

Shariefuddin Pirzada; T. A. Naikoo; F. A. Dar
Signed degree sets in signed graphs

Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 3, 843–848

Persistent URL: <http://dml.cz/dmlcz/128210>

Terms of use:

© Institute of Mathematics AS CR, 2007

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

SIGNED DEGREE SETS IN SIGNED GRAPHS

S. PIRZADA, T. A. NAIKOO and F. A. DAR, Srinagar

(Received June 1, 2005)

Abstract. The set D of distinct signed degrees of the vertices in a signed graph G is called its signed degree set. In this paper, we prove that every non-empty set of positive (negative) integers is the signed degree set of some connected signed graph and determine the smallest possible order for such a signed graph. We also prove that every non-empty set of integers is the signed degree set of some connected signed graph.

Keywords: signed graphs

MSC 2000: 05C20

1. INTRODUCTION

All graphs in this paper are finite, undirected, without loops and multiple edges. A signed graph G is a graph in which each edge is assigned a positive or a negative sign. These were first introduced by Harary [3]. The signed degree of a vertex v_i in a signed graph G is denoted by $\text{sdeg}(v_i)$ (or simply by d_i) and is defined as the number of positive edges incident with v_i less the number of negative edges incident with v_i . So, if v_i is incident with d_i^+ positive edges and d_i^- negative edges, then $\text{sdeg}(v_i) = d_i^+ - d_i^-$. A signed degree sequence $\sigma = [d_1, d_2, \dots, d_n]$ of a signed graph G is formed by listing the vertex signed degrees in non-increasing order. A sequence $\sigma = [d_1, d_2, \dots, d_n]$ of integers is graphical if σ is a signed degree sequence of some signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \dots, d_n]$ is a standard sequence if σ is non-increasing, $\sum_{i=1}^n d_i$ is even, $d_1 > 0$, each $|d_i| < n$, and $|d_1| \geq |d_n|$.

The following result, due to Chartrand et al. [1], gives a necessary and sufficient condition for a sequence of integers to be graphical, which is similar to Hakimi's result for degree sequences [2].

Theorem 1.1. Let $\sigma = [d_1, d_2, \dots, d_n]$ be a standard sequence. Then, σ is graphical if and only if there exist integers r and s with $d_1 = r - s$ and $0 \leq s \leq \frac{1}{2}(n - 1 - d_1)$ such that

$$\sigma' = [d_2 - 1, d_3 - 1, \dots, d_{r+1} - 1, d_{r+2}, d_{r+3}, \dots, d_{n-s}, d_{n-s+1} + 1, \dots, d_n + 1]$$

is graphical.

The next characterization for signed degrees in signed graphs is given by Yan et al. [5].

Theorem 1.2. A standard integral sequence $\sigma = [d_1, d_2, \dots, d_n]$ is graphical if and only if

$$\sigma'_m = [d_2 - 1, \dots, d_{d_1+m+1} - 1, d_{d_1+m+2}, \dots, d_{n-m}, d_{n-m+1} + 1, \dots, d_n + 1]$$

is graphical, where m is the maximum non-negative integer such that $d_{d_1+m+1} > d_{n-m+1}$.

In [4], Kapoor et al. proved that every non-empty set of distinct positive integers is the degree set of a connected graph and determined the smallest order for such a graph.

2. MAIN RESULTS

First we have the following definition.

Definition. The set D of distinct signed degrees of the vertices in a signed graph G is called its signed degree set.

Now, we obtain the following results.

Theorem 2.1. Every non-empty set D of positive integers is the signed degree set of some connected signed graph and the minimum order of such a signed graph is $N + 1$, where N is the maximum integer in the set D .

Proof. Let D be a signed degree set and $n_0(D)$ denotes the minimum order of a signed graph G realizing D . Since N is the maximum integer in D , therefore there is a vertex in G which is adjacent to at least N other vertices with a positive sign. Then, $n_0(D) \geq N + 1$. Now, if there exists a signed graph of order $N + 1$ with D as signed degree set, then $n_0(D) = N + 1$. The existence of such a signed graph is obtained by using induction on the number of elements of D .

Let $D = \{d_1, d_2, \dots, d_n\}$, where $d_1 < d_2 < \dots < d_n$, be a set of positive integers. For $n = 1$, let G be a complete graph on $d_1 + 1$ vertices, that is K_{d_1+1} , in which each edge is assigned a positive sign. Then,

$$\text{sdeg}(v) = (d_1 + 1 - 1) - 0 = d_1, \quad \text{for all } v \in V(G).$$

Therefore, G is a signed graph with signed degree set $D = \{d_1\}$.

For $n = 2$, let G_1 be a complete graph on d_1 vertices, that is K_{d_1} , in which each edge is assigned a positive sign and let G_2 be a null graph on $d_2 - d_1 + 1 > 0$ vertices, that is $\overline{K}_{d_2-d_1+1}$. Join every vertex of G_1 to each vertex of G_2 with a positive edge, so that we obtain a signed graph G on $d_1 + d_2 - d_1 + 1 = d_2 + 1$ vertices with

$$\text{sdeg}(u) = (d_1 - 1) + (d_2 - d_1 + 1) - 0 = d_2, \quad \text{for all } u \in V(G_1),$$

and

$$\text{sdeg}(v) = (0) + (d_1) - 0 = d_1, \quad \text{for all } v \in V(G_2).$$

Therefore, the signed degree set of G is $D = \{d_1, d_2\}$.

For $n = 3$, let G_1 be a complete graph on d_1 vertices, that is K_{d_1} , in which each edge is assigned a positive sign, G_2 be a complete graph on $d_2 - d_1 + 1 > 0$ vertices, that is $\overline{K}_{d_2-d_1+1}$, in which each edge is assigned a positive sign, and G_3 be a null graph on $d_3 - d_2 > 0$ vertices, that is $\overline{K}_{d_3-d_2}$. Join every vertex of G_1 to each vertex of G_2 with a positive edge and join every vertex of G_1 to each vertex of G_3 with a positive edge, so that we obtain a signed graph G on $d_1 + d_2 - d_1 + 1 + d_3 - d_2 = d_3 + 1$ vertices with

$$\text{sdeg}(u) = (d_1 - 1) + (d_2 - d_1 + 1) + (d_3 - d_2) - 0 = d_3, \quad \text{for all } u \in V(G_1),$$

$$\text{sdeg}(v) = (d_2 - d_1 + 1 - 1) + (d_1) - 0 = d_2, \quad \text{for all } v \in V(G_2),$$

and

$$\text{sdeg}(w) = (0) + (d_1) - 0 = d_1, \quad \text{for all } w \in V(G_3).$$

Therefore, the signed degree set of G is $D = \{d_1, d_2, d_3\}$.

Assume that the result holds for k . We show that the result is true for $k + 1$.

Let $D = \{d_1, d_2, \dots, d_k, d_{k+1}\}$ be a set of $k + 1$ positive integers with $d_1 < d_2 < \dots < d_k < d_{k+1}$. Clearly, $0 < d_2 - d_1 < d_3 - d_1 < \dots < d_k - d_1$. Therefore, by induction hypothesis, there is a signed graph G_1 realizing the signed degree set $D_1 = \{d_2 - d_1, d_3 - d_1, \dots, d_k - d_1\}$ on $d_k - d_1 + 1$ vertices as $|V(D_1)| < k$. Let G_2 be a complete graph on d_1 vertices, that is K_{d_1} , in which each edge is assigned a positive sign and G_3 be a null graph on $d_{k+1} - d_k > 0$ vertices, that is $\overline{K}_{d_{k+1}-d_k}$.

Join every vertex of G_2 to each vertex of G_1 with a positive edge and join every vertex of G_2 to each vertex of G_3 with a positive edge, so that we obtain a signed graph G on $d_k - d_1 + 1 + d_1 + d_{k+1} - d_k = d_{k+1} + 1$ vertices with

$$\begin{aligned} \text{sdeg}(u) &= (d_i - d_1) + (d_1) - 0 = d_i, \quad \text{for all } u \in V(G_1) \text{ where } 2 \leq i \leq k, \\ \text{sdeg}(v) &= (d_1 - 1) + (d_k - d_1 + 1) + (d_{k+1} - d_k) - 0 = d_{k+1}, \quad \text{for all } v \in V(G_2), \end{aligned}$$

and

$$\text{sdeg}(w) = (0) + (d_1) - 0 = d_1, \quad \text{for all } w \in V(G_3).$$

Therefore, the signed degree set of G is $D = \{d_1, d_2, \dots, d_k, d_{k+1}\}$. Clearly, by construction, all the signed graphs are connected. Hence, the result follows.

Theorem 2.2. *Every non-empty set D of negative integers is the signed degree set of some connected signed graph and the minimum order of such a graph is $|M|+1$, where M is the minimum integer in the set D .*

Proof. Let D be a signed degree set and let $m_0(D)$ denote the minimum order of a signed graph G realizing D . Since $|M|$ is the maximum integer in D , therefore there is a vertex in G which is adjacent to at least $|M|$ other vertices with a negative sign. Then, $m_0(D) \geq |M| + 1$. Now, if there exists a signed graph of order $|M| + 1$ with D as signed degree set, then $m_0(D) = |M| + 1$.

Let $D = \{-d_1, -d_2, \dots, -d_n\}$, $-d_1 > -d_2 > \dots > -d_n$, be a set of negative integers where d_1, d_2, \dots, d_n are positive integers. Now, $D_1 = \{d_1, d_2, \dots, d_n\}$ is a set of positive integers with $d_1 < d_2 < \dots < d_n$. By Theorem 2.1, there exists a connected signed graph G_1 on $d_n + 1 = |-d_n| + 1$ vertices with signed degree set $D_1 = \{d_1, d_2, \dots, d_n\}$. Now, construct a signed graph G from G_1 by interchanging positive edges with negative edges. Then, G is a connected signed graph on $|-d_n| + 1$ vertices with signed degree set $D = \{-d_1, -d_2, \dots, -d_n\}$. This proves the result.

Theorem 2.3. *Every non-empty set D of integers is the signed degree set of some connected signed graph.*

Proof. Let D be a set of n integers. We have the following cases.

Case I. D is a set of positive (negative) integers. Then, the result follows by Theorem 2.1 (Theorem 2.2).

Case II. $D = \{0\}$. Then, a null graph G on one vertex, that is K_1 , has signed degree set $D = \{0\}$.

Case III. D is a set of non-negative (non-positive) integers. Let $D = D_1 \cup \{0\}$, where D_1 is a set of positive (negative) integers. Then, by Theorem 2.1 (Theorem 2.2), there is a signed graph G_1 with signed degree set D_1 . Let G_2 be a null

graph on two vertices, that is \overline{K}_2 . Let $e = uv$ be an edge in G_1 with positive (negative) sign and let $x, y \in V(G_2)$. Add the positive (negative) edges ux and vy , and the negative (positive) edges uy and vx , so that we obtain a connected signed graph G with signed degree set D . We note that addition of such edges do not effect the signed degrees of the vertices of G_1 , and the vertices x and y have signed degrees zero each.

Case IV. D is a set of non-zero integers. Let $D = D_1 \cup D_2$, where D_1 is a set of positive integers and D_2 is a set of negative integers. Then, by Theorem 2.1 and Theorem 2.2, there are connected signed graphs G_1 and G_2 with signed degree sets D_1 and D_2 . Let $e_1 = uv$ be an edge in G_1 with positive sign and $e_2 = xy$ be an edge in G_2 with negative sign. Add the positive edges ux and vy , and the negative edges uy and vx , so that we obtain a connected signed graph G with signed degree set D . We note that addition of such edges do not effect the signed degrees of the vertices of G_1 and G_2 .



Case V. D is a set of integers. Let $D = D_1 \cup D_2 \cup \{0\}$, where D_1 and D_2 are the sets of positive and negative integers respectively. Then, by Theorem 2.1 and Theorem 2.2, there are connected signed graphs G_1 and G_2 with signed degree sets D_1 and D_2 . Let G_3 be a null graph on one vertex, that is K_1 . Let $e_1 = uv$ be an edge in G_1 with positive sign, and let $x \in V(G_2)$ and $y \in V(G_3)$. Add the positive edges uy and vx , and the negative edges ux and vy , so that we obtain a connected signed graph G with signed degree set D . We note that addition of such edges do not effect the signed degrees of the vertices of G_1 and G_2 , and the vertex y has signed degree zero. This completes the proof. \square

Theorem 2.4. *If G is a signed graph with vertex set V , where $|V| = r$, and signed degree set $\{d_1, d_2, \dots, d_n\}$. Then, for each $k \geq 1$, there is a signed graph with kr vertices and signed degree set $\{d_1, d_2, \dots, d_n\}$.*

Proof. For each $i, 1 \leq i \leq k$, let G_i be a copy of G with vertex set V_i . Define a signed graph H with vertex set $W = \bigcup_{i=1}^k V_i$ where $V_i \cap V_j = \emptyset$ ($i \neq j$) and the edges of H are the edges of G_i for all i , where $1 \leq i \leq k$. Then, H is a signed graph on kr vertices with signed degree set $\{d_1, d_2, \dots, d_n\}$. \square

References

- [1] *G. Chartrand, H. Gavlas, F. Harary and M. Schultz:* On signed degrees in signed graphs. Czech. Math. J. 44 (1994), 677–690. zbl
- [2] *S. L. Hakimi:* On the realizability of a set of integers as degrees of the vertices of a graph. SIAM J. Appl. Math. 10 (1962), 496–506. zbl
- [3] *F. Harary:* On the notion of balance in a signed graph. Michigan Math. J. 2 (1953), 143–146. zbl

- [4] *S. F. Kapoor, A. O. Polimeni and C. E. Wall*: Degree sets for graphs. *Fund. Math.* 65 (1977), 189–194. 
- [5] *J. H. Yan, K. W. Lih, D. Kuo and G. J. Chang*: Signed degree sequences of signed graphs. *J. Graph Theory* 26 (1997), 111–117. 

Authors' address: S. Pirzada, T. A. Naikoo, F. A. Dar, Department of Mathematics, University of Kashmir, Srinagar-190006, India, e-mail: sdpirzada@yahoo.co.in, tariqnaikoo@rediffmail.com, sfarooqdar@yahoo.co.in.