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A COMPLETION OF $\mathbb{Z}$ IS A FIELD

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Abstract. We define various ring sequential convergences on $\mathbb{Z}$ and $\mathbb{Q}$. We describe their properties and properties of their convergence completions. In particular, we define a convergence $\mathbb{L}_1$ on $\mathbb{Z}$ by means of a nonprincipal ultrafilter on the positive prime numbers such that the underlying set of the completion is the ultraproduct of the prime finite fields $\mathbb{Z}/(p)$. Further, we show that $(\mathbb{Z}, \mathbb{L}_1^*)$ is sequentially precompact but fails to be strongly sequentially precompact; this solves a problem posed by D. Dikranjan.

Keywords: sequential convergence, convergence ring, completion of a convergence ring

MSC 2000: 54A20 13J99

1. Introduction

We introduce some ring sequential convergences on $\mathbb{Z}$ and $\mathbb{Q}$. Our main result is a ring convergence $\mathbb{L}_1$ on $\mathbb{Z}$ such that the completion of $(\mathbb{Z}, \mathbb{L}_1)$ is a field which consists of an ultraproduct of prime finite fields $\mathbb{Z}/(p)$. We extend this convergence to some ring convergences on $\mathbb{Q}$. These convergences are not given by any ring topology.

We recall Ostrowski's theorem: Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to a p-adic absolute value or to the usual absolute value. As a counterpoint, one can desire a ring topology on $\mathbb{Z}$ or $\mathbb{Q}$ for which there exists an infinite set of primes $D = \{q_1, q_2, \ldots, q_n, \ldots\}$ such that

$$\lim_{n\to\infty} (q_1 q_2 \ldots q_n) = 0. \tag{1}$$

We get this by means of the filter of ideals

$$(q_1) \supset (q_1 q_2) \supset \ldots \supset (q_1 q_2 \ldots q_n) \supset \ldots$$

as a basis of zero neighborhoods for a ring topology on $\mathbb{Z}$. However, the completion of $\mathbb{Z}$ with this ring topology has zero divisors. The idea is to use an ultrafilter in the set of prime numbers to define a ring convergence that also satisfies (1), but now the completion of $\mathbb{Z}$ is a field. In the last section we present a completion of $\mathbb{Z}$ which is an ultraproduct of the rings of p-adic integers.

In some situations the topologies are not suitable, and then one can employ sequential convergences fruitfully. For instance, in spaces of functions with the pointwise convergence it was done in [2].

1.1. Basic definitions and properties.

Information about sequential convergences, $\mathscr{L}_0$-groups and $\mathscr{L}_0$-rings can be found in the papers [7], [9], [10], [11], [15] and in the book [14, § 20]. All rings and fields considered are commutative. The set of all strictly increasing mappings from $\mathbb{N}$ into $\mathbb{N}$ is denoted by MON. Let X be a set. For each sequence $S = \langle S(n)\rangle \in X^{\mathbb{N}}$ and each $s \in$ MON, the composite $S \circ s = \langle S(s(n))\rangle$ denotes the subsequence that corresponds to s. If X is equipped with an algebraic structure, then the operations in $X^{\mathbb{N}}$ are defined pointwise. For each $x \in X$, the corresponding constant sequence is denoted by $\langle x\rangle$.

A sequential convergence, or simply a convergence, $\mathbb{L}$ in X is a collection $\mathbb{L} \subseteq X^{\mathbb{N}} \times X$ of sequences and their limits. The expression $(S, x) \in \mathbb{L}$, or $S \in \mathbb{L}^{\leftarrow}(x)$, means that the sequence S converges to x. We always assume the following axioms:

(F) if $S \in \mathbb{L}^{\leftarrow}(x)$, then $S \circ s \in \mathbb{L}^{\leftarrow}(x)$ for each subsequence $S \circ s$,

(S) $\langle x\rangle \in \mathbb{L}^{\leftarrow}(x)$ for each constant sequence $\langle x\rangle$,

(H) the uniqueness of limits, i.e., if $S \in \mathbb{L}^{\leftarrow}(x)$ and $S \in \mathbb{L}^{\leftarrow}(y)$, then $x = y$.

A convergence fulfilling these axioms is called an $\mathscr{L}_0$-convergence. The pair $(X, \mathbb{L})$ is called an $\mathscr{L}_0$-space. We do not usually assume the Urysohn axiom:

(U) given $S \in X^{\mathbb{N}}$ and $x \in X$, if for each $s \in$ MON there exists $t \in$ MON such that $(S \circ s \circ t, x) \in \mathbb{L}$, then $(S, x) \in \mathbb{L}$.

If this axiom is satisfied we call the convergence an $\mathscr{L}_0^*$-convergence.

If X is a ring, the algebraic operations are sequentially continuous if the following axiom is fulfilled:

(Lr) if $(S, x), (T, y) \in \mathbb{L}$, then $(S - T, x - y) \in \mathbb{L}$ and $(ST, xy) \in \mathbb{L}$.

That is, the convergence and the ring operations are compatible. In this case, $(X, \mathbb{L})$ is called an $\mathscr{L}_0$-ring. If X is a field we consider the following axiom:

(Lf) if $(S, x), (T, y) \in \mathbb{L}$, then $(S - T, x - y) \in \mathbb{L}$ and $(ST, xy) \in \mathbb{L}$, and if $x \neq 0$ and $S(n) \neq 0$ for all $n \in \mathbb{N}$, then $(S^{-1}, x^{-1}) \in \mathbb{L}$.

In case it is satisfied, we call $(X, \mathbb{L})$ an $\mathscr{L}_0$-field.

If $(X, \mathbb{L})$ is an $\mathscr{L}_0$-space, then the Urysohn modification $\mathbb{L}^*$ of $\mathbb{L}$ is defined as follows: $(S, x) \in \mathbb{L}^*$ whenever each subsequence of S contains another subsequence

$\mathbb{L}$-converging to x. In this situation $(X, \mathbb{L}^*)$ is an $\mathscr{L}_0^*$-space. If $(X, \mathbb{L})$ is an $\mathscr{L}_0$-ring (field), then the Urysohn modification preserves the continuity of algebraic operations. In an $\mathscr{L}_0$-ring (field) the convergence is homogeneous and the set of zero sequences $\mathbb{L}^{\leftarrow}(0)$ determines the convergence.

Let $(X, \mathbb{L})$ be an $\mathscr{L}_0$-space. For each subset $A \subseteq X$ we define its closure $\operatorname{cl} A$ to be the set of all limits of convergent sequences ranging in A. This closure operator is not necessarily idempotent, and consequently need not produce a topology. For every ordinal number α and for every subset A we define α-$\operatorname{cl} A$ as follows:

- $1\text{-}\operatorname{cl} A = \operatorname{cl} A$,
- $\alpha\text{-}\operatorname{cl} A = \operatorname{cl}(\beta\text{-}\operatorname{cl} A)$ if $\alpha = \beta + 1$,
- $\alpha\text{-}\operatorname{cl} A = \cup_{\beta<\alpha}(\beta\text{-}\operatorname{cl} A)$ if α is a limit ordinal.

Then each α-$\operatorname{cl} A$ is a closure operator for X (see [8]). Let ω_1 be the first uncountable ordinal number. For each subset $A \subseteq X$ we have $\operatorname{cl}(\omega_1\text{-}\operatorname{cl} A) = \omega_1\text{-}\operatorname{cl} A$; therefore ω_1-$\operatorname{cl} A$ is idempotent and hence topological. The sequential order of $(X, \mathbb{L})$ is the least ordinal number $\sigma \geqslant 1$ such that $\operatorname{cl}(\sigma\text{-}\operatorname{cl} A) = \sigma\text{-}\operatorname{cl} A$ for each subset $A \subseteq X$. Clearly $1 \leqslant \sigma \leqslant \omega_1$. If $X = \operatorname{cl} A$, then A is said to be *closure dense*, and if $X = \omega_1$-$\operatorname{cl} A$, then A is said to be *topologically dense*.

Let $(X, \mathbb{L})$ be an $\mathscr{L}_0$-ring; a sequence S in X is said to be $\mathbb{L}$-Cauchy if $S \circ t - S \circ s \in \mathbb{L}^{\leftarrow}(0)$ for all $s, t \in \mathrm{MON}$. This is equivalent to saying that $S - S \circ s \in \mathbb{L}^{\leftarrow}(0)$ for all $s \in \mathrm{MON}$. As usual, if all $\mathbb{L}$-Cauchy sequences in X converge, then $(X, \mathbb{L})$ is said to be complete.

Let $(X', \mathbb{L}')$ be an $\mathscr{L}_0$-ring, let X be a subring of X' and let $\mathbb{L} = \mathbb{L}'|_X$. Then $(X', \mathbb{L}')$ is said to be an *extension* of $(X, \mathbb{L})$.

Let $(X, \mathbb{L})$ be an $\mathscr{L}_0$-ring. Then its *completion* is an extension $(X', \mathbb{L}')$ which is complete and such that X is topologically dense in $(X', \mathbb{L}')$. If X is closure dense in $(X', \mathbb{L}')$, we call it a *dense completion.* The completion, in case it exists, is far from being unique. The following condition is necessary for an $\mathscr{L}_0$-ring to have a completion:

($\mathrm{C_r}$) $S, T \in X^{\mathbb{N}}$, if S is $\mathbb{L}$-Cauchy and $T \in \mathbb{L}^{\leftarrow}(0)$, then $ST \in \mathbb{L}^{\leftarrow}(0)$.

We consider a stronger condition: An $\mathscr{L}_0$-ring $(X, \mathbb{L})$ is *bounded* if $ST \in \mathbb{L}^{\leftarrow}(0)$ whenever $S \in X^{\mathbb{N}}$ and $T \in \mathbb{L}^{\leftarrow}(0)$. A field cannot be bounded.

We say that a completion $(\hat{X}, \hat{\mathbb{L}})$ of an $\mathscr{L}_0$-ring $(X, \mathbb{L})$ is its categorical ring completion (in the category of $\mathscr{L}_0$-rings) if the following universal property holds: for each continuous homomorphism f from $(X, \mathbb{L})$ into a complete $\mathscr{L}_0$-ring $(Y, \mathbb{M})$ there exists a unique continuous homomorphism $\hat{f}$ from $(\hat{X}, \hat{\mathbb{L}})$ into $(Y, \mathbb{M})$ such that $f(x) = \hat{f}(x)$ for all $x \in X$. We have the same concept in the category of $\mathscr{L}_0^*$-rings. In [2] it was shown that, if an $\mathscr{L}_0^*$-ring has a completion, then it has the categorical one.

Let X be a set with two $\mathscr{L}_0$-convergences $\mathbb{L}_1 \subseteq \mathbb{L}_2$. We say that $\mathbb{L}_1$ is finer than $\mathbb{L}_2$, and $\mathbb{L}_2$ is coarser than $\mathbb{L}_1$.

Let X be a ring; an $\mathscr{L}_0$-ring convergence $\mathbb{L}$ in X is *coarse* if there is no $\mathscr{L}_0$-ring convergence strictly larger than $\mathbb{L}$. In this case $\mathbb{L}$ satisfies the Urysohn axiom, that is, $\mathbb{L} = \mathbb{L}^*$. The concept of the coarse ring convergence is the counterpart of the concept of the minimal ring topology in the theory of topological rings.

The following results are easy to prove (cf. [6], [11]).

Lemma 1. *Let X be a ring and let $\mathbb{L}$ be an $\mathscr{L}_0$-ring convergence on X. Then $\mathscr{A} = \mathbb{L}^{\leftarrow}(0)$ has the following properties:*

(i) *$\mathscr{A}$ is a subring (without unit) of the ring $X^{\mathbb{N}}$.*

(ii) *If $S \in \mathscr{A}$ then $S \circ s \in \mathscr{A}$ for each $s \in$ MON.*

(iii) *$\langle x\rangle\mathscr{A} \subseteq \mathscr{A}$ for each $x \in X$.*

(iv) *$\langle x\rangle \notin \mathscr{A}$ whenever $x \neq 0$.*

(v) *$\mathbb{L}^{\leftarrow}(x) = \mathscr{A} + \langle x\rangle$ for each $x \in X$.*

Lemma 2. *Let X be a ring and let $\mathscr{A}$ be a subset of $X^{\mathbb{N}}$ satisfying conditions* (i)–(iv). *Then there is an $\mathscr{L}_0$-ring convergence $\mathbb{L}$ on X such that $\mathbb{L}^{\leftarrow}(0) = \mathscr{A}$.*

Lemma 3. *Let X be a field and let $\mathbb{L}$ be an $\mathscr{L}_0$-ring convergence on X. Then $\mathbb{L}$ is an $\mathscr{L}_0$-field convergence if and only if the following condition is satisfied:*

(vi) *If each $S \in \mathbb{L}^{\leftarrow}(0)$ with $S(n) \neq -1$ for all $n \in \mathbb{N}$, then $S(n)/(1+S(n)) \in \mathbb{L}^{\leftarrow}(0)$.*

In some articles a different terminology is used: An $\mathscr{L}_0$-convergence is called an FSH-convergence, an $\mathscr{L}_0^*$-convergence is called an FUSH-convergence, an $\mathscr{L}_0^*$-ring is called an FLUSH-ring, ...

2. The ring convergence on $\mathbb{Z}$

The expression "almost all $n \in \mathbb{N}$" means all natural numbers, except possibly a finite number of them.

Let $\mathbb{P}$ be the set of positive prime numbers. Let $\mathscr{U}$ be a nonprincipal ultrafilter in $\mathbb{P}$. We define an $\mathscr{L}_0$-ring convergence $\mathbb{L}_1$ on $\mathbb{Z}$: a sequence $(a_n)_{n\in\mathbb{N}}$ converges to zero if the set of primes $p \in \mathbb{P}$ such that p divides a_n for almost all $n \in \mathbb{N}$ belongs to $\mathscr{U}$. It is easy to check that $\mathbb{L}_1^{\leftarrow}(0)$ satisfies conditions (i)–(iv) in Lemma 2. It is obvious that $\mathbb{L}_1$ is a bounded ring convergence on $\mathbb{Z}$, and consequently, fulfils the condition $(\mathrm{C_r})$.

We will use the following elementary result:

Lemma 4. *Let $S \in \mathbb{Z}^{\mathbb{N}}$ be any sequence of integers. Then it has a subsequence β that satisfies the following condition: each prime p which divides infinitely many terms $\beta(n)$ divides almost all of them.*

P r o o f. We shall construct successive subsequences of S, from which we get the required subsequence β. Let $q_1, \dots, q_i$ be the prime numbers that divide $S(1)$. We define a subsequence $S_{1,1}$ as follows: $S_{1,1}(1) = S(1)$. If q_1 divides only finitely many terms $S(n)$, then $S_{1,1}(n) = S(n)$ for all n. If q_1 divides infinitely many terms $S(n)$, then we define $S_{1,1}$ to be the subsequence consisting just of these terms. Now we define another subsequence $S_{1,2}$ as follows: $S_{1,2}(1) = S(1)$. If q_2 divides only finitely many terms $S_{1,1}(n)$, then $S_{1,2}(n) = S_{1,1}(n)$ for all n. If q_2 divides infinitely many terms $S_{1,1}(n)$, then we define $S_{1,2}$ to be the subsequence consisting just of these terms. We proceed analogously with $q_3, \dots, q_i$. After i steps we have defined $S_{1,i}$.

Now let $p_1, \dots, p_j$ be the primes that divide $S_{1,i}(2)$. We define a subsequence $S_{2,1}$ as follows: $S_{2,1}(1) = S(1)$, $S_{2,1}(2) = S_{1,i}(2)$. If p_1 divides only finitely many terms $S_{1,i}(n)$, then $S_{2,1}(n) = S_{1,i}(n)$ for all n. If p_1 divides infinitely many terms $S_{1,i}(n)$, then we define $S_{2,1}$ to be the subsequence consisting just of these terms. We continue in the same way and, after j steps, we get $S_{2,j}$. Now let $t_1, \dots, t_k$ be the primes which divide $S_{2,j}(3)$. In the same manner we get the subsequences $S_{3,1}, \dots, S_{3,t}$, and so on.

Finally, we define β, a subsequence of S, as follows: $\beta(n) = S_{n,1}(n)$ for all n. It satisfies the required condition. $\square$

It is easy to check that a sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ is $\mathbb{L}_1$-Cauchy if and only if the following condition is satisfied: the set of primes p for which there exists k_p such that $p \mid a_n - a_m$ for all $n, m \geqslant k_p$ belongs to $\mathscr{U}$.

According to [4], a convergence group $(G, \mathscr{L}_0)$ is *sequentially precompact* if each sequence has a Cauchy subsequence. A convergence group $(G, \mathscr{L}_0)$ is *strongly sequentially precompact* if there exists a sequentially compact group containing $(G, \mathscr{L}_0)$ as a convergence subgroup. (In [17] the expression “sequentially precompact” means the latter of the two concepts).

Lemma 5. *The $\mathscr{L}_0$-ring $(\mathbb{Z}, \mathbb{L}_1)$ is sequentially precompact.*

P r o o f. Let $S = (S(n))_{n\in\mathbb{N}}$ be any sequence of integers. We take a set of primes $B = \{q_1, q_2, \dots\} \in \mathscr{U}$. We are taking successive subsequences $S_1, S_2, \dots$ of S, with S_m being a subsequence of S_{m-1}, in the following manner:

Let $S_1(1) = S(1)$. There exists $i \in \{0, 1, \dots, q_1 - 1\}$ and an infinite subset $T_1 \subseteq \mathbb{N} \setminus \{1\}$ such that

$$S(n) \equiv i \bmod (q_1) \quad \text{for all } n \in T_1.$$

Let $t \in \mathrm{MON}$ be defined by $t(1) = 1$ and $\{t(n) \colon n \geqslant 2\} = T_1$. Put $S_1 = S \circ t$. If we have constructed S_{m-1}, we define S_m as follows: $S_m(n) = S_{m-1}(n)$ for $n = 1, 2, \dots, m$. There exist $i \in \{0, 1, \dots, q_m - 1\}$ and an infinite subset $T_m \subseteq \mathbb{N} \setminus$

$\{1, \ldots, m\}$ such that

$$S_{m-1}(n) \equiv i \bmod (q_m) \quad \text{for all } n \in T_m.$$

Let $v \in \mathrm{MON}$ be defined by $v(n) = n$ for $n = 1, \ldots, m$ and $\{v(n) \colon n \geqslant m+1\} = T_m$. Put $S_m = S_{m-1} \circ v$. Finally, let Y be the subsequence of S defined by $Y(n) = S_m(n)$ for $n \leqslant m$. It is also a subsequence of each S_m. Each $q_n \in B$ satisfies that $q_n \mid Y(j) - Y(k)$ for all $j, k \geqslant n$. Consequently, Y is an $\mathbb{L}_1$-Cauchy sequence. $\square$

Lemma 6. *Let $\alpha, \beta \in \mathbb{Z}^{\mathbb{N}}$. There exists $v \in \mathrm{MON}$ such that both $\alpha \circ v$ and $\beta \circ v$ are Cauchy sequences in $(\mathbb{Z}, \mathbb{L}_1)$.*

Proof. There exists $s \in \mathrm{MON}$ such that $\alpha \circ s$ is $\mathbb{L}_1$-Cauchy. For $\beta \circ s$ there exists $t \in \mathrm{MON}$ such that $\beta \circ s \circ t$ is $\mathbb{L}_1$-Cauchy. Thus both $\alpha \circ (s \circ t)$ and $\beta \circ (s \circ t)$ are $\mathbb{L}_1$-Cauchy. $\square$

Lemma 7. *Each nonzero ideal $(a) \subseteq \mathbb{Z}$ is closure dense in $\mathbb{Z}$.*

Proof. Let $b \in \mathbb{Z} \setminus (a)$. Let $A = \{q_1, q_2, \ldots\} \in \mathscr{U}$ be the set of primes which do not divide a. We choose a sequence $(b_n)_{n \in \mathbb{N}}$ that satisfies

$$b_n \equiv 0 \bmod (a), \quad b_n \equiv b \bmod (q_1 \ldots q_n) \quad \text{for all } n \in \mathbb{N}.$$

The sequence $(b_n)_{n \in \mathbb{N}} \subseteq (a)$ converges to b. Therefore $\mathrm{cl}(a) = \mathbb{Z}$. $\square$

Lemma 8. *Let $(a_n)_{n \in \mathbb{N}}$ be an $\mathbb{L}_1$-Cauchy sequence not converging to zero. Then there exists another $\mathbb{L}_1$-Cauchy sequence $(b_n)_{n \in \mathbb{N}}$ such that $(a_n b_n)_{n \in \mathbb{N}}$ converges to 1.*

Proof. We consider the sets

$$B = \{p \in \mathbb{P} \colon p \mid a_n \quad \text{for almost all } n \in \mathbb{N}\} \notin \mathscr{U},$$
$$E = \{p \in \mathbb{P} \colon p \mid a_n - a_m \quad \text{for all } n, m \geqslant k_p\} \in \mathscr{U}.$$

Let $F = E \cap (\mathbb{P} \setminus B) \in \mathscr{U}$; we write $F = \{q_1, q_2, \ldots\}$. For each $q_i \in F$ there exists $n_i \in \mathbb{N}$ such that $q_i \nmid a_n$ and $q_i \mid a_n - a_m$ for all $n, m \geqslant n_i$. We choose a sequence $(b_n)_{n \in \mathbb{N}}$ which satisfies the following conditions:

$$\begin{aligned} b_n = 1 \quad & \text{for } n < n_1, \\ a_n b_n \equiv 1 \bmod (q_1) \quad & \text{for } n_1 \leqslant n < n_2, \\ \vdots \\ a_n b_n \equiv 1 \bmod (q_1 q_2 \ldots q_i) \quad & \text{for } n_i \leqslant n < n_{i+1}. \end{aligned}$$

It is clear that $(b_n)_{n \in \mathbb{N}}$ is an $\mathbb{L}_1$-Cauchy sequence and that $(a_n b_n)_{n \in \mathbb{N}}$ converges to 1. $\square$

We extend the $\mathscr{L}_0$-ring convergence $\mathbb{L}_1$ from $\mathbb{Z}$ to an $\mathscr{L}_0$-ring convergence $\mathbb{L}_2$ on $\mathbb{Q}$ in the following easy way:

A sequence $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ $\mathbb{L}_2$-converges to zero if there exists $(a_n)_{n\in\mathbb{N}} \in \mathbb{L}_1^{\leftarrow}(0) \subseteq \mathbb{Z}^{\mathbb{N}}$ and $c \in \mathbb{N}$ such that $\alpha_n = a_n/c$ for all $n \in \mathbb{N}$. The following result is an obvious consequence of Lemmas 2 and 3.

Lemma 9. *$\mathbb{L}_2$ is an $\mathscr{L}_0$-ring convergence on $\mathbb{Q}$. It is not a field convergence. Further, $\mathbb{L}_2|_{\mathbb{Z}} = \mathbb{L}_1$. Any other $\mathscr{L}_0$-ring convergence $\mathbb{L}_b$ on $\mathbb{Q}$ that satisfies $\mathbb{L}_b|_{\mathbb{Z}} = \mathbb{L}_1$ is coarser than $\mathbb{L}_2$, i.e., $\mathbb{L}_2 \subseteq \mathbb{L}_b$.*

We will construct a dense completion of $(\mathbb{Z}, \mathbb{L}_1)$. We denote the ring of $\mathbb{L}_1$-Cauchy sequences by C_1. Two Cauchy sequences $S, T \in C_1$ are equivalent if $S - T \in \mathbb{L}_1^{\leftarrow}(0)$. Algebraically, the completion $\hat{\mathbb{Z}}$ is the ring of classes of equivalent Cauchy sequences. That is, $\hat{\mathbb{Z}}$ is the quotient ring of the ring C_1 by the ideal of null sequences $\mathbb{L}_1^{\leftarrow}(0)$. This quotient exists since $(\mathbb{Z}, \mathbb{L}_1)$ satisfies the condition $(\mathrm{C_r})$.

Lemma 10. *The completion ring $\hat{\mathbb{Z}}$ is a field of zero characteristic.*

P r o o f. It is enough to consider Lemma 8 and the fact that $\mathbb{Z} \subseteq \hat{\mathbb{Z}}$. □

We will use an ultraproduct of prime finite fields. The reader may consult [3], [13], [18]. Let $\mathscr{U}$ be the ultrafilter on $\mathbb{P}$ used for defining $\mathbb{L}_1$. For each prime $p \in \mathbb{P}$ we take the corresponding prime finite field $\mathbb{F}_p = \mathbb{Z}/(p) = GF(p)$. We take the product $\prod\limits_{p\in\mathbb{P}} \mathbb{F}_p$ of all of them. In this product we consider the equivalence relation

$$(a_p)_{p\in\mathbb{P}} \sim (b_p)_{p\in\mathbb{P}} \stackrel{\text{def}}{\Longleftrightarrow} \{p \in \mathbb{P} \colon a_p = b_p\} \in \mathscr{U}.$$

The ultraproduct is the quotient ring of equivalence classes by this relation. It is a field, which we denote by

$$\mathbb{F} = \frac{\prod_{p\in\mathbb{P}} \mathbb{F}_p}{\mathscr{U}}.$$

We can also take the maximal ideal $M = \{(a_p)_{p\in\mathbb{P}} \colon \{p \in \mathbb{P} \colon a_p = 0\} \in \mathscr{U}\}$ in $\prod\limits_{p\in\mathbb{P}} \mathbb{F}_p$, and then the ultraproduct is the quotient

$$\mathbb{F} = \frac{\prod_{p\in\mathbb{P}} \mathbb{F}_p}{M}.$$

This field has the cardinality of continuum.

Theorem 11. *The completion ring $\hat{\mathbb{Z}}$ is isomorphic to the field ultraproduct $\mathbb{F} = \prod\limits_{p\in\mathbb{P}} \mathbb{F}_p/\mathscr{U}$.*

P r o o f. Let C_1 be the ring of $\mathbb{L}_1$-Cauchy sequences. We define a ring homomorphism

$$f\colon\, C_1 \longrightarrow \frac{\prod_{p\in\mathbb{P}}\mathbb{F}_p}{\mathscr{U}}$$

as follows: for $\alpha=(a_n)_{n\in\mathbb{N}}\in C_1$ there exists $B\in\mathscr{U}$ such that each $p\in B$ satisfies $p\mid a_n-a_m$ for all $n,m\geqslant k_p$ for a certain constant k_p. We choose $\beta=[b_p]_{p\in\mathbb{P}}\in\mathbb{F}$ fulfilling the following condition: if $p\in B$, then $b_p=\overline{a_n}$, the class of a_n mod(p) for $n\geqslant k_p$; if $p\notin B$, then $b_p=\overline{0}$. We set $f(\alpha)=\beta$. It is easy to check that f is a homomorphism, f is onto, and $\ker(f)=\mathbb{L}_1^{\leftarrow}(0)$. Therefore

$$\hat{\mathbb{Z}}=\frac{C_1}{\mathbb{L}_1^{\leftarrow}(0)}\cong\mathbb{F}.$$

$\square$

Now, we have to define an $\mathscr{L}_0$-ring convergence on $\hat{\mathbb{Z}}$. We follow a process similar to that used in [9], [10], [12] to construct $\varrho\mathbb{Q}$, the categorical $\mathscr{L}_0^*$-ring completion of $\mathbb{Q}$.

Let (α_n) be an $\mathbb{L}_2$-Cauchy sequence. It is easy to see that there exist an $\mathbb{L}_1$-Cauchy sequence and $c\in\mathbb{N}$ such that $\alpha_n=a_n/c$ for all n. The $\mathbb{L}_2$-limit of (α_n) is the class $[a_n]c^{-1}$ in $\hat{\mathbb{Z}}$. Let $\{1\}\cup B$ be a basis of $\hat{\mathbb{Z}}$ as a linear space over $\mathbb{Q}$. Let $\mathbb{L}_3$ be the set of all pairs $(S,x)\in\hat{\mathbb{Z}}^{\mathbb{N}}\times\hat{\mathbb{Z}}$ such that S is of the form

$$S(n)=S_0(n)+S_1(n)b_1+\ldots+S_k(n)b_k\quad\text{for all } n\in\mathbb{N},\tag{2}$$

where $k\in\mathbb{N}$, $b_1,\ldots,b_k\in B$, S_i is a Cauchy sequence in $(\mathbb{Q},\mathbb{L}_2)$ for $i=0,1,\ldots,k$, and $x=x_0+x_1b_1+\ldots+x_kb_k$, where x_i is the $\mathbb{L}_2$-limit of the Cauchy sequence S_i, $i=,0,1,\ldots,k$. The following result is easy to prove:

Lemma 12. *$\mathbb{L}_3$ does not depend on the choice of the basis $\{1\}\cup B$. $\mathbb{L}_3$ is an $\mathscr{L}_0$-ring convergence on $\hat{\mathbb{Z}}$. Further, $\mathbb{L}_3|_{\mathbb{Q}}=\mathbb{L}_2$ and $\mathbb{L}_3|_{\mathbb{Z}}=\mathbb{L}_1$.*

This leads to our main result.

Theorem 13. *The $\mathscr{L}_0$-ring $(\hat{\mathbb{Z}},\mathbb{L}_3)$ is the categorical $\mathscr{L}_0$-ring completion of the convergence rings $(\mathbb{Z},\mathbb{L}_1)$ and $(\mathbb{Q},\mathbb{L}_2)$.*

P r o o f. First, we prove that $(\hat{\mathbb{Z}},\mathbb{L}_3)$ is complete. Let S be an $\mathbb{L}_3$-Cauchy sequence in $\hat{\mathbb{Z}}$. There exists a finite set $\{b_1,\ldots,b_k\}\subset B$ such that $S(n)$ belongs to the $\mathbb{Q}$-linear subspace $\langle 1,b_1,\ldots,b_k\rangle$ for all $n\in\mathbb{N}$. Otherwise, we could find a subsequence $S\circ s$ such that $\{S(n)-S(s(n))\colon\, n\in\mathbb{N}\}$ would not be included in any finite linear subspace of $\hat{\mathbb{Z}}$, and hence $S-S\circ s\notin\mathbb{L}_3^{\leftarrow}(0)$. Therefore S is of the form

$$S(n)=S_0(n)+S_1(n)b_1+\ldots+S_k(n)b_k\qquad\text{for all } n\in\mathbb{N},$$

where each S_i, $i = 0, 1, \ldots, k$ is an $\mathbb{L}_2$-Cauchy sequence. Thus S has a limit in $(\hat{\mathbb{Z}}, \mathbb{L}_3)$. Consequently, $(\hat{\mathbb{Z}}, \mathbb{L}_3)$ is complete.

As $\mathbb{Z} \subset \mathbb{Q} \subset \hat{\mathbb{Z}}$ and both $\mathbb{L}_3|_{\mathbb{Q}} = \mathbb{L}_2$, $\mathbb{L}_3|_{\mathbb{Z}} = \mathbb{L}_1$, we have proved that $(\hat{\mathbb{Z}}, \mathbb{L}_3)$ is a dense completion of the convergence rings $(\mathbb{Z}, \mathbb{L}_1)$ and $(\mathbb{Q}, \mathbb{L}_2)$.

Finally, since $\mathbb{L}_2$ is the finest $\mathscr{L}_0$-ring convergence on $\mathbb{Q}$ such that $\mathbb{L}_2|_{\mathbb{Z}} = \mathbb{L}_1$, it suffices to prove that $(\hat{\mathbb{Z}}, \mathbb{L}_3)$ is the categorical $\mathscr{L}_0$-ring completion of $(\mathbb{Q}, \mathbb{L}_2)$. Let $(K, \mathbb{M})$ be a complete $\mathscr{L}_0$-ring, and let $f\colon \mathbb{Q} \longrightarrow K$ be a continuous ring homomorphism. Then K contains a field isomorphic to $\hat{\mathbb{Z}}$. We identify $(\hat{\mathbb{Z}}, \mathbb{L}_3)$ with its image under f. Since $\mathbb{L}_2 \subseteq \mathbb{M}|_{\mathbb{Q}}$, it is clear that each sequence of the form (2) converges in $(K, \mathbb{M})$. That is, $\mathbb{L}_3 \subseteq \mathbb{M}|_{\hat{\mathbb{Z}}}$. $\square$

3. The Urysohn modification of $\mathbb{L}_1$

In this section, we shall see that the $\mathscr{L}_0$-ring convergence $\mathbb{L}_1$ does not satisfy the Urysohn axiom (U); hence we shall study its Urysohn modification. First, we enunciate two easy properties of $(\mathbb{Z}, \mathbb{L}_1)$.

Lemma 14. *The $\mathscr{L}_0$-ring $(\mathbb{Z}, \mathbb{L}_1)$ possesses the following two properties:*

- *Let $s, t \in \mathrm{MON}$ such that $\mathbb{N}$ is the disjoint union of $\{s(n)\colon n \in \mathbb{N}\}$ and $\{t(n)\colon n \in \mathbb{N}\}$ and let $S \in \mathbb{Z}^{\mathbb{N}}$. If both $S \circ s$, $S \circ t \in \mathbb{L}_1^{\leftarrow}(x)$, then $S \in \mathbb{L}_1^{\leftarrow}(x)$.*
- *If $(a_n)_{n\in\mathbb{N}} \in \mathbb{L}_1^{\leftarrow}(x)$ and the sequence $(b_n)_{n\in\mathbb{N}}$ is obtained from the sequence (a_n) by finite repetition of its elements, then $(b_n) \in \mathbb{L}_1^{\leftarrow}(x)$.*

If $(X, \mathbb{L})$ is an $\mathscr{L}_0^*$-space (it fulfils the Urysohn axiom), then it has the two properties from the above lemma. Nevertheless, $(\mathbb{Z}, \mathbb{L}_1)$ does not satisfy the Urysohn axiom. Consider for each $n \in \mathbb{N}$ the set

$$A_n = \{p \in \mathbb{P}\colon p < n \text{ and } p^r \neq n \text{ for each } r \in \mathbb{N}\}.$$

We define a sequence of integers $T = (\tau_n)_{n\in\mathbb{N}}$ as

$$\tau_n = \prod_{p \in A_n} p.$$

For instance, $\tau_9 = 2 \cdot 5 \cdot 7 = 70$. For each prime p, there are infinitely many $n \in \mathbb{N}$ such that $p \nmid \tau_n$. Therefore $(\tau_n) \notin \mathbb{L}_1^{\leftarrow}(0)$. Let us note that every subsequence of T has a subsequence which $\mathbb{L}_1$-converges to zero. Let $s \in \mathrm{MON}$ and consider $T \circ s$. If there exists a prime p such that $B_p = \{p^t\colon t \in \mathbb{N}\} \cap \{s(n)\colon n \in \mathbb{N}\}$ is an infinite set, we take $v \in \mathrm{MON}$ satisfying $B_p = \{s(v(n))\colon n \in \mathbb{N}\}$. Then it is clear that $T \circ s \circ v \in \mathbb{L}_1^{\leftarrow}(0)$. If the set B_p is finite for every prime p, then each prime p divides almost all terms of the sequence $T \circ s$ and $T \circ s \in \mathbb{L}_1^{\leftarrow}(0)$.

We denote by $\mathbb{L}_1^*$ the Urysohn modification of $\mathbb{L}_1$, which is an $\mathscr{L}_0^*$-ring convergence in $\mathbb{Z}$.

Lemma 15. *Let $(R, \mathbb{L})$ be a bounded $\mathscr{L}_0$-ring. Let $\mathbb{L}^*$ be the Urysohn modification of $\mathbb{L}$. Then $(R, \mathbb{L}^*)$ is a bounded $\mathscr{L}_0^*$-ring.*

Proof. Let $S \in R^{\mathbb{N}}$ and $T \in \mathbb{L}^{*\leftarrow}(0)$. For each $s \in \mathrm{MON}$, there exists $t \in \mathrm{MON}$ such that $T \circ s \circ t \in \mathbb{L}^{\leftarrow}(0)$. Then $(ST) \circ s \circ t = (S \circ s \circ t)(T \circ s \circ t) \in \mathbb{L}^{\leftarrow}(0)$; hence $ST \in \mathbb{L}^{*\leftarrow}(0)$. □

Therefore, $(\mathbb{Z}, \mathbb{L}_1^*)$ is bounded, and consequently, satisfies the condition $(\mathrm{C_r})$. Since $\mathbb{L}_1 \subset \mathbb{L}_1^*$, hence $(\mathbb{Z}, \mathbb{L}_1^*)$ is also sequentially precompact, and each nonzero ideal in $\mathbb{Z}$ is closure dense. Now, we present a more characteristic property of $\mathbb{L}_1^*$.

Lemma 16. *$\mathbb{L}_1^*$ is a coarse convergence in the class of $\mathscr{L}_0$-ring convergences on $\mathbb{Z}$ satisfying the condition $(\mathrm{C_r})$.*

Proof. We reason by way of contradiction. We assume that there exists an $\mathscr{L}_0$-ring convergence $\mathbb{M}$ on $\mathbb{Z}$ which satisfies the condition $(\mathrm{C_r})$ and $\mathbb{L}_1^* \subsetneq \mathbb{M}$. Then we have a sequence $\beta \in \mathbb{M}^{\leftarrow}(0) \setminus \mathbb{L}_1^{*\leftarrow}(0)$. There exists $s \in \mathrm{MON}$ such that no subsequence of $\beta \circ s$ belongs to $\mathbb{L}_1^{\leftarrow}(0)$. Applying Lemma 4, we get $t \in \mathrm{MON}$ such that the subsequence $\beta \circ s \circ t$ has the following property: each prime p which divides infintely many terms of $\beta \circ s \circ t$, divides almost all the terms. Let B be the set of those primes which divide infinitely many terms of $\beta \circ s \circ t$; clearly $B \notin \mathscr{U}$. Thus $E = \mathbb{P} \setminus B \in \mathscr{U}$. It is easy to define a sequence $\alpha \in \mathbb{Z}^{\mathbb{N}}$ which satisfies that $\alpha(n)$ and $\beta(s(t(n)))$ are coprime for all $n \in \mathbb{N}$, and E is the set of primes which divide almost all terms of α. Hence $\alpha \in \mathbb{L}_1^{\leftarrow}(0)$. There are two sequences of integers γ and δ such that

$$1 = \gamma(n)\alpha(n) + \delta(n)\beta(s(t(n))) \quad \text{for all } n \in \mathbb{N}.$$

Since $(\mathbb{Z}, \mathbb{L}_1)$ is sequentially precompact, there exists $v \in \mathrm{MON}$ such that $\delta \circ v$ is $\mathbb{L}_1$-Cauchy, and so $\mathbb{M}$-Cauchy. By hypothesis $\mathbb{M}$ satisfies the condition $(\mathrm{C_r})$, hence

$$(\delta \circ v)(\beta \circ s \circ t \circ v) \in \mathbb{M}^{\leftarrow}(0).$$

Further,

$$(\gamma \circ v)(\alpha \circ v) \in \mathbb{L}_1^{\leftarrow}(0) \subset \mathbb{M}^{\leftarrow}(0).$$

Finally, we get the contradiction

$$\langle 1 \rangle = (\gamma \circ v)(\alpha \circ v) + (\delta \circ v)(\beta \circ s \circ t \circ v) \in \mathbb{M}^{\leftarrow}(0).$$

□

Corollary 17. *The convergence $\mathbb{L}_1^*$ is coarse in the class of bounded $\mathscr{L}_0$-ring convergences on $\mathbb{Z}$.*

We denote by C_1 the ring of $\mathbb{L}_1$-Cauchy sequences and by C_1^* the ring of $\mathbb{L}_1^*$-Cauchy sequences. We are going to describe the relationship between C_1 and C_1^*, and between the corresponding completions $\hat{\mathbb{Z}}$ and $\hat{\mathbb{Z}}^*$. The following result is covered by Proposition 3 in [9].

Lemma 18.
(a) $\mathbb{L}_1^{\leftarrow}(0) = C_1 \cap \mathbb{L}_1^{*\leftarrow}(0)$.
(b) $C_1 + \mathbb{L}_1^{*\leftarrow}(0) = C_1^*$.
(c) *The respective completion rings $\hat{\mathbb{Z}}$ and $\hat{\mathbb{Z}}^*$ are (algebraically) isomorphic.*

Proof. (a) The inclusion $\mathbb{L}_1^{\leftarrow}(0) \subseteq C_1 \cap \mathbb{L}_1^{*\leftarrow}(0)$ is obvious. We prove the other inclusion: let $\alpha \in \mathbb{L}_1^{*\leftarrow}(0) \cap C_1$, then there exists $s \in \mathrm{MON}$ such that $\alpha \circ s \in \mathbb{L}_1^{\leftarrow}(0)$. Besides, $\alpha - \alpha \circ s \in \mathbb{L}_1^{\leftarrow}(0)$. Therefore $\alpha = (\alpha - \alpha \circ s) + \alpha \circ s \in \mathbb{L}_1^{\leftarrow}(0)$.

(b) The inclusion $C_1 + \mathbb{L}_1^{*\leftarrow}(0) \subseteq C_1^*$ is obvious. We check the other inclusion. Let $\beta \in C_1^*$. There exists $s \in \mathrm{MON}$ such that $\beta \circ s \in C_1$ (by virtue of Lemma 5). Since $\beta - \beta \circ s \in \mathbb{L}_1^{*\leftarrow}(0)$, we conclude that $\beta = \beta \circ s + (\beta - \beta \circ s) \in C_1 + \mathbb{L}_1^{*\leftarrow}(0)$.

(c) $(\mathbb{Z}, \mathbb{L}_1^*)$ satisfies the condition $(\mathrm{C_r})$. We have the ring homomorphism

$$
\begin{aligned}
f\colon C_1 &\longrightarrow \frac{C_1^*}{\mathbb{L}_1^{*\leftarrow}(0)},\\
S &\longrightarrow S + \mathbb{L}_1^{*\leftarrow}(0).
\end{aligned}
$$

Using parts (a) and (b), we conclude that f is onto and $\ker(f) = \mathbb{L}_1^{\leftarrow}(0)$. Therefore

$$
\hat{\mathbb{Z}} = \frac{C_1}{\mathbb{L}_1^{\leftarrow}(0)} \cong \frac{C_1^*}{\mathbb{L}_1^{*\leftarrow}(0)} = \hat{\mathbb{Z}}^*.
$$

□

In what follows we identify $\hat{\mathbb{Z}} = \hat{\mathbb{Z}}^* = \mathbb{F}$. The $\mathscr{L}_0^*$-convergence $\mathbb{L}_1^*$ is coarse in the class of $\mathscr{L}_0$-ring convergences in $\mathbb{Z}$ satisfying the condition $(\mathrm{C_r})$. In addition, every nonzero ideal is dense in $(\mathbb{Z}, \mathbb{L}_1^*)$. There is an analogous situation [1, Theorem 3.5.3] in the theory of topological rings:

"Let (R,τ) be a Hausdorff topological commutative domain, and let the topology τ be minimal in the class of all Hausdorff ring topologies. For the completion $(\hat{R},\hat{\tau})$ of the topological ring (R,τ) to be a topological field it is necessary and sufficient that all the non-zero ideals of R be totally dense."

Let $\mathbb{L}_2^*$ be the Urysohn modification of the convergence $\mathbb{L}_2$ defined on $\mathbb{Q}$. It is clear that a sequence α belongs to $\mathbb{L}_2^{*\leftarrow}(0)$ if and only if there exist a sequence

$(a_n)_{n\in\mathbb{N}} \in \mathbb{L}_1^{*\leftarrow}(0)$ and $c \in \mathbb{N}$ such that $\alpha(n) = a_n/c$ for all $n \in \mathbb{N}$. Consequently, $\mathbb{L}_2^*|_{\mathbb{Z}} = \mathbb{L}_1^*$, and $\mathbb{L}_2^*$ is the finest $\mathscr{L}_0^*$-ring convergence in $\mathbb{Q}$ which satisfies $\mathbb{L}_2^*|_{\mathbb{Z}} = \mathbb{L}_1^*$.

Let us quote a result from [9] by Koutník and Novák:

"Let K be a field. Let $(K, \mathbb{L})$ be an $\mathscr{L}_0^*$-ring which satisfies the condition $(\mathrm{C_r})$ and also the following condition:

(C_q) Let α be a sequence no subsequence of which is $\mathbb{L}$-Cauchy; then there exist $s, t \in \mathrm{MON}$ such that no subsequence of the sequence $\alpha \circ s - \alpha \circ t$ is $\mathbb{L}$-Cauchy.

Then $(K, \mathbb{L})$ has the categorical completion in the category of $\mathscr{L}_0^*$-rings."

Lemma 19. *The $\mathscr{L}_0^*$-ring $(\mathbb{Q}, \mathbb{L}_2^*)$ satisfies the condition* $(\mathrm{C_q})$.

Proof. Let $\alpha = (a_n/b_n)$ be a sequence no subsequence of which is $\mathbb{L}_2^*$-Cauchy (neither it is $\mathbb{L}_2$-Cauchy). We assume that a_n/b_n is a reduced fraction and $b_n \in \mathbb{N}$ for all n. Since $(\mathbb{Z}, \mathbb{L}_1)$ is sequentially precompact, the subset $\{a/c\colon a \in \mathbb{Z}\} \subset \mathbb{Q}$ is also sequentially precompact for each $c \in \mathbb{N}$. Thus, the set $\{n \in \mathbb{N}\colon b_n = c\}$ is finite for all $c \in \mathbb{N}$. Hence there exist $s, t \in \mathrm{MON}$ such that $nb_{s(n)} < b_{t(n)}$ for all n. It is clear that no subsequence of $(\alpha \circ s - \alpha \circ t)$ is $\mathbb{L}_2^*$-Cauchy. □

Now, we proceed in a way similar to [9], [10], [12]. The categorical $\mathscr{L}_0^*$-ring completion of $(\mathbb{Q}, \mathbb{L}_2^*)$ is the following: Let C_2^* be the ring of $\mathbb{L}_2^*$-Cauchy sequences. Then $\mathbb{F} \cong C_2^*/\mathbb{L}_2^{*\leftarrow}(0)$. Let $\{1\} \cup B$ be a basis of $\mathbb{F}$ as a linear space over $\mathbb{Q}$. Let $\widetilde{\mathbb{L}_2^*}$ be the set of all pairs $(S, x) \in \mathbb{F}^{\mathbb{N}} \times \mathbb{F}$ such that S is of the form

$$S(n) = S_0(n) + S_1(n)b_1 + \ldots + S_k(n)b_k \quad \text{for all } n \in \mathbb{N},$$

where $k \in \mathbb{N}$, $b_1, \ldots, b_k \in B$, S_i is a Cauchy sequence in $(\mathbb{Q}, \mathbb{L}_2^*)$ for $i = 0, 1, \ldots, k$, and $x = x_0 + x_1b_1 + \ldots + x_kb_k$, where x_i is the $\mathbb{L}_2^*$-limit of the Cauchy sequence S_i, $i = 0, 1, \ldots, k$. Now we take the Urysohn modification of this convergence, i.e., $\bigl(\widetilde{\mathbb{L}_2^*}\bigr)^*$. The completion is $\bigl(\mathbb{F}, \bigl(\widetilde{\mathbb{L}_2^*}\bigr)^*\bigr)$.

We quote Proposition 3 in [9] which clarifies the above:

"Let K be a field. Let $(K, \mathbb{L})$ be an $\mathscr{L}_0$-ring satisfying the condition $(\mathrm{C_r})$, such that the $\mathscr{L}_0^*$-ring $(K, \mathbb{L}^*)$ satisfies the condition $(\mathrm{C_q})$ and every $\mathbb{L}^*$-Cauchy sequence has a subsequence which is $\mathbb{L}$-Cauchy. Let $\bigl(\widetilde{K}, \widetilde{\mathbb{L}}\bigr)$ be the categorical $\mathscr{L}_0$-ring completion of $(K, \mathbb{L})$, and let $\bigl(\widetilde{K}, \bigl(\widetilde{\mathbb{L}^*}\bigr)^*\bigr)$ be the categorical $\mathscr{L}_0^*$-ring completion of $(K, \mathbb{L}^*)$. Then $\bigl(\widetilde{K}, \bigl(\widetilde{\mathbb{L}^*}\bigr)^*\bigr)$ is the Urysohn modification of $\bigl(\widetilde{K}, \widetilde{\mathbb{L}}\bigr)$."

We apply this result to the convergences described above.

Corollary 20. *The convergence $\bigl(\widetilde{\mathbb{L}_2^*}\bigr)^*$ on $\mathbb{F}$ is the Urysohn modification of $\mathbb{L}_3$. (Hence we denote it by $\mathbb{L}_3^*$).*

Lemma 21. *$(\mathbb{F}, \mathbb{L}_3^*)$ is the categorical $\mathscr{L}_0^*$-ring completion of $(\mathbb{Z}, \mathbb{L}_1^*)$.*

P r o o f. It suffices to notice that, if a complete $\mathscr{L}_0^*$-ring is an extension of $(\mathbb{Z}, \mathbb{L}_1^*)$, then it is also an extension of $(\mathbb{Q}, \mathbb{L}_2^*)$. $\square$

Lemma 22. *The $\mathscr{L}_0^*$-ring $(\mathbb{Z}, \mathbb{L}_1^*)$ is not strongly sequentially precompact.*

P r o o f. It is enough to show that its categorical $\mathscr{L}_0^*$-ring completion $\big(\hat{\mathbb{Z}}, \mathbb{L}_3^*\big)$ is not sequentially compact. $\hat{\mathbb{Z}}$ is a field. Let (a_n) be a sequence of nonzero elements which converges to zero in $\big(\mathbb{Z}, \mathbb{L}_1^*\big)$, then $(1/a_n)$ has no convergent subsequence in $\big(\hat{\mathbb{Z}}, \mathbb{L}_3^*\big)$. $\square$

Since $(\mathbb{Z}, \mathbb{L}_1^*)$ is sequentially precompact, we have a negative answer in the category of $\mathscr{L}_0^*$-rings to the question 4.4 posed by Dikranjan [4] for $\mathscr{L}_0^*$-groups:
"Is every sequentially precompact group strongly sequentially precompact?"

4. $\mathscr{L}_0$-field convergences on $\mathbb{Q}$

In this section we present two $\mathscr{L}_0$-field convergences on $\mathbb{Q}$ obtained from the ring convergences in the previous sections.

Let R be an integral domain, and let K be its field of fractions. Let $\mathbb{L}$ be an $\mathscr{L}_0$-ring convergence on R. Let C be the set of $\mathbb{L}$-Cauchy sequences. We assume that if two sequences satisfy $\alpha, \beta \in C$ and $\alpha, \beta \notin \mathbb{L}^{\leftarrow}(0)$, then their product $\alpha\beta \notin \mathbb{L}^{\leftarrow}(0)$. Let $\mathscr{A}$ be the set of sequences $(a_n/b_n) \in K^{\mathbb{N}}$ such that $a_n, b_n \in R$ for all n, $(a_n) \in \mathbb{L}^{\leftarrow}(0)$, $(b_n) \in C$, $(b_n) \notin \mathbb{L}^{\leftarrow}(0)$.

Lemma 23. *Under the above assumptions, if $(R, \mathbb{L})$ fulfils the condition* $(\mathrm{C_r})$, *then there exists an $\mathscr{L}_0$-field convergence $\mathbb{M}$ on K such that $\mathbb{M}^{\leftarrow}(0) = \mathscr{A}$. In addition, $\mathbb{L} \subseteq \mathbb{M}|_R$.*

P r o o f. It is easy to check that $\mathscr{A}$ possesses properties (i)–(iv) and (vi) from Lemmas 2 and 3. The inclusion is obvious. $\square$

Note that, if $(R, \mathbb{L})$ is complete, then $\mathbb{M}^{\leftarrow}(0)$ consists of the sequences $(a_n/b_n)_{n\in\mathbb{N}}$ such that $a_n, b_n \in R$, $(a_n) \in \mathbb{L}^{\leftarrow}(0)$, $(b_n) \in \mathbb{L}^{\leftarrow}(x)$ with $x \neq 0$. If, in addition, R is a field, then $\mathbb{M}$ is the field convergence constructed in [6, Lemma 1.2.5], [10, Theorem 5.6].

We apply the above construction to $(\mathbb{Z}, \mathbb{L}_1)$ and we get the $\mathscr{L}_0$-field $(\mathbb{Q}, \mathbb{M}_1)$.

Lemma 24. *Under the above assumptions we have $\mathbb{M}_1|_{\mathbb{Z}} = \mathbb{L}_1$.*

P r o o f. The inclusion $\mathbb{L}_1 \subseteq \mathbb{M}_1|_{\mathbb{Z}}$ is obvious. We show the other one. We take a sequence $(a_n/b_n) \in \mathbb{M}_1^{\leftarrow}(0)$ such that $a_n/b_n \in \mathbb{Z}$ for all n. We reason by way of contradiction. We assume that $(a_n/b_n) \notin \mathbb{L}_1^{\leftarrow}(0)$. There exists $D \in \mathscr{U}$ such that, for each prime $p \in D$, there exists an infinite subset $S_p \subseteq \mathbb{N}$ such that p does not divide a_n/b_n for all $n \in S_p$. Since $(a_n) \in \mathbb{L}_1^{\leftarrow}(0)$ and $(b_n) \in C_1$, we have $A = \{p \in \mathbb{P}\colon\ p \mid a_n$ for almost all $n\} \in \mathscr{U}$ and $B = \{p \in \mathbb{P}\colon\ p \mid b_n - b_m$ for all $n, m \geqslant k_p\} \in \mathscr{U}$. Note that $D \cap A \cap B \in \mathscr{U}$. For each $p \in D \cap A \cap B$ and for each $n \in S_p$ we have $p \mid a_n$ and $p \nmid (a_n/b_n)$, hence $p \mid b_n$ for all $n \in S_p$. If $p \in B$ divides infinitely many b_n, then it divides almost all b_n. Therefore, each $p \in D \cap A \cap B$ divides almost all b_n, and so $(b_n) \in \mathbb{L}_1^{\leftarrow}(0)$. We have arrived at a contradiction. □

It is obvious that $\mathbb{L}_2 \subsetneq \mathbb{M}_1$. We shall consider another $\mathscr{L}_0^*$-field convergence $\mathbb{M}$ on $\mathbb{Q}$.

Definition 25. Let $(R, \mathbb{L})$ be an $\mathscr{L}_0$-ring. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ is bounded away from zero if $a_n \neq 0$ for all n, and no subsequence of (a_n) converges to zero.

There is an analogous concept in the theory of topological rings; see [16, p. 72] and [19, p. 125].

Lemma 26. *If two sequences $\beta = (b_n)$ and $\delta = (d_n)$ in $(\mathbb{Z}, \mathbb{L}_1^*)$ are bounded away from zero, then their product $\beta\delta = (b_n d_n)$ is also bounded away from zero.*

P r o o f. We reason by way of contradiction. We suppose that there exists $s \in \mathrm{MON}$ such that $(\beta\delta) \circ s \in \mathbb{L}_1^{*\leftarrow}(0)$. We apply Lemma 6 and get $v \in \mathrm{MON}$ such that $\beta \circ s \circ v, \delta \circ s \circ v \in C_1 \subset C_1^*$. We have $\beta \circ s \circ v, \delta \circ s \circ v \notin \mathbb{L}_1^{*\leftarrow}(0)$, and their product satisfies $(\beta \circ s \circ v)(\delta \circ s \circ v) = (\beta\delta) \circ s \circ v \in \mathbb{L}_1^{*\leftarrow}(0)$, which is a contradiction. □

Let $\mathscr{B}$ be the set of rational sequences $(a_n/b_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ such that $a_n, b_n \in \mathbb{Z}$ for all n, $(a_n) \in \mathbb{L}_1^{*\leftarrow}(0)$, and (b_n) is bounded away from zero. The following result is easy to prove:

Lemma 27. *Let $a_n, b_n \in \mathbb{Z}$, and let a_n/b_n be a reduced fraction for all n. If $(a_n)_{n\in\mathbb{N}} \in \mathbb{L}_1^{*\leftarrow}(0)$, then $(a_n/b_n)_{n\in\mathbb{N}} \in \mathscr{B}$.*

Lemma 28. *Let $a_n, b_n \in \mathbb{Z}$ such that $(a_n/b_n)_{n\in\mathbb{N}} \in \mathscr{B}$. If for each n we have $d_n \in \mathbb{Z}$ that divides a_n and b_n, then*

$$\Big(\frac{a_n/d_n}{b_n/d_n}\Big)_{n\in\mathbb{N}} \in \mathscr{B}.$$

P r o o f. Both sequences (b_n/d_n) and (d_n) are bounded away from zero. We check that $(a_n/d_n) \in L_1^{*\leftarrow}(0)$. Let $(a_n/d_n) \circ u$ be a subsequence of (a_n/d_n). There exists $s \in \mathrm{MON}$ such that $(a_n) \circ u \circ s \in L_1^{\leftarrow}(0)$. We apply Lemma 4 to the sequence $(d_n) \circ u \circ s$ and get a subsequence $(d_n) \circ u \circ s \circ t$ such that each prime which divides infinitely many terms of $(d_n) \circ u \circ s \circ t$ divides almost all of them. Therefore the subsequence $(a_n/d_n) \circ u \circ s \circ t \in L_1^{\leftarrow}(0)$. □

Theorem 29. *There exists an $\mathscr{L}_0^*$-field convergence $\mathbb{M}$ on $\mathbb{Q}$ such that $\mathbb{M}^{\leftarrow}(0) = \mathscr{B}$. Besides, $\mathbb{M}|_{\mathbb{Z}} = \mathbb{L}_1^*$. Further $\mathbb{M}$ is the Urysohn modification of $\mathbb{M}_1$ (that is, $\mathbb{M} = \mathbb{M}_1^*$).*

Proof. Using Lemma 26 and the fact that $(\mathbb{Z}, \mathbb{L}_1^*)$ is bounded, we easily check that $\mathscr{B}$ satisfies conditions (i)–(iv) and (vi) in Lemmas 2 and 3. Hence $\mathbb{M}$ is an $\mathscr{L}_0$-field convergence on $\mathbb{Q}$.

It is obvious that $\mathbb{L}_1^* \subseteq \mathbb{M}|_{\mathbb{Z}}$. We show the other inclusion: We reason by way of contradiction. Let $\gamma = (a_n/b_n)_{n\in\mathbb{N}} \in \mathbb{M}^{\leftarrow}(0)$ with $a_n/b_n \in \mathbb{Z}$ for all n. We denote $\alpha = (a_n)_{n\in\mathbb{N}}$, $\beta = (b_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$. Then $\alpha \in \mathbb{L}_1^{*\leftarrow}(0)$ and β is bounded away from zero. We suppose that $\gamma \notin \mathbb{L}_1^{*\leftarrow}(0)$. Hence there exists a sequence $s \in \mathrm{MON}$ such that $\gamma \circ s \notin \mathbb{L}_1^{\leftarrow}(0)$, neither do its subsequences, but $\alpha \circ s \in \mathbb{L}_1^{\leftarrow}(0)$. We recall Lemma 4; there exists $v \in \mathrm{MON}$ such that, if a prime p divides infinitely many terms of $\gamma \circ s \circ v$, then it divides almost all of them. Since $\gamma \circ s \circ v \notin \mathbb{L}_1^{\leftarrow}(0)$, we have

$$D = \{p \in \mathbb{P}\colon\ p \mid \gamma(s(v(n))) \text{ for almost all } n\} \notin \mathscr{U}.$$

As $\alpha \circ s \circ v \in \mathbb{L}_1^{\leftarrow}(0)$, we have

$$A = \{p \in \mathbb{P}\colon\ p \mid \alpha(s(v(n))) \text{ for almost all } n\} \in \mathscr{U}.$$

Taking into account that $\beta \circ s \circ v \notin \mathbb{L}_1^{\leftarrow}(0)$, we have

$$B = \{p \in \mathbb{P}\colon\ p \mid \beta(s(v(n))) \text{ for almost all } n\} \notin \mathscr{U}.$$

But $B \supseteq A \cap (\mathbb{P} \setminus D) \in \mathscr{U}$; we have a contradiction.

Let us show that $\mathbb{M}$ is the Urysohn modification of $\mathbb{M}_1$. Obviously $\mathbb{M}_1 \subseteq \mathbb{M}$. Let $\gamma = (a_n/b_n) \in \mathbb{M}^{\leftarrow}(0)$, where $a_n, b_n \in \mathbb{Z}$ for all n. Let $\alpha = (a_n)$ and $\beta = (b_n)$. For each $s \in \mathrm{MON}$ there exists $t \in \mathrm{MON}$ such that $\beta \circ s \circ t \in C_1 \subset C_1^*$ (consider Lemma 5) and $\beta \circ s \circ t \notin \mathbb{L}_1^{\leftarrow}(0)$; therefore $\gamma \circ s \circ t \in \mathbb{M}_1^{\leftarrow}(0)$. We have proved that $\mathbb{M} \subseteq \mathbb{M}_1^*$. Let us show the other inclusion: let $\gamma = (a_n/b_n) \in \mathbb{M}_1^{*\leftarrow}(0)$, where a_n/b_n is a reduced fraction for all n. It is easy to check that $(a_n) \in \mathbb{L}_1^{*\leftarrow}(0)$ and that (b_n) is bounded away from zero. Consequently $\gamma \in \mathbb{M}^{\leftarrow}(0)$, hence $\mathbb{M}_1^* \subseteq \mathbb{M}$. $\square$

Since the condition $(\mathrm{C_r})$ is necessary for an $\mathscr{L}_0$-ring to have a completion, the following result is significant.

Lemma 30. *The $\mathscr{L}_0^*$-ring $(\mathbb{Q}, \mathbb{M})$ does not satisfy the condition $(\mathrm{C_r})$.*

Proof. We are going to construct a sequence $\gamma = (d_n b_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ such that $\gamma \in \mathbb{L}_1^{\leftarrow}(0) \subset \mathbb{M}^{\leftarrow}(0)$ and $1/\gamma = (1/(d_n b_n))_{n\in\mathbb{N}}$ is $\mathbb{M}$-Cauchy. As their product $\langle 1\rangle \notin \mathbb{M}^{\leftarrow}(0)$, the lemma is proved.

We consider a set of primes $D = \{q_1, \dots, q_n, \dots\} \in \mathscr{U}$, and let $b_n = q_1 \dots q_n$ for all n. Then $(b_n)_{n\in\mathbb{N}} \in \mathbb{L}_1^{\leftarrow}(0) \subset \mathbb{M}^{\leftarrow}(0)$. We will define inductively another sequence $(d_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ such that $(1/(d_n b_n))$ is $\mathbb{M}$-Cauchy. Since $(\mathbb{Z}, \mathbb{L}_1)$ is bounded, we have $(d_n b_n) \in \mathbb{L}_1^{\leftarrow}(0) \subset \mathbb{M}^{\leftarrow}(0)$.

We set $d_1 = 1$. We assume, by induction hypothesis, that we have defined $d_1, \dots, d_n$ satisfying

$$\begin{aligned} &q_i \nmid d_n && \text{for } i = 1, \dots, n, \\ &q_n \dots q_{i+1} d_n \equiv d_i \mod(q_1^2 \dots q_i^2) && \text{for } i = 1, \dots, n-1. \end{aligned}$$

At the next step we define d_{n+1} such that

$$\begin{aligned} &q_{n+1} \nmid d_{n+1}, \\ &q_{n+1} d_{n+1} \equiv d_n \mod(q_1^2 \dots q_n^2). \end{aligned}$$

As a consequence, d_{n+1} also satisfies

$$\begin{aligned} &q_i \nmid d_{n+1} && \text{for } i = 1, \dots, n+1, \\ &q_{n+1} \dots q_{i+1} d_{n+1} \equiv d_i \mod(q_1^2 \dots q_i^2) && \text{for } i = 1, \dots, n. \end{aligned}$$

Now, for $i < n$, we compute

$$\text{(3)} \qquad \frac{1}{d_i b_i} - \frac{1}{d_n b_n} = \frac{q_n \dots q_{i+1} d_n - d_i}{b_n d_i d_n} = \frac{(q_n \dots q_{i+1} d_n - d_i)/(q_1 \dots q_i)}{(q_{i+1} \dots q_n) d_i d_n}.$$

In the last fraction the numerator and denominator are integers. We note that q_j does not divide the denominator $(q_{i+1} \dots q_n) d_i d_n$ for $j = 1, \dots, i$. On the other hand, $q_1 \dots q_i$ divides the numerator of (3). Consequently, as i and n increase, the numerator of (3) converges to zero in $(\mathbb{Z}, \mathbb{L}_1)$, and the denominator becames bounded away from zero. Hence $\gamma - \gamma \circ s \in \mathbb{M}^{\leftarrow}(0)$ for each $s \in \mathrm{MON}$. Therefore γ is an $\mathbb{M}$-Cauchy sequence. $\square$

Corollary 31. *The $\mathscr{L}_0^*$-ring $(\mathbb{Q}, \mathbb{M})$ has no completion.*

In [5, § 2] it is proved that each coarser $\mathscr{L}_0$-ring convergence in $\mathbb{Q}$ coarse than the usual metric convergence does not satisfy the condition $(\mathrm{C_r})$.

5. The convergence $\mathbb{L}_1^*$ is not topological

Consider the natural functor from the category of topological spaces into the category of sequential convergence spaces. Let $(X,\mathscr{T})$ be a topological space. Define $(S,x) \in \mathscr{L}(\mathscr{T}) \subseteq X^{\mathbb{N}} \times X$ whenever S converges in $(X,\mathscr{T})$ to x. Then $\mathscr{L}(\mathscr{T})$ is a sequential convergence satisfying the Urysohn axiom. Therefore there is no ring topology $\mathscr{T}$ on $\mathbb{Z}$ such that $\mathbb{L}_1 = \mathscr{L}(\mathscr{T})$. In addition, we shall prove that there is no ring topology $\mathscr{T}$ on $\mathbb{Z}$ such that $\mathbb{L}_1^* = \mathscr{L}(\mathscr{T})$.

Let $(X,\mathscr{T}_1)$ be a first countable topological space. A set $B \subseteq X$ is closed if and only if every sequence $(b_n)_{n\in\mathbb{N}} \subseteq B$ which converges to $b \in X$ has its limit $b \in B$. Consequently, if $\mathscr{T}_2$ is another topology on X such that $\mathscr{L}(\mathscr{T}_1) \subseteq \mathscr{L}(\mathscr{T}_2)$, then $\mathscr{T}_1 \supseteq \mathscr{T}_2$.

Lemma 32. *There is no ring topology $\mathscr{T}$ on $\mathbb{Z}$ such that $\mathscr{L}(\mathscr{T}) = \mathbb{L}_1^*$.*

Proof. We reason by way of contradiction. We suppose that there exists a ring topology $\mathscr{T}$ on $\mathbb{Z}$ such that $\mathscr{L}(\mathscr{T}) = \mathbb{L}_1^*$. We take a set D in the ultrafilter $\mathscr{U}$,

$$D = \{q_1, q_2, \dots, q_n, \dots\} \in \mathscr{U}.$$

We consider the ring topology $\mathscr{T}_D$ on $\mathbb{Z}$ given by the filter of ideals

$$(q_1) \supset (q_1 q_2) \supset \dots \supset (q_1 q_2 \dots q_n) \supset \dots$$

as a fundamental system of zero neighborhoods. It is clear that the $\mathscr{L}_0$-ring convergence corresponding to $\mathscr{T}_D$ satisfies $\mathscr{L}(\mathscr{T}_D) \subsetneq \mathbb{L}_1^*$. Since $\mathscr{T}_D$ is first countable we have $\mathscr{T}_D \supsetneq \mathscr{T}$. It is easy to prove that the infimum of the topologies $\mathscr{T}_D$ in the lattice of ring topologies on $\mathbb{Z}$ is the trivial one. We get that $\{\emptyset, \mathbb{Z}\} \supseteq \mathscr{T}$, which is absurd. □

In a similar manner we can prove that the other $\mathscr{L}_0^*$-ring convergences in this article, $\mathbb{L}_2^*$, $\mathbb{M}$, are not given by any ring topology on $\mathbb{Q}$.

6. A completion of $\mathbb{Z}$ is an ultraproduct of p-adic rings

Finally, we mention briefly another $\mathscr{L}_0$-ring convergence on $\mathbb{Z}$ which we call $\mathbb{L}_a$. For each prime $p \in \mathbb{P}$, let $|\ |_p$ be the corresponding p-adic valuation on $\mathbb{Z}$. Let $\mathscr{U}$ be a nonprincipal ultrafilter in $\mathbb{P}$. A sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ belongs to $\mathbb{L}_a^{\leftarrow}(0)$ if and only if the set of primes $p \in \mathbb{P}$ such that

$$\lim_{n\to\infty} |a_n|_p = 0$$

belongs to $\mathscr{U}$. The completion of $(\mathbb{Z}, \mathbb{L}_a)$ is isomorphic to an ultraproduct of the rings of p-adic integers $\mathbb{Z}_p$. That is,

$$\hat{\mathbb{Z}} \cong \frac{\prod_{p\in\mathbb{P}} \mathbb{Z}_p}{\mathscr{U}}.$$

Similarly, there is a completion of $\mathbb{Q}$ that is an ultraproduct of the fields of p-adic numbers $\mathbb{Q}_p$.

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