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STEADY STATE COEXISTENCE SOLUTIONS OF REACTION-DIFFUSION COMPETITION MODELS

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Abstract. Two species of animals are competing in the same environment. Under what conditions do they coexist peacefully? Or under what conditions does either one of the two species become extinct, that is, is either one of the two species excluded by the other? It is natural to say that they can coexist peacefully if their rates of reproduction and self-limitation are relatively larger than those of competition rates. In other words, they can survive if they interact strongly among themselves and weakly with others. We investigate this phenomena in mathematical point of view.

In this paper we concentrate on coexistence solutions of the competition model

$$\begin{cases} \Delta u + u(a - g(u,v)) = 0, \\ \Delta v + v(d - h(u,v)) = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases}$$

This system is the general model for the steady state of a competitive interacting system. The techniques used in this paper are elliptic theory, super-sub solutions, maximum principles, implicit function theorem and spectrum estimates. The arguments also rely on some detailed properties of the solution of logistic equations.

Keywords: elliptic theory, maximum principles

MSC 2000: 35J55, 35J60

1. Introduction

A lot of research has been focused on reaction-diffusion equations modeling various systems in mathematical biology, especially the elliptic steady states of competitive and predator-prey interacting processes with various boundary conditions. In earlier literature, investigations into mathematical biology models were concerned with studying those with homogeneous Neumann boundary conditions. Later on, the

more important Dirichlet problems, which allow flux across the boundary, became the subject of study.

Suppose two species of animals, rabbits and squirrels for instance, are competing in a bounded domain Ω. Let $u(x,t)$ and $v(x,t)$ be densities of the two habitats in the place x of Ω at time t. Then we have the following biological interpretation of terms.

(A) The partial derivatives $u_t(x,t)$ and $v_t(x,t)$ mean the rate of change of densities with respect to time t.

(B) The laplacians $\Delta u(x,t)$ and $\Delta v(x,t)$ stand for the diffusion or migration rates.

(C) The rates of self-reproduction of each species of animals are expressed as multiples of some positive constants a, d and current densities $u(x,t)$, $v(x,t)$, i.e. $au(x,t)$ and $\mathrm{d}v(x,t)$ which will increase the rate of change of densities in (A), where $a>0$, $d>0$ are called the self-reproduction constants.

(D) The rates of self-limitation of each species of animals are multiples of some positive constants b, f and the frequency of encounters among themselves $u^2(x,t)$, $v^2(x,t)$, i.e. $bu^2(x,t)$ and $fv^2(x,t)$ which will decrease the rate of change of densities in (A), where $b>0$, $f>0$ are called the self-limitation constants.

(E) The rates of competition of each species of animals are multiples of some positive constants c, e and the frequency of encounters of each species with the other $u(x,t)v(x,t)$, i.e. $cu(x,t)v(x,t)$ and $eu(x,t)v(x,t)$ which will decrease the rate of change of densities in (A), where $c>0$, $e>0$ are called the competition constants.

(F) We assume that none of the species of animals is staying on the boundary of Ω.

Combining all those together, we have the dynamic model

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + au(x,t) - bu^2(x,t) - cu(x,t)v(x,t), & \\ v_t(x,t) = \Delta v(x,t) + \mathrm{d}v(x,t) - fv^2(x,t) - eu(x,t)v(x,t) & \text{in } \Omega\times[0,\infty), \\ u(x,t) = v(x,t) = 0 & \text{for } x\in\partial\Omega, \end{cases}$$

or equivalently,

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) - cv(x,t)), & \\ v_t(x,t) = \Delta v(x,t) + v(x,t)(d - fv(x,t) - eu(x,t)) & \text{in } \Omega\times[0,\infty), \\ u(x,t) = v(x,t) = 0 & \text{for } x\in\partial\Omega. \end{cases}$$

Here we are interested in the time independent, positive solutions, i.e. the positive solutions $u(x)$, $v(x)$ of

$$\begin{cases} \Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0, & \\ \Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, & \end{cases}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so its existence means the two species of animals can live peacefully and forever.

A lot of work about the existence and uniqueness of the coexistence state of the above steady state model has already been done during the last decade. (See [1], [2], [3], [6], [7], [9], [10].)

In this paper we study rather general types of the system. We are concerned with the existence and uniqueness of positive coexistence when the relative growth rates are nonlinear, more precisely, the existence and uniqueness of a positive steady state of

$$\begin{cases} \Delta u + u(a - g(u,v)) = 0, \\ \Delta v + v(d - h(u,v)) = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases}$$

where a, d are positive constants, g, h are C^1 functions, Ω is a bounded domain in $\mathbb{R}^n$ and u, v are densities of the two competitive species.

The following are questions raised in the general model with nonlinear growth rates.

Problem 1: Under what conditions do the species coexist? Under what conditions do they have a unique steady state? When does either one of the species become extinct?

Problem 2: Assuming that they can coexist and the coexistence state is unique at a fixed self-reproduction (a, d), can they still coexist regardless of a slight change of that self-reproduction?

Problem 3: This is a generalization of Problem 2. If we have existence and uniqueness of the coexistence state on the left boundary of a closed convex region Γ for the reproduction (a, d), can we extend the region Γ to an open set including Γ without losing the uniqueness?

We will need some information on the solutions of the logistic equation. (cf. [9])

$$\begin{cases} \Delta u + uf(u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, & u > 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and

(A) f is a strictly decreasing C^1 function,

(B) there exists $c_0 > 0$ such that $f(u) \leqslant 0$ for $u \geqslant c_0$.

(1) If $f(0) > \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary condition, then the above equation has a unique positive solution.

(2) If $f(0) < \lambda_1$, then $u \equiv 0$ is the only nonnegative solution of the above equation.

In the case (1), we denote this unique positive solution as θ_f. The main property of this positive solution is that θ_f is larger provided f is larger, i.e. $\theta_g \leqslant \theta_f$ if $g \leqslant f$. In Section 2, some sufficient conditions guaranteeing the existence and uniqueness of positive solutions are obtained, and we can also see that there is no positive solution for small self-reproduction rates, mainly by using upper-lower solutions and spectrum estimates, which solves Problem 1. In Sections 3 and 4, we answer Problems 2 and 3 using elliptic theory, maximum principles and implicit function theorem.

2. Existence, nonexistence and uniqueness of steady state

We consider the elliptic system

$$\begin{cases} \Delta u + u(a - g(u,v)) = 0, \\ \Delta v + v(d - h(u,v)) = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

Here Ω is a bounded, smooth domain in $\mathbb{R}^n$ and

(U1) $g, h \in C^1$ are strictly increasing functions with respect to u, v,

(U2) there exist $k_1, k_2 > 0$ such that $g(u,0) > a$ for $u \geqslant k_1$ and $h(0,v) > d$ for $v \geqslant k_2$.

If there were no competition between the species, that is, if we consider

$$\begin{cases} \Delta u + u(a - g(u,0)) = 0, \\ \Delta v + v(d - h(0,v)) = 0 \quad \text{in } \Omega, \\ u = v = 0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}$$

then the condition $a > \lambda_1$, $d > \lambda_1$ (i.e. reproductions are relatively large) were sufficient to guarantee the existence of a positive density solution $\theta_{a-g(\cdot,0)}$, $\theta_{d-h(0,\cdot)}$. But, if there is some competition between them, then as we see in the following Theorem 2.1, we should have larger lower bounds for reproduction rates a and d, i.e. we have stronger conditions $a > \lambda_1 + g(0,k_2)$, $d > \lambda_1 + h(k_1,0)$ to guarantee their coexistence. (I.e. the reproductions should be much larger.)

The following theorem is the main result:

Theorem 2.1. *(A) If $a > \lambda_1 + g(0,k_2)$ and $d > \lambda_1 + h(k_1,0)$, then (1) has a positive solution (u,v) with*

$$\theta_{a-g(\cdot,k_2)} < u < \theta_{a-g(\cdot,0)}, \qquad \theta_{d-h(k_1,\cdot)} < v < \theta_{d-h(0,\cdot)}.$$

Conversely, any positive solution (u,v) to (1) must satisfy these inequalities.

(B) *If* $a > \lambda_1 + g(0, k_2)$ *and* $d > \lambda_1 + h(k_1, 0)$ *and*

$$4 \inf_B \Big(\frac{\partial g}{\partial u}\Big) \inf_B \Big(\frac{\partial h}{\partial v}\Big) > \sup \frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}} \Big(\sup \Big(\frac{\partial g}{\partial v}\Big)\Big)^2 + \sup \frac{\theta_{dh(0,\cdot)}}{\theta_{ag(\cdot,k_2)}} \Big(\sup \Big(\frac{\partial h}{\partial u}\Big)\Big)^2 + 2 \sup \Big(\frac{\partial g}{\partial v}\Big) \sup \Big(\frac{\partial h}{\partial u}\Big),$$

where $B = [0, k_1] \times [0, k_2]$, *then* (1) *has a unique coexistence state.*

(C) *If* $a \leqslant \lambda_1$ *or* $d \leqslant \lambda_1$, *then* (1) *has no positive solution.*

Biologically, we can interpret the conditions in Theorem 2.1 as follows. The constants a, d and functions g, h describe how species 1 (u) and 2 (v) interact among themselves and with each other. Hence, both the conditions in (A) and (B) imply that species 1 interacts strongly among themselves and weakly with species 2. Similarly for species 2, they interact more strongly among themselves than they do with species 1. The inequalities in the assertion (A) imply that the densities with competitions (u and v) are less than those without competition ($\theta_{a-g(\cdot,0)}$ and $\theta_{d-h(0,\cdot)}$). Furthermore, (C) says that if one of the species has small reproduction, then it may be extinct, which means that the two species can not coexist.

Proof. (A) The proof of the existence has already been done in [10]. Here we concentrate on proving the inequalities for the solution. Let $\bar{u} = \theta_{a-g(\cdot,0)}$, $\bar{v} = \theta_{d-h(0,\cdot)}$. Then since g is increasing, we have

$$\begin{aligned} \Delta\bar{u} + \bar{u}(a - g(\bar{u}, \bar{v})) &= \Delta\bar{u} + \bar{u}(a - g(\bar{u}, 0) + g(\bar{u}, 0) - g(\bar{u}, \bar{v})) \\ &= \bar{u}(g(\bar{u}, 0) - g(\bar{u}, \bar{v})) < 0. \end{aligned}$$

Similarly, we have

$$\Delta\bar{v} + \bar{v}(d - h(\bar{u}, \bar{v})) < 0.$$

So, $(\bar{u}, \bar{v})$ is an upper solution of (1).

Let $\underline{u} = \theta_{a-g(\cdot,k_2)}, \underline{v} = \theta_{d-h(k_1,\cdot)}$. Then by the Maximum Principle we obtain

$$\begin{cases} \underline{u} \leqslant \theta_{a-g(\cdot,0)} \leqslant k_1, \\ \underline{v} \leqslant \theta_{d-h(0,\cdot)} \leqslant k_2. \end{cases}$$

Since g is increasing, we get

$$\begin{aligned} \Delta\underline{u} + \underline{u}(a - g(\underline{u}, \underline{v})) &= \Delta\underline{u} + \underline{u}(a - g(\underline{u}, k_2) + g(\underline{u}, k_2) - g(\underline{u}, \underline{v})) \\ &= \underline{u}(g(\underline{u}, k_2) - g(\underline{u}, \underline{v})) \geqslant 0. \end{aligned}$$

Similarly, we get

$$\Delta\underline{v} + \underline{v}(d - h(\underline{u}, \underline{v})) \geqslant 0.$$

Therefore, $(\underline{u}, \underline{v})$ is a lower solution of (1).

Furthermore, $\underline{u} < \bar{u}$ and $\underline{v} < \bar{v}$ in Ω and $\underline{u} = \bar{u} = \underline{v} = \bar{v} = 0$ on $\partial\Omega$.

So, by the upper-lower solution method, (1) has a solution (u, v) with

$$\theta_{a-g(\cdot,k_2)} < u < \theta_{a-g(\cdot,0)}, \qquad \theta_{d-h(k_1,\cdot)} < v < \theta_{d-h(0,\cdot)}.$$

Suppose (u, v) is a positive solution to (1). By the Mean Value Theorem, there is v^* such that

$$g(u, v) = g(u, 0) + \frac{\partial g(u, v^*)}{\partial v}\, v.$$

Then

$$\Delta u + u(a - g(u, 0)) = \frac{\partial g(u, v^*)}{\partial v} uv > 0 \text{ in } \Omega.$$

Hence, u is a subsolution to

$$\begin{cases} \Delta z + z(a - g(z, 0)) = 0 & \text{in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Any sufficiently large positive constant is a upper solution to

$$\begin{cases} \Delta z + z(a - g(z, 0)) = 0 & \text{in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Therefore, by the upper-sub solution method, we have

$$u \leqslant \theta_{a-g(\cdot,0)}. \tag{2}$$

The same argument shows

$$v \leqslant \theta_{d-h(0,\cdot)}. \tag{3}$$

For sufficiently small $\varepsilon > 0$,

$$\begin{aligned} &\varepsilon\Delta\theta_{d-h(0,\cdot)} + \varepsilon\theta_{d-h(0,\cdot)}(d - h(0, \varepsilon\theta_{d-h(0,\cdot)})) \\ &\qquad = \varepsilon[\Delta\theta_{d-h(0,\cdot)} + \theta_{d-h(0,\cdot)}(d - h(0, \varepsilon\theta_{d-h(0,\cdot)}))] \\ &\qquad > \varepsilon[\Delta\theta_{d-h(0,\cdot)} + \theta_{d-h(0,\cdot)}(d - h(0, \theta_{d-h(0,\cdot)}))] = 0 \text{ in } \Omega, \end{aligned}$$

and so $\varepsilon\theta_{d-h(0,\cdot)}$ is a sub solution to

$$\begin{cases} \Delta z + z(d - h(0, z)) = 0 & \text{in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Since $d - h(0, k_2) < 0$, k_2 is a upper solution to

$$\begin{cases} \Delta z + z(d - h(0, z)) = 0 & \text{in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Hence, by the upper-sub solution method again,

$$\theta_{d-h(0,\cdot)} \leqslant k_2.$$

So,

$$g(u, v) \leqslant g(u, \theta_{d-h(0,\cdot)}) \leqslant g(u, k_2)$$

since $g(u, z)$ is increasing. Therefore,

$$\Delta u + u(a - g(u, k_2)) \leqslant \Delta u + u(a - g(u, v)) = 0 \quad \text{in } \Omega.$$

Hence, u is a upper solution to

$$\begin{cases} \Delta z + z(a - g(z, k_2)) = 0 & \text{in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Let φ_1 be the first eigenvector of

$$\begin{cases} \Delta u + \lambda_1 z = 0 \text{ in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Then for sufficiently small $\varepsilon > 0$,

$$a - g(\varepsilon\varphi_1, k_2) - \lambda_1 > 0 \quad \text{in } \Omega,$$

and

$$\begin{aligned} &\Delta(\varepsilon\varphi_1) + \varepsilon\varphi_1(a - g(\varepsilon\varphi_1, k_2)) \\ &\qquad = \varepsilon[\Delta(\varepsilon\varphi_1) + \varphi_1(a - g(\varepsilon\varphi_1, k_2))]\varepsilon(\Delta\varphi_1 + \lambda_1\varphi_1) = 0 \quad \text{in } \Omega. \end{aligned}$$

Consequently, $\varepsilon\varphi_1$ is a sub solution to

$$\begin{cases} \Delta z + z(a - g(z, k_2)) = 0 & \text{in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Hence, by the upper-sub solution method again,

$$\theta_{a-g(\cdot,k_2)} \leqslant u. \tag{4}$$

The same argument shows

$$\theta_{d-h(k_1,\cdot)} \leqslant v. \tag{5}$$

From (2) to (5), we have

$$\theta_{a-g(\cdot,k_2)} \leqslant u \leqslant \theta_{a-g(\cdot,0)}, \qquad \theta_{d-h(k_1,\cdot)} \leqslant v \leqslant \theta_{d-h(0,\cdot)}. \tag{6}$$

Consequently, for any positive solution (u, v) of (1), the inequalities (6) hold.

(B) Suppose (u_1, v_1) and (u_2, v_2) are positive solutions to (1). Let $p = u_1 - u_2$ and $q = v_1 - v_2$. Then

$$\begin{aligned}
\Delta p + (a - g(u_1, v_1))p &= \Delta u_1 - \Delta u_2 + (a - g(u_1, v_1))(u_1 - u_2) \\
&= -\Delta u_2 - (a - g(u_1, v_1))u_2 \\
&= -\Delta u_2 - u_2(a - g(u_2, v_2) + g(u_2, v_2) - g(u_1, v_1)) \\
&= -u_2(g(u_2, v_2) - g(u_1, v_1)) \\
&= -u_2(g(u_2, v_2) - g(u_1, v_2) + g(u_1, v_2) - g(u_1, v_1)) \\
&= -u_2\Big(\frac{\partial g(\tilde{x}, v_2)}{\partial u}(-p) + \frac{\partial g(u_1, \bar{x})}{\partial v}(-q)\Big) \\
&= u_2\Big(p\frac{\partial g(\tilde{x}, v_2)}{\partial u} + q\frac{\partial g(u_1, \bar{x})}{\partial v}\Big) \quad \text{in } \Omega,
\end{aligned}$$

where $\tilde{x}, \bar{x}$ are from Mean Value Theorem depending on u_1, u_2, v_1, v_2. Hence,

$$\Delta p + (a - g(u_1, v_1))p - u_2\Big(p\frac{\partial g(\tilde{x}, v_2)}{\partial u} + q\frac{\partial g(u_1, \bar{x})}{\partial v}\Big) = 0 \quad \text{in } \Omega. \tag{7}$$

Similarly, we can get

$$\Delta q + (d - h(u_2, v_2))q - v_1\Big(p\frac{\partial h(\tilde{y}, v_1)}{\partial u} + q\frac{\partial h(u_2, \bar{y})}{\partial v}\Big) = 0 \quad \text{in } \Omega, \tag{8}$$

where $\tilde{y}$, $\bar{y}$ are from Mean Value Theorem depending on u_1, u_2, v_1, v_2. Since $\lambda_1(a - g(u_1, v_1)) = 0$, by the Variational Characterization of the first eigenvalue we obtain

$$\int_\Omega z(-\Delta z - (a - g(u_1, v_1))z)\,\mathrm{d}x \geqslant 0 \tag{9}$$

for any $z \in C^2(\bar{\Omega})$ and $z|_{\partial\Omega} = 0$. The same argument shows that

$$\int_\Omega w(-\Delta w - (d - h(u_2, v_2))w)\,\mathrm{d}x \geqslant 0 \tag{10}$$

for any $w \in C^2(\bar{\Omega})$ and $w|_{\partial\Omega} = 0$. From (7) and (8) we have

$$\begin{cases} -p\Delta p - (a - g(u_1, v_1))p^2 + u_2 p\Big(p\dfrac{\partial g(\tilde{x}, v_2)}{\partial u} + q\dfrac{\partial g(u_1, \bar{x})}{\partial v}\Big) = 0, \\ -q\Delta q - (d - h(u_2, v_2))q^2 + v_1 q\Big(p\dfrac{\partial h(\tilde{y}, v_1)}{\partial u} + q\frac{\partial h(u_2, \bar{y})}{\partial v}\Big) = 0 \qquad \text{in } \Omega. \end{cases}$$

Using (9) and (10), we conclude

$$\int_\Omega \Big[u_2 p\Big(p\frac{\partial g(\tilde{x}, v_2)}{\partial u} + q\frac{\partial g(u_1, \bar{x})}{\partial v}\Big) + v_1 q\Big(p\frac{\partial h(\tilde{y}, v_1)}{\partial u} + q\frac{\partial h(u_2, \bar{y})}{\partial v}\Big)\Big] \leqslant 0.$$

Hence,

$$\int_\Omega \Big[u_2\frac{\partial g(\tilde{x}, v_2)}{\partial u}p^2 + \Big(u_2\frac{\partial g(u_1, \bar{x})}{\partial v} + v_1\frac{\partial h(\tilde{y}, v_1)}{\partial u}\Big)pq + v_1\frac{\partial h(u_2, \bar{y})}{\partial v}q^2\Big] \leqslant 0.$$

Therefore, $p \equiv q \equiv 0$ if we can show that

$$\Big(u_2\frac{\partial g(u_1, \bar{x})}{\partial v} + v_1\frac{\partial h(\tilde{y}, v_1)}{\partial u}\Big)^2 - 4u_2 v_1\frac{\partial g(\tilde{x}, v_2)}{\partial u}\frac{\partial h(u_2, \bar{y})}{\partial v} < 0 \quad \text{in } \Omega,$$

which is true if

$$\begin{aligned} u_2^2\Big(\frac{\partial g(u_1, \bar{x})}{\partial v}\Big)^2 + v_1^2\Big(\frac{\partial h(\tilde{y}, v_1)}{\partial u}\Big)^2 + 2u_2 v_1\frac{\partial g(u_1, \bar{x})}{\partial v}\frac{\partial h(\tilde{y}, v_1)}{\partial u} \\ - 4u_2 v_1\frac{\partial g(\tilde{x}, v_2)}{\partial u}\frac{\partial h(u_2, \bar{y})}{\partial v} < 0 \quad \text{in } \Omega, \end{aligned}$$

i.e.,

$$\begin{aligned} 4u_2 v_1\frac{\partial g(\tilde{x}, v_2)}{\partial u}\frac{\partial h(u_2, \bar{y})}{\partial v} > u_2^2\Big(\frac{\partial g(u_1, \bar{x})}{\partial v}\Big)^2 + v_1^2\Big(\frac{\partial h(\tilde{y}, v_1)}{\partial u}\Big)^2 \\ + 2u_2 v_1\frac{\partial g(u_1, \bar{x})}{\partial v}\frac{\partial h(\tilde{y}, v_1)}{\partial u} \quad \text{in } \Omega, \end{aligned}$$

or

$$\begin{aligned} 4\frac{\partial g(\tilde{x}, v_2)}{\partial u}\frac{\partial h(u_2, \bar{y})}{\partial v} > \frac{u_2}{v_1}\Big(\frac{\partial g(u_1, \bar{x})}{\partial v}\Big)^2 + \frac{v_1}{u_2}\Big(\frac{\partial h(\tilde{y}, v_1)}{\partial u}\Big)^2 \\ + 2\frac{\partial g(u_1, \bar{x})}{\partial v}\frac{\partial h(\tilde{y}, v_1)}{\partial u} \quad \text{in } \Omega. \end{aligned}$$

This is the case from the hypothesis of the theorem and (6), and so the uniqueness is proved.

(C) Assume $a \leqslant \lambda_1$. Suppose (u, v) is a nonnegative solution to (1). Then since g is an increasing function with respect to u and v,

$$\begin{aligned}\Delta u + u(a - g(u,0)) &= \Delta u + u(a - g(u,v) + g(u,v) - g(u,0)) \\ &= u(g(u,v) - g(u,0)) \geqslant 0.\end{aligned}$$

Therefore, u is a sub solution to

$$\begin{cases} \Delta u + u(a - g(u,0)) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Any constant larger than k_1 is a upper solution to

$$\begin{cases} \Delta u + u(a - g(u,0)) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Hence, by the upper-lowersolution method, there is a solution $\bar{u}$ of

$$\begin{cases} \Delta u + u(a - g(u,0)) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

such that $0 \leqslant u \leqslant \bar{u}$. But, since $a \leqslant \lambda_1$, $\bar{u} \equiv 0$, and so $u \equiv 0$.

3. Uniqueness under small perturbation of reproduction rates

We consider the model

$$\begin{cases} \Delta u + u(a - g(u,v)) = 0, \\ \Delta v + v(d - h(u,v)) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \tag{11}$$

Here Ω is a bounded, smooth domain in $\mathbb{R}^n$ and

(P1) $g, h \in C^1$ are strictly increasing functions with respect to u and v, and $g(0,0) = h(0,0) = 0$,

(P2) there are $k_1, k_2 > 0$ such that $g(u,0) > a > \lambda_1$ for $u \geqslant k_1$ and $h(0,v) > d > \lambda_1$ for $v \geqslant k_2$.

The following theorem is the main result.

Theorem 3.1. *Suppose*

(A) $a > \lambda_1(g(0, \theta_{d-h(0,\cdot)}))$, $d > \lambda_1(h(\theta_{a-g(\cdot,0)}, 0))$,

(B) (11) *has a unique coexistence state* (u, v),

(C) *the Fréchet derivative of* (11) *at* (u, v) *is invertible.*

Then there is a neighborhood V *of* (a, d) *in* $\mathbb{R}^2$ *such that if* $(a_0, d_0) \in V$, *then* (11) *with* $(a, d) = (a_0, d_0)$ *has a unique coexistence state.*

Theorem 3.1 looks like a consequence of Implicit Function Theorem. However, the inverse function theorem only guaranteed the uniqueness locally. Theorem 3.1 yields the global uniqueness. The techniques we will use includes naturally Implicit Function Theorem and a priori estimates on solutions of (11).

Biologically, the first condition in this theorem indicates that the rates of self-reproduction are large. The condition of invertibility of the Fréchet derivative also illustrates that the rates of self-limitation are relatively larger than those of competition which will appear in Theorem 3.3. Then the conclusion says that a small perturbation of reproduction rates does not cause the loss of existence and uniqueness of a positive steady state, i.e. the species can still coexist peacefully even if there is some slight change of the reproduction rates.

Proof. Since the Fréchet derivative of (11) at (u, v) is invertible, by the Implicit Function Theorem there is a neighborhood V of (a, d) in $\mathbb{R}^2$ and a neighborhood W of (u, v) in $[C_0^{2+\alpha}(\bar{\Omega})]^2$ such that for all $(a_0, d_0) \in V$, there is a unique positive solution $(u_0, v_0) \in W$ of (11). Suppose the conclusion of the theorem is false. Then there are sequences $(a_n, d_n, u_n, v_n), (a_n, d_n, u_n^*, v_n^*)$ in $V \times [C_0^{2+\alpha}(\bar{\Omega})]^2$ such that (u_n, v_n) and (u_n^*, v_n^*) are positive solutions with $(a, d) = (a_n, d_n)$ and $(u_n, v_n) \neq (u_n^*, v_n^*)$ and $(a_n, d_n) \to (a, d)$. By the standard elliptic theory, $(u_n, v_n) \to (\bar{u}, \bar{v})$ and $(u_n^*, v_n^*) \to (u^*, v^*)$ in $C^{2,\alpha}$, and $(\bar{u}, \bar{v})$, (u^*, v^*) are solutions of (11). We claim $\bar{u} > 0$, $\bar{v} > 0$, $u^* > 0$, $v^* > 0$. It is enough to show that $\bar{u}$ and $\bar{v}$ are not identically zero because of the Maximum Principle. Suppose not, then by the Maximum Principle again, one of the following cases should occur: (1) $\bar{u}$ is identically zero and $\bar{v} > 0$. (2) $\bar{u} > 0$ and $\bar{v}$ is identically zero. (3) $\bar{u}$ is identically zero and $\bar{v}$ is identically zero.

Without loss of generality, assume $\bar{u}$ is identically zero.

Let $\tilde{u}_n = u_n/\|u_n\|_\infty, \tilde{v}_n = v_n$ for all $n \in \mathbb{N}$. Then

$$\begin{cases} \Delta\tilde{u}_n + \tilde{u}_n(a_n - g(u_n, \tilde{v}_n)) = 0, \\ \Delta\tilde{v}_n + \tilde{v}_n(d_n - h(u_n, \tilde{v}_n)) = 0 \quad \text{in } \Omega. \end{cases}$$

From the elliptic theory, $\tilde{u}_n \to \tilde{u}$ and

$$\begin{cases} \Delta\tilde{u} + \tilde{u}(a - g(0, \bar{v})) = 0, \\ \Delta\bar{v} + \bar{v}(d - h(0, \bar{v})) = 0 \quad \text{in } \Omega, \end{cases}$$

since g, h are continuous, i.e., $a = \lambda_1(g(0, \bar{v}))$.

(1) If $\overline{v} \equiv 0$, then by the monotonicity of g and λ_1, $a = \lambda_1(g(0,\overline{v})) = \lambda_1(g(0,0)) \leqslant \lambda_1(g(0,\theta_{d-h(0,\cdot)}))$, which contradicts our assumption.

(2) If $\overline{v}$ is not identically zero, then $\overline{v} = \theta_{d-h(0,\cdot)}$ and so $a = \lambda_1(g(0,\overline{v})) = \lambda_1(g(0,\theta_{d-h(0,\cdot)}))$, which is also a contradiction to our assumption. Consequently, $(\bar{u},\overline{v})$ and (u^*,v^*) are coexistence states for (a,d). But, since the coexistence state with respect to (a,d) is unique, $(\bar{u},\overline{v}) = (u^*,v^*) = (u,v)$. But, since $(u_n,v_n) \neq (u_n^*,v_n^*)$, this contradicts the Implicit Function Theorem.

The proof of the theorem also tells us that if one of the species becomes extinct, in other words, if one is excluded by the other, then that means the reproduction rates are small, i.e. the condition of reproduction rates (A) is reasonable.

Theorem 3.2. *If $(a_n,d_n,u_n,v_n) \to (a,d,u,v)$ and if $u \equiv 0$ or $v \equiv 0$, then $a \leqslant \lambda_1(g(0,\theta_{d-h(0,\cdot)}))$ or $d \leqslant \lambda_1(h(\theta_{a-g(\cdot,0)},0))$.*

The condition, invertibility of the Fréchet derivative, in Theorem 3.1 is too artificial. Now we turn out attention to get conditions guaranteing the invertibility of the Fréchet derivative.

Theorem 3.3. *Suppose (u,v) is a positive solution to* (11). *If*

$$4\inf\frac{\partial g(x,y)}{\partial x}\inf\frac{\partial h(x,y)}{\partial y}uv > [\sup\frac{\partial g(x,y)}{\partial y}u + \sup\frac{\partial h(x,y)}{\partial x}v]^2,$$

then the Fréchet derivative of (11) *at (u,v) is invertible.*

Proof. The Fréchet derivative at (u,v) is

$$A = \begin{pmatrix} -\Delta + g(u,v) + u\dfrac{\partial g(u,v)}{\partial u} - a & u\dfrac{\partial g(u,v)}{\partial v} \\ v\dfrac{\partial h(u,v)}{\partial u} & -\Delta + h(u,v) + v\dfrac{\partial h(u,v)}{\partial v} - d \end{pmatrix}.$$

We need to show that $N(A) = \{0\}$ by the Fredholm alternative. If

$$\begin{cases} -\Delta\varphi + \Big(g(u,v) + u\dfrac{\partial g(u,v)}{\partial u} - a\Big)\varphi + \dfrac{\partial g(u,v)}{\partial v}u\psi = 0, \\ -\Delta\psi + \dfrac{\partial h(u,v)}{\partial u}v\varphi + \Big(h(u,v) + v\dfrac{\partial h(u,v)}{\partial v} - d\Big)\psi = 0, \end{cases}$$

then

$$\int_\Omega \Big[|\nabla\varphi|^2 + \Big(g(u,v) + u\frac{\partial g(u,v)}{\partial u} - a\Big)\varphi^2 + \frac{\partial g(u,v)}{\partial v}u\varphi\psi\Big] = 0,$$
$$\int_\Omega \Big[|\nabla\psi|^2 + \frac{\partial h(u,v)}{\partial u}v\varphi\psi + \Big(h(u,v) + v\frac{\partial h(u,v)}{\partial v} - d\Big)\psi^2\Big] = 0.$$

Since $\lambda_1(g(u,v)-a)=\lambda_1(h(u,v)-d)=0$, we have

$$\int_\Omega [|\nabla\varphi|^2+(g(u,v)-a)\varphi^2]\geqslant 0,$$
$$\int_\Omega [|\nabla\psi|^2+(h(u,v)-d)\psi^2]\geqslant 0.$$

Hence,

$$\int_\Omega \Big(u\frac{\partial g(u,v)}{\partial u}\varphi^2+\frac{\partial g(u,v)}{\partial v}u\varphi\psi\Big)\leqslant 0,$$
$$\int_\Omega \Big(\frac{\partial h(u,v)}{\partial u}v\varphi\psi+\frac{\partial h(u,v)}{\partial v}v\psi^2\Big)\leqslant 0.$$

Therefore,

$$\int_\Omega \Big[u\frac{\partial g(u,v)}{\partial u}\varphi^2+\Big(\frac{\partial g(u,v)}{\partial v}u+\frac{\partial h(u,v)}{\partial u}v\Big)\varphi\psi+\frac{\partial h(u,v)}{\partial v}v\psi^2\Big]\leqslant 0.$$

Consequently, if

$$4\inf\Big(\frac{\partial g(x,y)}{\partial x}\Big)\inf\Big(\frac{\partial h(x,y)}{\partial y}\Big)uv>\Big[\Big(\sup\Big(\frac{\partial g(x,y)}{\partial y}\Big)\Big)u+\Big(\sup\Big(\frac{\partial h(x,y)}{\partial x}\Big)\Big)v\Big]^2,$$

then the integrand on the left hand side is a positive definite form in Ω, which means $\varphi\equiv\psi\equiv 0$. Therefore, the above Fréchet derivative A is invertible.

Combining Theorems 2.1, 3.1 and 3.3, we have the following Corollary which is actually the main result in this section.

Corollary 3.4. *Suppose*

(A) $a>\lambda_1+g(0,k_2)$, $d>\lambda_1+h(k_1,0)$, *and*

(B) $$4\inf_B\frac{\partial g(x,y)}{\partial x}\inf_B\frac{\partial h(x,y)}{\partial y}>\Big[\sup\frac{\partial g(x,y)}{\partial y}+\sup\frac{\partial h(x,y)}{\partial x}\sup\frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}}\Big]\times\Big[\sup\frac{\partial g(x,y)}{\partial y}\sup\frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}}+\sup\frac{\partial h(x,y)}{\partial x}\Big],$$

where $B=[0,k_1]\times[0,k_2]$.

Then there is a neighborhood V *of* (a,d) *in* $\mathbb{R}^2$ *such that if* $(a_0,d_0)\in V$, *then* (11) *with* $(a,d)=(a_0,d_0)$ *has a unique coexistence state.*

Proof. From $\theta_{a-g(,0)}<k_1$, $\theta_{d-h(0,)}<k_2$, and the monotonicity of $g(0,\cdot)$, $h(\cdot,0)$ we have

$$\begin{cases} a>\lambda_1+g(0,k_2)\geqslant\lambda_1(g(0,\theta_{d-h(0,\cdot)})),\\ d>\lambda_1+h(k_1,0)\geqslant\lambda_1(h(\theta_{a-g(\cdot,0)},0)).\end{cases}$$

Further,

$$
\begin{aligned}
4\inf_B \frac{\partial g(x,y)}{\partial x}\inf_B \frac{\partial h(x,y)}{\partial y} &> \Big[\sup \frac{\partial g(x,y)}{\partial y} + \sup \frac{\partial h(x,y)}{\partial x}\sup \frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(0,k_2)}}\Big] \\
&\quad \times \Big[\sup \frac{\partial g(x,y)}{\partial y}\sup \frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}} + \sup \frac{\partial h(x,y)}{\partial x}\Big] \\
&= \Big[\sup \frac{\partial g(x,y)}{\partial y}\Big]^2 \sup \frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}} + \sup \frac{\partial g(x,y)}{\partial y}\sup \frac{\partial h(x,y)}{\partial x} \\
&\quad + \sup \frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}}\sup \frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}}\sup \frac{\partial g(x,y)}{\partial y}\sup \frac{\partial h(x,y)}{\partial x} \\
&\quad + \Big[\sup \frac{\partial h(x,y)}{\partial x}\Big]^2 \sup \frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}} \\
&\geqslant \Big[\sup \frac{\partial g(x,y)}{\partial y}\Big]^2 \sup \frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}} + 2\sup \frac{\partial g(x,y)}{\partial y}\sup \frac{\partial h(x,y)}{\partial x} \\
&\quad + \Big[\sup \frac{\partial h(x,y)}{\partial x}\Big]^2 \sup \frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}}
\end{aligned}
$$

since $\theta_{a-g(\cdot,0)} > \theta_{a-g(\cdot,k_2)}, \theta_{d-h(0,\cdot)} > \theta_{d-h(k_1,\cdot)}$.

Therefore, (11) has a unique coexistence state (u,v) by Theorem 2.1. Furthermore, by the estimate of the solution in the proof of Theorem 2.1,

$$
\begin{aligned}
4\inf_B \frac{\partial g(x,y)}{\partial x}\inf_B \frac{\partial h(x,y)}{\partial y} &> \Big[\sup \frac{\partial g(x,y)}{\partial y} + \sup \frac{\partial h(x,y)}{\partial x}\sup \frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}}\Big] \\
&\quad \times \Big[\sup \frac{\partial g(x,y)}{\partial y}\sup \frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}} + \sup \frac{\partial h(x,y)}{\partial x}\Big] \\
&\geqslant \Big[\sup \frac{\partial g(x,y)}{\partial y} + \sup \frac{\partial h(x,y)}{\partial x}\frac{v}{u}\Big]\Big[\sup \frac{\partial g(x,y)}{\partial y}\frac{u}{v} + \sup \frac{\partial h(x,y)}{\partial x}\Big].
\end{aligned}
$$

Thus, we obtain

$$
4\inf_B \frac{\partial g(x,y)}{\partial x}\inf_B \frac{\partial h(x,y)}{\partial y}uv > \Big[\sup \frac{\partial g(x,y)}{\partial y}u + \sup \frac{\partial h(x,y)}{\partial x}v\Big]^2.
$$

Thus Theorem 3.3 implies that the Fréchet derivative of (11) at (u,v) is invertible. Therefore, the theorem follows from Theorem 3.1.

4. Uniqueness in a region of reproduction rates

Consider the model

$$
\begin{cases}
\Delta u + u(a - g(u,v)) = 0, \\
\Delta v + v(d - h(u,v)) = 0 \quad \text{in } \Omega, \\
u|_{\partial\Omega} = v|_{\partial\Omega} = 0.
\end{cases}
\tag{12}
$$

Here Ω is a bounded smooth domain in $\mathbb{R}^n$ and $g, h \in C^1$ are strictly increasing functions with respect to u and v, and $g(0,0) = h(0,0) = 0$.

The following theorem is the main result.

Theorem 4.1. *Suppose*

(A) Γ *is a closed, convex region in* $\mathbb{R}^2$ *such that for all* $(a,d) \in \Gamma$, $a > \lambda_1(g(0, \theta_{d-h(0,\cdot)}))$ *and* $d > \lambda_1(h(\theta_{a-g(\cdot,0)}, 0))$,

(B) *there exist* $c_0 > 0$ *and* $c_1 > 0$ *such that for all* $(a,d) \in \Gamma$, $g(x,0) > a > \lambda_1$ *for* $x > c_0$ *and* $h(0,y) > d > \lambda_1$ *for* $y > c_1$,

(C) (12) *has a unique positive solution for every* $(a,d) \in \partial_L\Gamma$, *where* $\partial_L\Gamma = \{(\lambda_d, d) \in \Gamma | \text{for any fixed } d, \lambda_d = \inf\{a|(a,d) \in \Gamma\}\}$,

(D) *for all* $(a,d) \in \Gamma$, *the Fréchet derivative of* (12) *at every positive solution to* (12) *is invertible.*

Then for all $(a,d) \in \Gamma$, (12) *has a unique positive solution. Furthermore, there is an open set* W *in* $\mathbb{R}^2$ *such that* $\Gamma \subseteq W$ *and for every* $(a,d) \in W$, (12) *has a unique positive solution.*

Theorem 4.1 goes even further than Theorem 3.1 which states uniqueness in the whole region of (a,d) whenever we have uniqueness on the left boundary and invertibility of the linearized operator at any particular solution inside the domain.

Proof. For each fixed d, let $\lambda^d = \sup\{a\colon (a,d) \in \Gamma\}$ and $\lambda_d = \inf\{a|(a,d) \in \Gamma\}$. We need to show that for every a such that $\lambda_d \leqslant a \leqslant \lambda^d$, (12) has a unique positive solution. Since (12) with $(a,d) = (\lambda_d, d)$ has a unique positive solution (u,v) and the Fréchet derivative of (12) at (u,v) is invertible by Theorem 3.1, there is an open neighborhood V of (λ_d, d) in $\mathbb{R}^2$ such that if $(a_0, d_0) \in V$, then (12) with $(a,d) = (a_0, d_0)$ has a unique positive solution. Let $\lambda_s = \sup\{\lambda \geqslant \lambda_d\colon$ (12) has a unique coexistence state for $\lambda_d \leqslant a \leqslant \lambda\}$. We need to show that $\lambda_s \geqslant \lambda^d$. Suppose $\lambda_s < \lambda^d$. By the definition of λ_s, there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_s^-$ and a sequence (u_n, v_n) of the unique positive solutions of (12) with $(a,d) = (\lambda_n, d)$. Then by the elliptic theory, there is (u_0, v_0) such that (u_n, v_n) converges to (u_0, v_0) uniformly and (u_0, v_0) is the solution to (12) with $(a,d) = (\lambda_s, d)$. We claim that u_0 is not identically zero and v_0 is not identically zero. Suppose this

is false. Then by the Maximum Principle, one of the following cases should occur: (1) u_0 is identically zero and v_0 is not identically zero. (2) u_0 is not identically zero and v_0 is identically zero. (3) Both u_0 and v_0 are identically zero. The argument is similar to what we had in the previous section.

(1) Suppose u_0 is identically zero. Let $\tilde{u}_n = u_n/\|u_n\|_\infty$ and $\tilde{v}_n = v_n$ for all $n \in \mathbb{N}$. Then

$$\begin{cases} \Delta\tilde{u}_n + \tilde{u}_n(\lambda_n - g(u_n, \tilde{v}_n)) = 0, \\ \Delta\tilde{v}_n + \tilde{v}_n(d - h(u_n, \tilde{v}_n)) = 0 \qquad \text{in } \Omega. \end{cases}$$

We know $\tilde{u}_n \to \tilde{u}$ from the elliptic theory, and

$$\begin{cases} \Delta\tilde{u} + \tilde{u}(\lambda_s - g(0, v_0)) = 0, \\ \Delta v_0 + v_0(d - h(0, v_0)) = 0 \quad \text{in } \Omega, \end{cases}$$

since g, h are continuous. Hence, $v_0 = \theta_{d-h(0,\cdot)}$ and $\lambda_s = \lambda_1(g(0, v_0))$. If v_0 is identically zero, then by the monotonicity of g and λ_1 we have $\lambda_s = \lambda_1(g(0, v_0)) = \lambda_1(g(0, 0)) \leqslant \lambda_1(g(0, \theta_{d-h(0,\cdot)})) < \lambda_d$, which is impossible. If v_0 is not identically zero, then $v_0 = \theta_{d-h(0,\cdot)}$ and so $\lambda_s = \lambda_1(g(0, v_0)) = \lambda_1(g(0, \theta_{d-h(0,\cdot)})) < \lambda_d$, which is also impossible.

(2) Suppose v_0 is identically zero.

Let $\tilde{u}_n = u_n$ and $\tilde{v}_n = v_n/\|v_n\|_\infty$ for all $n \in \mathbb{N}$. Then

$$\begin{cases} \Delta\tilde{u}_n + \tilde{u}_n(\lambda_n - g(\tilde{u}_n, v_n)) = 0, \\ \Delta\tilde{v}_n + \tilde{v}_n(d - h(\tilde{u}_n, v_n)) = 0 \qquad \text{in } \Omega. \end{cases}$$

Again $\tilde{v}_n \to \tilde{v}$ by the elliptic theory, and

$$\begin{cases} \Delta u_0 + u_0(\lambda_s - g(u_0, 0)) = 0, \\ \Delta\tilde{v} + \tilde{v}(d - h(u_0, 0)) = 0 \qquad \text{in } \Omega \end{cases}$$

since g, h are continuous. Hence, $d = \lambda_1(h(u_0, 0))$. If u_0 is identically zero, then by the monotonicity of h and λ_1 we have $d = \lambda_1(h(u_0, 0)) = \lambda_1(h(0, 0)) \leqslant \lambda_1(h(\theta_{\lambda_s - g(\cdot,0)}, 0)) < \lambda_1(h(\theta_{\lambda^d - g(\cdot,0)}, 0))$, which is impossible since $(\lambda^d, d) \in \Gamma$. If u_0 is not identically zero, then $u_0 = \theta_{\lambda_s - g(\cdot,0)}$ and so $d = \lambda_1(h(u_0, 0)) = \lambda_1(h(\theta_{\lambda_s - g(\cdot,0)}, 0)) < \lambda_1(h(\theta_{\lambda^d - g(\cdot,0)}, 0))$, which is also impossible since $(\lambda^d, d) \in \Gamma$. Consequently, $u_0 > 0$, $v_0 > 0$ in Ω, that is, (u_0, v_0) is a coexistence of (12) with $(a, d) = (\lambda_s, d)$. Since $(\lambda_s, d) \in \Gamma$, by the assumption, the Fréchet derivative of (12) with $(a, d) = (\lambda_s, d)$ at (u_0, v_0) is invertible. Hence, by the Implicit Function Theorem, there is an open neighborhood U of λ_s and an open neighborhood V of (u_0, v_0) such that if $a \in U$, then (12) has a unique coexistence state in V. But, by the definition of λ_s, there is a sequence $\{\lambda'_n\} \subseteq U$ such that $\lambda'_n \to \lambda_s^+$ and there is

a sequence $\{(u_n', v_n')\}$ of coexistence states of (12) with $(a,d) = (\lambda_n', d)$ such that $(u_n', v_n') \notin V$ for all $n \in \mathbb{N}$. By the elliptic theory again, $u_n' \to u_0'$, $v_n' \to v_0'$ and by the same argument as above, $(u_0', v_0') \notin V$ is also a coexistence of (12) with $(a,d) = (\lambda_s, d)$. Since $(\lambda_s, d) \in \Gamma$, by the assumption again, the Fréchet derivative of (12) at (u_0', v_0') is invertible. Hence, by the Implicit Function Theorem again, there is an open neighborhood U' of λ_s and an open neighborhood V' of (u_0', v_0') such that if $a \in U'$, then (12) has a unique coexistence state in V'. Consequently, there are points on the left hand side of λ_s such that (12) has two different coexistence states. That is a contradiction to the definition of λ_s. Hence, $\lambda_s \geqslant \lambda^d$ and the first part of the theorem is proved. Furthermore, by the assumption, for each $(a,d) \in \Gamma$, the Fréchet derivative of (12) at the unique solution (u,v) is invertible. Hence, Theorem 3.1 concludes that there is an open neighborhood $V_{(a,d)}$ of (a,d) in $\mathbb{R}^2$ such that if $(a_0, d_0) \in V_{(a,d)}$, then (12) with reproduction rates (a_0, d_0) has a unique coexistence state. Let $W = \bigcup_{(a,d)\in\Gamma} V_{(a,d)}$. Then W is an open set in $\mathbb{R}^2$ such that $\Gamma \subseteq W$ and for each $(a_0, d_0) \in W$, (12) has a unique coexistence state.

Apparently, Theorem 4.1 generalizes Theorem 3.1 and consequently, we have the following result which is actually the main conclusion in this section.

Corollary 4.2. *Suppose*

(A) Γ *is a closed, convex region in* $\mathbb{R}^2$,

(B) *there exist* $k_1, k_2 > 0$ *such that for all* $(a,d) \in \Gamma$, $a > \lambda_1 + g(0,k_2)$, $d > \lambda_1 + h(k_1, 0)$, $a - g(k_1, 0) < 0$, $d - h(0, k_2) < 0$,

(C) $$4 \inf_B \frac{\partial g(x,y)}{\partial x} \inf_B \frac{\partial h(x,y)}{\partial y} > \Big[\sup \frac{\partial g(x,y)}{\partial y} + \sup \frac{\partial h(x,y)}{\partial x} \sup_{(a,d)\in\Gamma} \frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}}\Big] \times \Big[\sup \frac{\partial g(x,y)}{\partial y} \sup_{(a,d)\in\Gamma} \frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}} + \sup \frac{\partial h(x,y)}{\partial x}\Big],$$

where $B = [0,k_1] \times [0,k_2]$.

Then there is an open set W *in* $\mathbb{R}^2$ *such that* $\Gamma \subseteq W$ *and for every* $(a,d) \in \Gamma$, (12) *has a unique positive solution.*

The condition (B) means Γ is a set of large self-reproduction rates, and the condition (C) implies that the self-limitation rates are relatively larger than competition rates. Then the conclusion says that the existence and uniqueness of a coexistence state are guaranteed on Γ and the region Γ can be extended to a larger set without losing the uniqueness.

Proof. From $\theta_{a-g(\cdot,0)} < k_1$, $\theta_{d-h(0,\cdot)} < k_2$ and the monotonicity of $g(0,\cdot)$, $h(\cdot,0)$ we have

$$\begin{cases} a > \lambda_1 + g(0,k_2) \geqslant \lambda_1(g(0, \theta_{d-h(0,\cdot)})), \\ d > \lambda_1 + h(k_1, 0) \geqslant \lambda_1(h(\theta_{a-g(\cdot,0)}, 0)) \end{cases}$$

for all $(a,d) \in \Gamma$.

By the condition (C), for every $(a,d)\in\partial\Gamma$,

$$\begin{aligned}
4\inf_B \frac{\partial g(x,y)}{\partial x}\inf_B \frac{\partial h(x,y)}{\partial y} &> \Big[\sup\frac{\partial g(x,y)}{\partial y}+\sup\frac{\partial h(x,y)}{\partial x}\sup\frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(0,k_2)}}\Big]\\
&\quad\times\Big[\sup\frac{\partial g(x,y)}{\partial y}\sup\frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}}+\sup\frac{\partial h(x,y)}{\partial x}\Big]\\
&=\Big[\sup\frac{\partial g(x,y)}{\partial y}\Big]^2\sup\frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}}+\sup\frac{\partial g(x,y)}{\partial y}\sup\frac{\partial h(x,y)}{\partial x}\\
&\quad+\sup\frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}}\sup\frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}}\sup\frac{\partial g(x,y)}{\partial y}\sup\frac{\partial h(x,y)}{\partial x}\\
&\quad+\Big[\sup\frac{\partial h(x,y)}{\partial x}\Big]^2\sup\frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}}\\
&\geqslant\Big[\sup\frac{\partial g(x,y)}{\partial y}\Big]^2\sup\frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}}+2\sup\frac{\partial g(x,y)}{\partial y}\sup\frac{\partial h(x,y)}{\partial x}\\
&\quad+\Big[\sup\frac{\partial h(x,y)}{\partial x}\Big]^2\sup\frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}}
\end{aligned}$$

since $\theta_{a-g(\cdot,0)}>\theta_{a-g(\cdot,k_2)}$, $\theta_{d-h(0,\cdot)}>\theta_{d-h(k_1,\cdot)}$. Therefore, by Theorem 2.1, (12) has a unique coexistence state for all $(a,d)\in\partial\Gamma$. Furthermore, by the estimate of the solution in the proof of Theorem 2.1, if (u,v) is a positive solution for $(a,d)\in\Gamma$, then

$$\begin{aligned}
4\inf_B \frac{\partial g(x,y)}{\partial x}\inf_B \frac{\partial h(x,y)}{\partial y} &> \Big[\sup\frac{\partial g(x,y)}{\partial y}+\sup\frac{\partial h(x,y)}{\partial x}\sup\frac{\theta_{d-h(0,\cdot)}}{\theta_{a-g(\cdot,k_2)}}\Big]\\
&\quad\times\Big[\sup\frac{\partial g(x,y)}{\partial y}\sup\frac{\theta_{a-g(\cdot,0)}}{\theta_{d-h(k_1,\cdot)}}+\sup\frac{\partial h(x,y)}{\partial x}\Big]\\
&\geqslant\Big[\sup\frac{\partial g(x,y)}{\partial y}+\sup\frac{\partial h(x,y)}{\partial x}\frac{v}{u}\Big][\sup\frac{\partial g(x,y)}{\partial y}\frac{u}{v}+\sup\frac{\partial h(x,y)}{\partial x}\Big].
\end{aligned}$$

Thus, we obtain

$$4\inf_B \frac{\partial g(x,y)}{\partial x}\inf_B \frac{\partial h(x,y)}{\partial y}uv>[\sup\frac{\partial g(x,y)}{\partial y}u+\sup\frac{\partial h(x,y)}{\partial x}v]^2.$$

This implies that if (u,v) is a positive solution of (12) for $(a,d)\in\Gamma$, then the Fréchet derivative of (12) at (u,v) is invertible by Theorem 3.3. Therefore, the theorem follows from Theorem 4.1.

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