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A NOTE ON ON-LINE RANKING NUMBER OF GRAPHS

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Abstract. A k-ranking of a graph $G=(V,E)$ is a mapping $\varphi\colon V\rightarrow\{1,2,\dots,k\}$ such that each path with endvertices of the same colour c contains an internal vertex with colour greater than c. The ranking number of a graph G is the smallest positive integer k admitting a k-ranking of G. In the on-line version of the problem, the vertices $v_1,v_2,\dots,v_n$ of G arrive one by one in an arbitrary order, and only the edges of the induced graph $G[\{v_1,v_2,\dots,v_i\}]$ are known when the colour for the vertex v_i has to be chosen. The on-line ranking number of a graph G is the smallest positive integer k such that there exists an algorithm that produces a k-ranking of G for an arbitrary input sequence of its vertices.

We show that there are graphs with arbitrarily large difference and arbitrarily large ratio between the ranking number and the on-line ranking number. We also determine the on-line ranking number of complete n-partite graphs. The question of additivity and heredity is discussed as well.

Keywords: on-line ranking number, complete n-partite graph, hereditary and additive properties of graphs

MSC 2000: 05C15, 05C85

1. Introduction and definitions

We consider only simple graphs, i.e. finite, undirected, without loops and multiple edges. We use standard graph terminology and notation. In particular, S_n denotes the star with center of degree n, P_n is the path of order n, K_n the complete graph of order n and $K_{m,n}$ the complete bipartite graph with one partition of order m and the other of order n (we recall that these partitions are uniquely determined up to the order of partition for $m=n$). The join of two graphs G and H, denoted by $G*H$, is the graph consisting of disjoint copies of G and H and all edges between $V(G)$ and $V(H)$.

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A *vertex k-ranking* (or briefly *k-ranking*) of G is a colouring $\varphi\colon V \to \{1,2,\ldots,k\}$ of the vertices of G such that each path with endvertices u, v of the same colour $\varphi(u) = \varphi(v) = c$ contains an internal vertex z with a colour $\varphi(z)$ greater than c. A graph is k-rankable if it admits at least one k-ranking. The *ranking number* $\chi_r(G)$ of G is the smallest positive integer k such that G is k-rankable.

We deal with the on-line version of this problem. In such a version vertices of a graph G are coming in an arbitrary order. The vertices of the graph are coloured one by one. At any moment only a local information concerning the edges between the vertices already present, say $v_1, v_2, \ldots, v_q$, is known. Moreover, we have no information on the relative position of the induced graph $G[\{v_1, v_2, \ldots, v_q\}]$ in the graph G. The assigned colours cannot be changed later on. The graph G is called *on-line k-rankable* if there is an algorithm that generates a k-ranking of G for every possible input sequence of the vertices of G. The *on-line ranking number* $\chi_r^*(G)$ is the smallest positive integer k such that G is on-line k-rankable. We remark that it can happen that a colour $c \leqslant k$ is not used by an algorithm for a specific input sequence (because of the strategy of the algorithm). But if the algorithm does not use the colour c for each input sequence of the vertices of the given graph then there exists a better algorithm that uses at least one colour less. The on-line ranking number was studied in [4], [5], [6] but the exact values and even estimations of its value are known for a few classes of graphs only.

In the second section we show that there are infinite classes of graphs with arbitrarily large difference and ratio between the ranking number and the on-line ranking number. The third section is devoted to complete n-partite graphs. We determine their ranking and on-line ranking numbers and provide a refinement of the asymptotic results obtained in section two. In the fourth section we treat the ranking number and the on-line ranking number with respect to heredity, induced-heredity and additivity.

2. The difference and ratio of ranking and on-line ranking number

It is already known that the difference between the chromatic number and the ranking number can be arbitrarily large. We prove an analogous result for the ranking number and the on-line ranking number.

Let us denote by S_n^*, $n \geqslant 3$, the graph obtained from S_n by inserting a vertex to every edge of S_n. Thus the resulting graph has one vertex of degree n (center of S_n^*), n vertices of degree 2 and n vertices of degree 1 (leaves of S_n^*). The ranking number and the on-line ranking number of $S_0^* = K_1$, $S_1^* = P_3$ and $S_2^* = P_5$ were already discussed in [4].

Theorem 2.1. *For every positive integer* n, $n > 2$ *the following holds:*

(i) $\chi_r(S_n^*) = 3$,

(ii) $\chi_r^*(S_n^*) = n + 1$.

P r o o f. (i) It is not so difficult to verify that $\chi_r(S_n^*) = 3$ for $n \geqslant 3$.

(ii) Let n be a given positive integer greater than 2. A path of length 1 that starts in a neighbour of the central vertex and ends in a leaf of S_n^* or vice versa will be called *a branch* of S_n^*.

Let the input sequence begin with $n + 1$ independent vertices. We cannot avoid the case that the central vertex receives the smallest colour, say k, among these $n + 1$ vertices. Then each branch has a vertex using a colour larger than k. For each branch, choose the larger colour from those used for its two vertices. Then all such colours are larger than k and must be distinct. This gives the lower bound $\chi_r^*(S_n^*) \geqslant n + 1$.

Consider now an algorithm based on the following rule: if the incoming vertex can be central then colour it with the smallest admissible colour distinct from 2, otherwise colour it with the first admissible colour. This algorithm is a slight modification of the First-Fit algorithm (sometimes also called Greedy colouring algorithm). One can easily see that after utilization of this algorithm, each branch of S_n^* contains a vertex coloured by colour 1. Furthermore, before recognizing the central vertex, the algorithm uses only colours 1, 2, 3 or 4. Moreover, the colour 4 is used only in the case when we have already coloured the central vertex but we have not been able to identify it (for example, if the input sequence starts with a path of length 3 or 4, then we know that one of its vertices is central but we do not know which one). Therefore the central vertex can obtain only colours 1, 3 or 4. If the central vertex is coloured with 3 or 4 then all vertices coming after the identification of the central vertex will be coloured by 1 or 2 and the algorithm uses 3 or 4 colours respectively ($4 \leqslant n+1$). If the central vertex is coloured with 1, then the algorithm needs exactly $n + 1$ colours. Hence in all cases the algorithm uses at most $n + 1$ colours and we have $\chi_r^*(S_n^*) \leqslant n + 1$. □

Corollary 2.2. *For an arbitrary positive integer* k *there exists a graph* G *such that* $\chi_r^*(G)/\chi_r(G) > k$.

Corollary 2.3. *For an arbitrary positive integer* k *there exists a graph* G *such that* $\chi_r^*(G) - \chi_r(G) \geqslant k$.

3. Complete n-partite graphs

In this section we establish the values of the ranking number and the on-line ranking number of the complete bipartite graphs $K_{m,n}$ and the complete s-partite graphs $K_{n_1,n_2,\ldots,n_s}$.

Theorem 3.1. $\chi_r(K_{m,n}) = \min\{m,n\} + 1$.

Proof. If G, H are two graphs then it is not so difficult to see that $\chi_r(G*H) = \min\{\chi_r(G) + |V(H)|, |V(G)| + \chi_r(H)\}$. Indeed, if we assign the same colour to at least two vertices from one of the original graphs, then all the vertices from the second graph must receive greater and pairwise different colours. Consequently, $\chi_r(K_{m,n}) = \min\{m,n\} + 1$. □

In order to determine the value of $\chi_r^*(K_{m,n})$ we introduce the following two algorithms.

ALGORITHM $\mathcal{A}$ (First Fit)

INPUT: m, n and a sequence $v_1, v_2, \ldots, v_{m+n}$ of vertices of $K_{m,n}$

1.) colour vertex v_1 with colour 1;
2.) FOR EACH $i \in \{2, 3, \ldots, m+n\}$ DO:
    IF v_i is adjacent to v_1 THEN colour v_i with the first unused colour from the set $\{1, 2, \ldots\}$
    ELSE colour v_i with colour 1.

We remark that, according to the definition of on-line ranking, with incoming vertices $v_1, v_2, \ldots, v_k$ we know also the structure of the subgraph of $K_{m,n}$ induced by $\{v_1, v_2, \ldots, v_k\}$.

ALGORITHM $\mathcal{B}$

INPUT: m, n and a sequence $v_1, v_2, \ldots, v_{m+n}$ of vertices of $K_{m,n}$

1.) LET $A = \{v_1\}$, $B = \emptyset$;
2.) colour vertex v_1 with colour 2;
3.) LET $q = \min\{m,n\}$;
    LET $i = 1$;
4.) WHILE ($|A| \leqslant q$ AND $|B| \leqslant q$) DO:
    LET $i = i + 1$;
    IF v_i is adjacent to a vertex from A THEN LET $B = B \cup \{v_i\}$
    ELSE LET $A = A \cup \{v_i\}$;
    IF ($|A| \leqslant q$ AND $|B| \leqslant q$) THEN colour v_i with colour $i + 1$
    ELSE colour vertex v_i with colour 1;

5.) LET $j = i$;
6.) WHILE $i < m + n$ DO:
LET $i = i + 1$;
IF v_i is adjacent to v_j THEN colour v_i with the first unused colour from $\{1, 2, \ldots\}$
ELSE colour v_i with colour 1.

One can easily see that both the algorithms produce a ranking of $K_{m,n}$.

Theorem 3.2. *Let m, n be positive integers, $m \leqslant n$. Then*

$$\chi_r^*(K_{m,n}) = \begin{cases} 2m + 1 & \text{if } n \geqslant 2m, \\ n + 1 & \text{if } n < 2m. \end{cases}$$

Proof. We prove the upper bound by analyzing ALGORITHM $\mathcal{A}$ and ALGORITHM $\mathcal{B}$.

ALGORITHM $\mathcal{A}$ assigns to the incoming vertex the first admissible colour. Therefore, if the first vertex is from the partition with cardinality m, then the algorithm uses colour 1 for this partition and the vertices from the second partition are coloured with colours $2, 3, \ldots, n + 1$. If the first vertex is from the partition with m vertices then the algorithm uses $m + 1$ colours. Hence in the worst case we use $n + 1$ colours.

ALGORITHM $\mathcal{B}$ does not use colour 1 until it is clear which partition is greater. Then it uses colour 1 for the greater partition and the first admissible colour for the incoming vertices from the second partition. Therefore in the worst case it uses $2m + 1$ colours.

Thus for $n \geqslant 2m$ we have the bound $\chi_r^*(K_{m,n}) \leqslant 2m + 1$ and for $n < 2m$ we have the bound $\chi_r^*(K_{m,n}) \leqslant n + 1$.

In order to prove lower bounds, let us distinguish two cases.

Case 1: $n \geqslant 2m$. If an algorithm accidentally assigns colour 1 to a vertex from the smaller partition of $K_{m,n}$, then it will use at least $n + 1 \geqslant 2m + 1$ colours. And this result is not better than the result of ALGORITHM $\mathcal{B}$. To obtain a better result we cannot assign colour 1 to a vertex from the smaller partition of $K_{m,n}$. Since we have to consider all possible input sequences of the vertices, in the worst case we must examine $2m + 1$ vertices in order to be able to decide whether a vertex belongs to the bigger partition or to the smaller one. Hence, if the sequence begins with m vertices from the first partition and m vertices from the second partition (they can be ordered arbitrarily), then we have to use $2m$ pairwise different colours to colour this part of the input sequence. Therefore for $2m \leqslant n$ there exists no algorithm that colours $K_{m,n}$ on-line and for a general input uses less than $2m + 1$ colours. It implies that $\chi_r^*(K_{m,n}) = 2m + 1$ for $2m \leqslant n$.

Case 2: $n < 2m$. Similarly as in the previous case, if we assign accidentally colour 1 to a vertex from the smaller partition, we need at least $n+1$ colours to colour $K_{m,n}$. Suppose now that there exists an algorithm that uses at most n colours. In the worst case, we can reliably decide whether a vertex belongs to the bigger partition after examination of $2m > n$ vertices. Before that we cannot use identical colours for different vertices because then (in the worst case) we need at least $2m > n$ colours. Thus for the general case we cannot obtain a better result than that provided by ALGORITHM $\mathcal{A}$. Therefore $\chi_r^*(K_{m,n}) = n+1$ for $n < 2m$. □

Corollary 3.3. *For arbitrary positive integers k, l there exists a graph G such that $\chi_r^*(G)/\chi_r(G) > k$ and $\chi_r(G)/\chi(G) > l$.*

P r o o f. Consider the graph $K_{m,n} \cup pK_1$ (the disjoint union of $K_{m,n}$ and p copies of K_1) where $m \leqslant n$ and p are positive integers satisfying $p + m \leqslant n$. It is easy to see that $\chi(K_{m,n} \cup pK_1) = 2$ and $\chi_r(K_{m,n} \cup pK_1) = m+1$. If an input sequence begins with $p + m \leqslant n + p$ isolated vertices, then we cannot decide whether such a vertex belongs to the smaller partition of $K_{m,n}$ or to a copy of K_1. If we use colour 1 for a "wrong" vertex from this part of the input sequence then we will use at least $n+1$ colours. Similarly, if we use at least twice colour k for "wrong" vertices from this part of the input sequence then we will use at least $n+k$ colours. Therefore, for the beginning of the input sequence an optimal algorithm uses $p+m$ different colours greater than 1. It implies that $\chi_r^*(K_{m,n} \cup pK_1) \geqslant p + m + 1$.

Let us put $p = m^2$ and $n = m^2 + m$. Then

$$\frac{\chi_r^*(K_{m,n} \cup pK_1)}{\chi_r(K_{m,n} \cup pK_1)} \geqslant m + (m+1)^{-1} \quad \text{and} \quad \frac{\chi_r(K_{m,n} \cup pK_1)}{\chi(K_{m,n} \cup pK_1)} \geqslant \frac{1}{2}(m+1).$$

□

Theorem 3.4. *Let $k \geqslant 2$ be an integer and let $n_1, n_2, \ldots, n_k$ be positive integers satisfying $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$. Let us put $r_1 = n_1$ and $r_i = n_i - n_{i-1}$ for $i = 2, 3, \ldots, k$. Then*

(i) $\chi_r(K_{n_1,n_2,\ldots,n_k}) = \sum\limits_{i=1}^{k} n_i - n_k + 1$,

(ii) $\chi_r^*(K_{n_1,n_2,\ldots,n_k}) = \sum\limits_{i=1}^{k} n_i - \max\limits_{i=1,\ldots,k} r_i + 1$.

P r o o f. (i) The proof of the first assertion proceeds in an analogous way as the proof of Theorem 3.1.

(ii) Let us consider the following algorithm:

ALGORITHM $\mathcal{C}$

INPUT: an integer $k \geqslant 2$, integers $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$,

a sequence $v_1, v_2, \ldots, v_{n_1+n_2+\ldots+n_k}$ of vertices of $K_{n_1,n_2,\ldots,n_k}$

1.) LET $n_0 = 0$;

LET $r_i = n_i - n_{i-1}$ for $i = 1, 2, \ldots, k$;

LET $R = \max\limits_{i=1,\ldots,k} r_i$;

LET $t = \min\{i : r_i = R, i = 1, 2, \ldots, k\}$;

LET $q = n_{t-1}$;

2.) LET $A_i = \emptyset$ for $i = 1, 2, \ldots, k$;

LET $l := 0$;

3.) WHILE $(\max\limits_{i=1,\ldots,k} |A_i| \leqslant q)$ DO:

LET $l := l + 1$;

LET $s = \min\{i : A_i = \emptyset$ or v_l is a neighbour of no vertex belonging to $A_i\}$;

LET $A_s := A_s \cup \{v_l\}$;

IF $(\max\limits_{i=1,\ldots,k} |A_i| \leqslant q)$ THEN colour v_l with colour $l + 1$

ELSE colour v_l with colour 1;

4.) LET $j = l$;

5.) WHILE $\left(l < \sum\limits_{i=1}^{k} n_i\right)$ DO:

$l := l + 1$;

IF v_l is adjacent to v_j THEN colour v_l with the first unused colour

ELSE colour v_l with colour 1.

ALGORITHM $\mathcal{C}$ uses colour 1 only if it recognizes a partition with at least n_t vertices (t is the index satisfying $t = \min\{i : r_i = R, i = 1, 2, \ldots, k\}$). Before that it uses mutually different colours. Step 5.) guarantees that all other incoming vertices from the partition with colour 1 are coloured with colour 1 as well and the vertices in the other partitions receive pairwise different colours.

It means that colour 1 is used at least $n_t - n_{t-1} = r_t$ times and the other vertices are coloured with colours $2, 3, \ldots, \sum\limits_{i=1}^{k} n_k - r_t + 1$. Hence, in the worst case (if the number $q = n_{t-1}$ is exceeded by a vertex from a partition of cardinality n_t) the algorithm uses exactly $\sum\limits_{i=1}^{k} n_k - r_t + 1$ colours.

Suppose that an input sequence begins with exactly n_1 vertices from each partition. If we use colour 1 for some of these vertices, or we use some colour twice, then in the worst case we need at least $\sum\limits_{i=1}^{k} n_i - n_1 + 1 = \sum\limits_{i=1}^{k} n_i - r_1 + 1$ colours. In the opposite case, consider that in the next part of the input sequence we have exactly $n_2 - n_1 = r_2$ new vertices from each partition of cardinality greater than n_1. Again, if we use colour 1 for some of these vertices or we use some colour twice, then we need at least $\sum_{i=1}^{k} n_i - (n_2 - n_1) + 1 = \sum_{i=1}^{k} n_i - r_2 + 1$ colours. By a similar argument we obtain that in the worst case we need at least $\sum_{i=1}^{k} n_i - \max_{i=1,\ldots,k} r_i + 1$ colours to produce a ranking of $K_{n_1,n_2,\ldots,n_k}$ on-line. □

4. Heredity and additivity

Ranking and on-line ranking provide a generalization of standard colouring of vertices of graphs. Various types of generalized (sometimes also called improper) colourings of graphs can be expressed in terms of the language of hereditary properties of graphs.

A *property of graphs* is a non-empty, isomorphism closed subclass of the class of all graphs. A property is *hereditary* (*induced-hereditary*) if it is closed under taking subgraphs (induced subgraphs). A property is called *additive* if it is closed under taking disjoint unions. For more details and some applications we refer the reader to [1], [2], [3], [7].

The next result shows that ranking and on-line ranking are of slightly different character.

Theorem 4.1.

(i) *The property* $\mathcal{R}_k = \{G\colon \chi_r(G) \leqslant k\}$ *is hereditary, induced-hereditary and additive.*

(ii) *The property* $\mathcal{R}_k^* = \{G\colon \chi_r^*(G) \leqslant k\}$ *is induced-hereditary but it is neither additive nor hereditary.*

Proof. (i) Let G be a graph with property $\mathcal{R}_k$ and H a subgraph of G. It is not difficult to see that any path in H is a path in G as well. Therefore any k-ranking of G is a k-ranking of H as well. It implies that $\chi_r(H) \leqslant \chi_r(G) \leqslant k$ and $\mathcal{R}_k$ is a hereditary property. Since every hereditary property is obviously also induced-hereditary, $\mathcal{R}_k$ is induced-hereditary too. The additivity of $\mathcal{R}_k$ follows from the fact that a disjoint union of two graphs cannot produce a new path in the resulted graph.

(ii) One can rather easily see that if $G \in \mathcal{R}_k^*$ and H is an induced subgraph of G then $\chi_r^*(H) \leqslant k$. Indeed, an algorithm that colours G on-line with at most k colours for any input sequence of vertices, colours H with at most k colours as well. The reason is that the vertices of H can form an initial part of the input sequence of the vertices of G (see also Corollary 2 in [4]).

Consider now the graph $K_{3,11}$ and its subgraph $K_{3,7} \cup 4K_1$. By Theorem 3.2 we know that $\chi_r^*(K_{3,11}) = 7$. By the proof of Corollary 3.3 we know that for the graph

$K_{3,7} \cup 4K_1$ the value of $\chi_r^*(K_{3,7} \cup 4K_1)$ is at least 8 (for $m = 3$, $n = 7$ and $p = 4$ we have $p + m = 7 \leqslant 7 = m$). Since $K_{3,7} \cup 4K_1 \subseteq K_{3,11}$ we obtain that $\mathcal{R}_k^*$ is not a hereditary property.

The fact that $\mathcal{R}_k^*$ is not additive follows from the results in [6], but it also follows from an example similar to the example constructed above. Indeed, $\chi_r^*(K_{3,7}) = 7$ and $\chi_r^*(4K_1) = 1$, but $\chi_r^*(K_{3,7} \cup 4K_1) \geqslant 8$. $\square$

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