g_i \leqslant h/n < g$ for each $i \in I$, then $ng_i \leqslant h < ng$ for each $i \in I$; but $\bigvee_{i \in I} ng_i = ng$, and so we arrive at a contradiction. For $\bigwedge_{i \in I} g_i$ we proceed analogously.

2.1. Theorem (cf. [3], [4]). *There exists a complete lattice ordered group G^D with the following properties:*

1) *G is regularly embedded into G^D;*
2) *if $h \in G^D$, then $h = \bigvee\{g \in G\colon\ g \leqslant h\}$;*
3) *if H is any complete lattice ordered group with the properties* 1) *and* 2)*, then there exists a unique isomorphism σ of G^D onto H such that $g\sigma = g$ for all $g \in G$;*
4) *if G contains no singular elements then G^D is a vector lattice;*
5) *if G is dense in a complete lattice ordered group H then G^D is the ℓ-ideal of H generated by G.*

G^D is called the Dedekind completion of G.

It is obvious that G^d has no singular elements; hence in view of 2.1, G^{dD} is a vector lattice and G is regularly embedded into G^{dD}. Thus we obtain as a corollary the main result of [10] (Theorem 2.1) saying that for each archimedean lattice ordered group G there exists a complete vector lattice V such that G is regularly embedded into V.

3. Direct product decompositions

Let G be as above. For $X \subseteq G$ we put

$$X^\delta = \{g \in G\colon\ |g| \wedge |x| = 0 \quad \text{for each } x \in X\};$$

X^δ is called the polar of G corresponding to the subset X. Each polar is a convex ℓ-subgroup of G.

The direct product of lattice ordered groups is defined in the usual way. For the direct product of lattice ordered groups $G_1, G_2, \ldots, G_n$ we apply the notation $G_1 \times G_2 \times \ldots \times G_n$.

The following result is well-known.

3.1. Lemma. *Let A be a convex ℓ-subgroup of G. Then A is a direct factor of G if and only if for each $0 \leqslant x \in G$ there exists $x^1 \in A$ such that*

$$x^1 = \bigvee\{t \in A^+\colon\ t \leqslant x\}.$$

If this condition is satisfied, then we have a direct product decomposition

$$G = A \times A^\delta$$

and x^1 is the component of the element x in the direct factor A; further, $A = A^{\delta\delta}$.

3.2. Lemma. *Assume that we have a direct product decomposition*

$$G = A \times B. \tag{1}$$

Then $G^d = A^d \times B^d$.

Proof. a) It is obvious that A^d is a subgroup of the group G^d. We consider the partial order on A^d which is inherited from G^d. Let $x/m \in A^d$. Put $y = x \vee 0$, $z = x \wedge 0$. Then $y, z \in A$, hence $y/n, z/n \in A^d$. We have

$$\frac{z}{n} \leqslant 0 \leqslant \frac{y}{n}, \quad \frac{z}{n} \leqslant \frac{x}{n} \leqslant \frac{y}{n}.$$

Therefore A^d is a directed group.

b) Let $x \in A$, $g \in G$, and $m, n \in \mathbb{N}$. Assume that

$$0 \leqslant \frac{g}{m} \leqslant \frac{x}{n}.$$

Then $0 \leqslant g$, $0 \leqslant x$ and

$$g \leqslant m\frac{x}{n} \leqslant mx,$$

whence $g \in A$ and $\frac{g}{m} \in A^d$. This yields that A^d is a convex subgroup of G^d.

c) From a) and b) we infer that A^d is a convex ℓ-subgroup of G^d.

d) Let $0 \leqslant x/n \in G^d$. Hence $0 \leqslant x$. In view of 3.1 there exists $x_1 \in G^+$ such that x_1 is the largest element of the set $\{a \in A^+ \colon a \leqslant x\}$.

We have $0 \leqslant x_1/n \leqslant x/n$, $x_1/n \in A^d$. Let $0 \leqslant y/m \in A^d$, $y/m \leqslant x/n$. Hence $0 \leqslant y$ and

$$ny \leqslant mx. \tag{2}$$

Thus $0 \leqslant ny \in A$.

For each $t \in G$ we denote by $t(A)$ the component of t in the direct factor A. Thus $x(A) = x_1$ and $y(A) = y$. Therefore in view of (2) we obtain

$$ny(A) = (ny)(A) \leqslant (mx)(A) = mx(A),$$
$$ny \leqslant mx_1, \quad \frac{y}{m} \leqslant \frac{x_1}{n}.$$

According to 3.1 we conclude that A^d is a direct factor of G^d. Analogously, B^d is a direct factor of G^d.

e) For $Z \subseteq G^d$ we put

$$Z^{\delta_1} = \{h \in G^d \colon |h| \wedge |z| = 0 \quad \text{for each } z \in Z\}.$$

Let $0 \leqslant x/n \in A^d$, $0 \leqslant y/m \in B^d$. Then $0 \leqslant x \in A$, $0 \leqslant y \in B$, whence $x \wedge y = 0$. Since $x/n \leqslant x$, $y/m \leqslant y$, we get

$$\frac{x}{n} \wedge \frac{y}{m} = 0.$$

This yields that $B^d \subseteq (A^\delta)^{\delta_1}$.

Let $0 \leqslant y/m \in (A^d)^{\delta_1}$. The polar $(A^d)^{\delta_1}$ of G^d is an ℓ-subgroup of G^d, hence

$$y = m\frac{y}{m} \in (A^d)^{\delta_1}.$$

Let $0 < x \in A$. Then $x \in A^d$, thus $x \wedge y = 0$. We obtain $y \in A^\delta$, therefore $y \in B$ and $y/m \in B^d$. Summarizing, $B^d = (A^d)^{\delta_1}$. Thus $G^d = A^d \times B^d$. $\square$

3.3. Proposition (cf. [11])**.** *Let* (1) *be valid. Then* $G^D = A^D \times B^D$.

3.4. Lemma. *Suppose that the set* $S(G)$ *is finite. Let* A *be the convex* ℓ*-subgroup of* G *which is generated by* $S(G)$*. Then*

(i) A *is a direct product of a finite number of linearly ordered groups;*

(ii) $G = A \times A^\delta$.

P r o o f. If $S(G) = \emptyset$, then the assertion is trivial. Suppose that $S(G)$ is nonempty, $S(G) = \{y_1, y_2, \ldots, y_n\}$. In this case the set $A(G)$ is also nonempty, $A(G) = \{x_1, x_2, \ldots, x_n\}$, $n \leqslant m$.

In view of [6], for each $i \in \{1, 2, \ldots, n\}$ there exists a linearly ordered group A_i such that

(i_1) A_i is a convex ℓ-subgroup of G which is generated by x_i,

(ii_1) $G = A_1 \times A_2 \times \ldots \times A_n \times B$, where $B = \{x_1, x_2, \ldots, x_n\}^\delta$.

It is clear that $A_1 \times A_2 \times \ldots \times A_n$ is the convex ℓ-subgroup of G which is generated by $S(G)$ and that $B = (A_1 \times A_2 \times \ldots \times A_n)^\delta$. □

4. Proofs of (A) and (B)

The following lemma is easy to verify, the proof will be omitted.

4.1. Lemma. *Let* X *be an archimedean linearly ordered group. Then both* X^δ *and* X^D *are linearly ordered.*

It is well-known that each linearly ordered group is completely distributive. Hence each direct product of linearly ordered groups is completely distributive as well.

Let G be as above.

4.2. Proposition. *If* G *is completely distributive, then* G^D *is completely distributive as well.*

P r o o f. This is a consequence of Theorem 2.2 in [8]. □

P r o o f o f (A). Let G be an archimedean lattice ordered group such that the set $S(G)$ is finite.

a) The implication (ii) ⇒ (i) is obviously valid.

b) Assume that the condition (i) is satisfied.

First suppose that the set $S(G)$ is empty. Then in view of 2.1, G^D is a vector lattice. Also, G is regularly embedded into G^D. Moreover, in view of 4.2, G^D is completely distributive. Thus (ii) holds.

Now suppose that $S(G) \neq \emptyset$. Hence $A(G)$ is nonempty and finite. Let us apply the same notation as in the proof of 3.4. Put $A_i^D = A_{i1}$ $(i = 1, 2, \ldots, n)$, $B^0 = B_1$. In view of 3.3 we have

$$G^D = A_{11} \times A_{21} \times \ldots \times A_{n1} \times B_1.$$

According to 4.1 and 4.2, G^D is completely distributive. Next, G is regularly embedded into G^D.

We set $A_{i1}^d = A_{i2}$ $(i = 1, 2, \ldots, n)$. Hence in view of 4.1, all A_{i2} are linearly ordered groups. Since B_1 is a vector lattice, we have $B_1^d = B_1$. Then Lemma 3.2 yields

$$G^{Dd} = A_{12} \times A_{22} \times \ldots \times A_{n2} \times B_1.$$

Further, G^{Dd} is completely distributive and G is regularly embedded into G^{Dd}.

Since G^{Dd} is divisible, in view of 2.1 we obtain that $V = G^{DdD}$ is a complete vector lattice. G is regularly embedded into V. According to 3.3,

$$V = A_{12}^D \times A_{22}^D \times \ldots \times A_{n2}^D \times B_1$$

since $B^D = B_1$. By 4.1, V is completely distributive. □

Now let α and β be infinite cardinals. Consider the following condition for a lattice ordered group X:

(c(β)) If $0 < x \in X$, then $\operatorname{card}[0, x] \leqslant \beta$.

4.3. Lemma. *Let X be an archimedean lattice ordered group satisfying the condition* c(β). *Then X^d satisfies this condition as well.*

Proof. This is an immediate consequence of the construction of X^d (cf. Section 2). □

4.4. Proposition. *Let X be an archimedean lattice ordered group. Assume that X is (α, β)-distributive and satisfies the condition* c(β). *Then X^D is (α, β)-distributive.*

Proof. This is a particular case of Theorem 2.2 in [8]. □

Proof of (B).

We proceed analogously as in the proof of (A) and apply the same notation. Clearly (ii) ⇒ (i). Suppose that (i) holds.

If $S(G) = \emptyset$, then it suffices to put $V = G^D$ and apply 4.4.

Let $S(G) \neq \emptyset$. Then B satisfies the condition c(β) and is (α, β)-distributive. Hence according to 4.4, B_1 is (α, β)-distributive.

Let V be as in the proof of (A). Thus V is a complete vector lattice, it is (α, β)-distributive and G is regularly embedded into V. Therefore (ii) holds. □

We remark that (B) could be applied for establishing a new version of the proof of (A).

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