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MINIMAL SUBMANIFOLDS IN $\mathbb{R}^4$ WITH A G.C.K. STRUCTURE

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Abstract. In this paper we obtain all invariant, anti-invariant and CR submanifolds in $(\mathbb{R}^4, g, J)$ endowed with a globally conformal Kähler structure which are minimal and tangent or normal to the Lee vector field of the g.c.K. structure.

Keywords: locally conformal Kähler structure, minimal submanifolds, invariant submanifolds, totally real submanifolds, CR-submanifolds

MSC 2000: 53B25, 53B35, 53C21, 53C42

1. Preliminaries

Although known since 1954 from P. Libermann's paper [5], locally conformal Kähler (l.c.K.) structures have been intensively studied only since 1976 after the impetus given by I. Vaisman in [6]. A great number of research papers has appeared since then studying the main properties of l.c.K. manifolds, generalized Hopf (g.H.) manifolds, the relations with contact metric manifolds and some important classifications of submanifolds in g.H. manifolds. In 1998, the monograph by S. Dragomir and L. Ornea [2] brought together all known results in this field at that moment. After the book, the geometers continued to study l.c.K. manifolds and many other interesting results have appeared so far.

Let (M, J, g) be a Hermitian manifold of dimension $2n$, where J denotes the complex structure and g the Hermitian metric. Then (M, J, g) is a *locally conformal Kähler* manifold if there is an open cover $\{U_i\}_{i\in I}$ of M and a family $\{f_i\}_{i\in I}$ of smooth functions $f_i\colon U_i \longrightarrow \mathbb{R}$ such that each local metric $g_i = \exp(-f_i)g|_{U_i}$ is Kählerian. Also, (M, J, g) is a *globally conformal Kähler* (g.c.K.) manifold if there is

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a smooth function $f\colon M \longrightarrow \mathbb{R}$ such that the metric $\exp(f)g$ is Kählerian. Let Ω be the Kähler 2-form associated with (J, g) (i.e. $\Omega(X,Y) = g(X, JY)$ for $X, Y \in \chi(M)$). Then the Hermitian manifold (M, J, g) is l.c.K. if and only if there exists a closed 1-form ω, globally defined on M, such that

$$d\Omega = \omega \wedge \Omega$$

(see [2] for more details). The closed 1-form ω is called the *Lee form* of the l.c.K. manifold M. Also, (M, J, g) is g.c.K. or Kähler if the Lee form ω is exact or $\omega = 0$, respectively. Thus any simply connected l.c.K. manifold is g.c.K.

For an l.c.K. manifold (M, J, g) we define the *Lee vector field* $B = \omega^{\#}$. Here $\#$ denotes the rising of indices with respect to g, namely $g(X, B) = \omega(X)$ for all $X \in \chi(M)$. It is very important that the Levi-Civita connections D^i of the local metrics $\{g_i\}_{i \in I}$ glue up to a globally defined torsion free linear connection D on M, called the *Weyl connection* of the l.c.K. manifold M and given by

$$D_X Y = \nabla_X Y - \frac{1}{2}\big(\omega(X)Y + \omega(Y)X - g(X,Y)B\big) \tag{1.1}$$

for any $X, Y \in \chi(M)$, where ∇ is the Levi-Civita connection of (M, g). Moreover, D satisfies $Dg = \omega \otimes g$ and $DJ = 0$. As a consequence, the Hermitian manifold (M, J, g) is l.c.K. if and only if

$$\nabla_X JY = J\nabla_X Y + \frac{1}{2}\big(\theta(Y)X - \omega(Y)JX - g(X,Y)A - \Omega(X,Y)B\big) \tag{1.2}$$

for any $X, Y \in \chi(M)$. Here $\theta = \omega \circ J$ and $A = -JB$ are the *anti-Lee form* and the *anti-Lee vector field*, respectively (see [2] for details).

2. 4-dimensional l.c.K. manifolds

In [4] a Hermitian structure on $\mathbb{R}^4$ is defined whose scalar curvature is constant and negative, but $\mathbb{R}^4$ with this Riemannian metric is not of constant curvature.

Let us consider on $\mathbb{R}^4$ global coordinates x, y, v, w and the Riemannian metric g whose matrix with respect to these coordinates is

$$g = \begin{pmatrix} 1 & 0 & -kx & 0 \\ 0 & \alpha & ky\alpha & kx \\ -kx & ky\alpha & \alpha\beta & k^2xy \\ 0 & kx & k^2xy & 1 \end{pmatrix}$$

where $\alpha = 1 + k^2x^2$ and $\beta = 1 + k^2y^2$. Note that for $k = 0$ one gets the 4-Euclidean space. If $k \neq 0$, then $\mathbb{R}^4(k)$ is of negative constant scalar curvature $-\frac{5}{2}k^2$.

Consider on $\mathbb{R}^4$ the $(1,1)$ tensor field J putting

$$J\frac{\partial}{\partial x}=\frac{\partial}{\partial w},\quad J\frac{\partial}{\partial y}=-ky\frac{\partial}{\partial y}+\frac{\partial}{\partial v},\quad J\frac{\partial}{\partial v}=-\beta\frac{\partial}{\partial y}+ky\frac{\partial}{\partial v},\quad J\frac{\partial}{\partial w}=-\frac{\partial}{\partial x}.$$

It is easy to prove that J is integrable and $g(JX,JY)=g(X,Y)$. So, $(\mathbb{R}^4,g,J)$ is a Hermitian manifold.

In the following consider the 1-form $\omega=k\,\mathrm{d}v$ and the Kähler 2-form Ω. One can easily check that the relation $d\Omega=\omega\wedge\Omega$ holds. Hence, $(\mathbb{R}^4,g,J)$ is a locally conformal Kähler manifold. Moreover, since $\mathbb{R}^4$ is simply connected, we get a globally conformal Kähler manifold with the Kähler metric $\tilde{g}=c\mathrm{e}^{-kv}g$, where c is a positive real number.

The Lee vector field is given explicitly as $B=k^2x\,\partial/\partial x-k^2y\,\partial/\partial y+k\,\partial/\partial v$. We immediately have the anti-Lee form $\theta=k\,dy+k^2y\,\mathrm{d}v$ and the anti-Lee vector field $A=k\,\partial/\partial y-k^2x\,\partial/\partial w$.

Let us remark that B is parallel if and only if M is the Euclidean space $(k=0)$. Yet, $\nabla_B B=0$ and $\nabla_B A=0$. Now, using (1.1), we obtain the Weyl connection D on $\mathbb{R}^4(k)$ given by

$$\begin{aligned}
D_{\partial/\partial x}\frac{\partial}{\partial x}&=-\frac{k^2x}{2}\frac{\partial}{\partial x}+\frac{k^2y}{2}\frac{\partial}{\partial y}-\frac{k}{2}\frac{\partial}{\partial v},\\
D_{\partial/\partial x}\frac{\partial}{\partial y}&=\frac{k^2x}{2}\frac{\partial}{\partial y}+\frac{k\gamma}{2}\frac{\partial}{\partial w},\\
D_{\partial/\partial x}\frac{\partial}{\partial v}&=-\frac{k\gamma}{2}\frac{\partial}{\partial x}+\frac{k^2x}{2}\frac{\partial}{\partial v}+\frac{k^2y\gamma}{2}\frac{\partial}{\partial w},\\
D_{\partial/\partial x}\frac{\partial}{\partial w}&=\frac{k}{2}\frac{\partial}{\partial y}-\frac{k^2x}{2}\frac{\partial}{\partial w},\\
D_{\partial/\partial y}\frac{\partial}{\partial y}&=\frac{k^2x\alpha}{2}\frac{\partial}{\partial x}-\frac{k^2y(2+\alpha)}{2}\frac{\partial}{\partial y}+\frac{k(2+\alpha)}{2}\frac{\partial}{\partial v},\\
D_{\partial/\partial y}\frac{\partial}{\partial v}&=\frac{k^3xy\alpha}{2}-\frac{k(\alpha\beta+2k^2y^2)}{2}\frac{\partial}{\partial y}+\frac{k^2y(2+\alpha)}{2}\frac{\partial}{\partial v}+\frac{k^2x\alpha}{2}\frac{\partial}{\partial w},\\
D_{\partial/\partial y}\frac{\partial}{\partial w}&=-\frac{k\gamma}{2}\frac{\partial}{\partial x}-\frac{k^3xy}{2}\frac{\partial}{\partial y}+\frac{k^2x}{2}\frac{\partial}{\partial v},\\
D_{\partial/\partial v}\frac{\partial}{\partial v}&=-\frac{k^2x\alpha\varrho}{2}\frac{\partial}{\partial x}-\frac{k^2y\beta(2+\alpha)}{2}\frac{\partial}{\partial y}+\frac{k[(\alpha+2)(\beta-2)+2]}{2}\frac{\partial}{\partial v},\\
D_{\partial/\partial v}\frac{\partial}{\partial w}&=-\frac{k^2y\gamma}{2}\frac{\partial}{\partial x}-\frac{k^2x\beta}{2}\frac{\partial}{\partial y}+\frac{k^3xy}{2}\frac{\partial}{\partial v}-\frac{k\gamma}{2}\frac{\partial}{\partial w},\\
D_{\partial/\partial w}\frac{\partial}{\partial w}&=\frac{k^2x}{2}\frac{\partial}{\partial x}-\frac{k^2y}{2}\frac{\partial}{\partial y}+\frac{k}{2}\frac{\partial}{\partial v}\quad(=\tfrac{1}{2}B),
\end{aligned}$$

where $\gamma=1-k^2x^2$ and $\varrho=1-k^2y^2$.

3. Invariant and totally real minimal submanifolds of $(\mathbb{R}^4, g, J)$

A proper *invariant* submanifold M in $(\mathbb{R}^4, g, J)$ is a 2-dimensional submanifold in $\mathbb{R}^4$ such that for all X tangent to M, JX is also tangent to M.

Theorem 1. *A proper invariant submanifold M in $(\mathbb{R}^4, g, J)$ is minimal if and only if it is given by the following implicit equations:*

$$\begin{cases} f_1(x,y,v,w) = w + kxy + C_1 = 0, \\ f_2(x,y,v,w) = x - C_2\mathrm{e}^{kv} = 0 \end{cases} \tag{3.3}$$

where C_1 and C_2 are real constants.

P r o o f. Let X, $Y = JX$ be unitary tangent vector fields on M. If h denotes the second fundamental form and H is the mean curvature, then $H = \frac{1}{2}\big(h(X,X) + h(JX,JX)\big)$ or, more precisely,

$$H = \tfrac{1}{2}\big(\mathrm{nor}(\nabla_X X) + \mathrm{nor}(\nabla_{JX} JX)\big)$$

where $\mathrm{nor}(V)$ means the normal part of the vector field V. By using (1.2) we get

$$\mathrm{nor}(\nabla_{JX} JX) = \mathrm{nor}(J\nabla_{JX} X) - \tfrac{1}{2}\,\mathrm{nor}(B) = -\,\mathrm{nor}(\nabla_X X) - \mathrm{nor}(B).$$

Thus, $H = -\frac{1}{2}\,\mathrm{nor}(B)$ and consequently M is a minimal invariant submanifold if and only if the Lee and anti-Lee vector fields are tangent to M (see also [3]).

Consider M given by $f_j(x,y,v,w) = 0$, $j = 1,2$, where $f_{1,2}$ are smooth functions on $\mathbb{R}^4$ verifying $\mathrm{rank}(D(f_1,f_2)/D(x,y,v,w)) = 2$ at every point of M. Since A and B are tangent to M, then they belong to $\ker(df_j)$, $j = 1,2$.

First, if $\partial f_1/\partial w$ and $\partial f_2/\partial w$ are both different from 0, then, by using the implicit function theorem, we can consider $f_j = w - F_j(x,y,v)$ with $F_j \in C^\infty(\mathbb{R}^3)$, $j = 1,2$. If $F = F_1 - F_2$ one obtains that M is given by $f_1 = 0$ and $F = 0$ which is false (in this case we have supposed that both functions depend on w).

Secondly, if $\partial f_j/\partial w = 0$ for $j = 1,2$, then $\partial f_j/\partial y = 0$ and $kx(\partial f_j/\partial x) - \partial f_j/\partial v = 0$, $j = 1,2$. It is obvious that $\partial f_j/\partial x \neq 0$ for $j = 1,2$ and consequently, by virtue of the implicit function theorem, we can conclude that $f_j = x - F_j(v)$, $j = 1,2$. This contradicts the fact that the Jacobian has rank 2 on M.

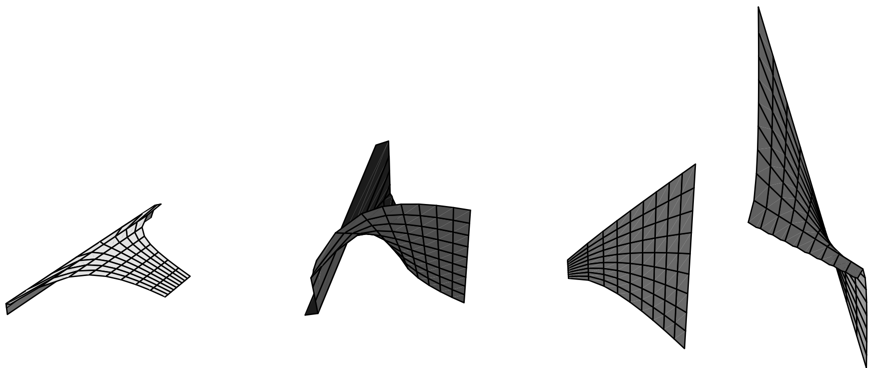
This means that we can suppose, without loss of the generality, that $\partial f_1/\partial w \neq 0$ and $\partial f_2/\partial w = 0$. Thus, as above, we can take $f_1 = w - F_1(x,y,v)$ and $f_2 = x - F_2(v)$ with F_1 and F_2 smooth functions satisfying

$$\frac{\partial F_1}{\partial y} + kF_2 = 0 \quad \text{and} \quad F_2' - kF_2 = 0 \quad \text{along the manifold.}$$

From here, we get the assertion. □

Remark 2. We want to know how this submanifold looks like. To do this we project it on the four coordinate planes. One obtains (when the two constants are equal to 1):

on $(x, y, v, 0)$: $y = -t/x$, $v = \ln x/k$ $(x > 0,\ t \in \mathbb{R})$,
on $(x, y, 0, w)$: $x = t$, $y = -w/kt$ $(t > 0,\ w \in \mathbb{R})$,
on $(x, 0, v, w)$: $v = \ln x/k$, $w = tx$ $(x > 0\ t \in \mathbb{R})$,
on $(0, y, v, w)$: $v = \ln t/k$, $w = -kty$ $(t > 0,\ y \in \mathbb{R})$.



A *totally real* submanifold M in $(\mathbb{R}^4, g, J)$ is a (q-dimensional) submanifold such that for all X tangent to M, JX is normal to M. It follows that q must be 1 or 2. It is obvious that any curve is a totally real submanifold, so we are interested in 2-dimensional totally real submanifolds in $\mathbb{R}^4$.

We will study only two cases, namely

a) B is normal to M and thus A is tangent;

b) B is tangent to M and thus A is a normal vector field.

Case a) Let $E_1 = \partial_y - kx\partial_w$ be the normalized vector of A. We are looking now for E_2 unitary, orthogonal to E_1 and tangent to M. From the relations $g(E_1, E_2) = 0$, $g(E_2, B) = 0$ and $g(E_2, E_2) = 1$ it follows that E_2 has the form $E_2 = \cos\psi\partial_x + \sin\psi\partial_w$, where ψ is a differentiable function on M. In order for a submanifold M tangent to E_1 and E_2 to exist, it is necessary to have the involutivity condition $[E_1, E_2] \in \operatorname{span}\{E_1, E_2\}$. We have

$$[E_1, E_2] = -\sin\psi(\psi_y - kx\psi_w)\partial_x + \cos\psi(\psi_y - kx\psi_w + k)\partial_w$$

(along M). Obviously it is possible to have only $[E_1, E_2] \| E_2$, which yields the following PDE:

$$\psi_y - kx\psi_w = -k\cos^2\psi \quad (\text{on } M). \tag{3.4}$$

Let us remark that $\psi = $ constant implies $\cos\psi = 0$.

Consider now that M is given by

$$M\colon \begin{cases} f_1(x,y,v,w)=0, \\ f_2(x,y,v,w)=0 \end{cases}$$

with f_1, f_2 smooth functions on $\mathbb{R}^4$ satisfying $\operatorname{rank} D(f_1,f_2)/D(x,y,v,w)=2$ at any point of M. Since E_1, E_2 are tangent to M we have

$$\text{(3.5)} \qquad \begin{cases} \dfrac{\partial f_i}{\partial y} - kx\dfrac{\partial f_i}{\partial w} = 0, \\ \cos\psi \, \dfrac{\partial f_i}{\partial x} + \sin\psi \, \dfrac{\partial f_i}{\partial w} = 0, \quad i=1,2 \ \text{(on } M). \end{cases}$$

Subcase I. If $\partial f_2/\partial w = 0$, then $\partial f_2/\partial y = 0$ and $\cos\psi\,\partial f_2/\partial x = 0$.

I.1. If $\cos\psi = 0$, we get $\partial f_1/\partial w = 0$. It follows that $\partial f_1/\partial y = 0$. Thus $f_1 = f_1(x,v)$ and $f_2 = f_2(x,v)$ with $\operatorname{Jac}(f_1,f_2) \neq 0$. Consequently, by virtue of the implicit function theorem, M is given by $x = \text{constant}$ and $v = \text{constant}$, that is, M is a portion of a 2-plane.

I.2. If $\cos\psi \neq 0$, it follows that $\partial f_2/\partial x = 0$ and hence $f_2 = f_2(v)$. Thus, the second equation can be replaced (by using the same argument) by $v = \text{constant}$. The Jacobian has rank 2, hence $(\partial f_1/\partial x)^2 + (\partial f_1/\partial y)^2 + (\partial f_1/\partial w)^2 \neq 0$. If $\partial f_1/\partial w = 0$, we get $\partial f_1/\partial y = 0$ and $\partial f_1/\partial x = 0$, which is impossible. Consequently, $\partial f_1/\partial w \neq 0$. From the implicit function theorem we can get $w = F(x,y)$ since v is constant. The PD equation (3.5) becomes

$$\text{(3.6)} \qquad \begin{cases} -F_y - kx = 0, \\ -\cos\psi F_x + \sin\psi = 0 \quad \text{along } M. \end{cases}$$

We obtain $F(x,y) = -kxy + \mathbf{q}(x)$, $\mathbf{q} \in C^\infty(I)$, $I \subset \mathbb{R}$ and $\psi = \psi(x,y) = \arctan(-ky + \mathbf{q}'(x))$; obviously ψ satisfies (3.4).

Finally, the submanifold M is given by $v = \text{constant}$ and $w = -kxy + \mathbf{q}(x)$.

Subcase II. Suppose $\partial f_i/\partial w \neq 0$ for $i = 1,2$. We can express M by

$$\begin{cases} w = F_1(x,y,v), \\ w = F_2(x,y,v). \end{cases}$$

After some computations, this subcase reduces to the previous one.

We can state

Theorem 3. *Let M be a 2-dimensional totally real submanifold in $(\mathbb{R}^4, g, J)$ normal to the Lee vector field B. Then, either*

A: *M is a portion of a 2-plane (namely $x =$ constant, $v =$ constant) or*

B: *M is given by*

$$M\colon \begin{cases} v = \text{constant}, \\ w = -kxy + \mathbf{q}(x), \quad \mathbf{q} \in C^\infty(I). \end{cases}$$

Remark 4. In the case A, $E_2 = \partial_w$ while in the case B,

$$E_2 = \frac{1}{\sqrt{1 + (\mathbf{q}'(x) - ky)^2}} \big(\partial_x + (\mathbf{q}'(x) - ky)^2 \partial_w\big).$$

Remark 5. The vector fields

$$\begin{aligned} V_1 &= kx\partial_x - ky\partial_y + \partial_v = \frac{1}{k}B, \\ V_2 &= -\sin\psi\,\partial_x + \cos\psi\,\partial_w \end{aligned}$$

are unitary, orthogonal and generate the normal bundle of M.

Theorem 6. *There is no minimal, totally real 2-dimensional submanifold M in $(\mathbb{R}^4, g, J)$ normal to the Lee vector field.*

P r o o f. After an easy computation one gets $\nabla_{E_1} E_1 = kV_1$, $\nabla_{E_2} E_2 = 0$ (in case A) and $\nabla_{E_2} E_2 = \frac{1}{2}k\sin 2\psi E_1 - k\cos^2\psi V_1 + (\mathbf{q}''(\mathbf{x})/\cos\psi)V_2$ (in case B).

It follows that the mean curvature is $H = \frac{1}{2}B$ (in case A) and $H = \frac{1}{2}\{k\sin^2\psi\, V_1 + (\mathbf{q}''(x)/\cos\psi)V_2\}$ (in case B).

Hence the conclusion. □

Case b). Let B be tangent to M (and A normal to M). Let $E_1 = kx\partial_x - ky\partial_y + \partial_v$ be the normalized vector of B. We are looking for E_2 unitary, orthogonal to E_1 and tangent to M. As in case a) we obtain $E_2 = \cos\psi\,\partial_x + \sin\psi\,\partial_w$ with ψ a smooth function on M. Since

$$\begin{aligned} [E_1, E_2] = &-(kx\sin\psi\,\psi_x - ky\sin\psi\,\psi_y + \sin\psi\,\psi_v + k\cos\psi)\partial_x \\ &+ (kx\psi_x - ky\psi_y + \psi_v)\cos\psi\,\partial_w, \end{aligned}$$

the involutivity condition yields the PDE

$$kx\psi_x - ky\psi_y + \psi_v = -\frac{k}{2}\sin 2\psi \quad (\text{along } M). \tag{3.7}$$

Notice that $\psi =$ constant implies $\sin 2\psi = 0$.

Consider M given by

$$M\colon \begin{cases} f_1(x,y,v,w)=0, \\ f_2(x,y,v,w)=0 \end{cases}$$

where f_1, f_2 are C^∞ functions on $\mathbb{R}^4$ verifying $\operatorname{rank}(D(f_1,f_2)/D(x,y,v,w))=2$ at any point of M. Since E_1, E_2 are tangent to M, it follows

$$\begin{cases} kx\,\partial f_i/\partial x - ky\,\partial f_i/\partial y + \partial f_i/\partial v = 0, \\ \cos\psi\,\partial f_i/\partial x + \sin\psi\,\partial f_i/\partial w = 0, \quad i=1,2 \ \text{(on } M). \end{cases} \tag{3.8}$$

Subcase I. If $\partial f_2/\partial w = 0$, then $\cos\psi\,\partial f_2/\partial x = 0$.

I.1. If $\cos\psi \neq 0$, it follows that $\partial f_2/\partial x = 0$ and $\partial f_2/\partial v = ky\,\partial f_2/\partial y$. Obviously $\partial f_2/\partial y \neq 0$. If we consider $f_2 = f_2(y,v) = y - G(v)$ we get $-G' = kG$ and hence the second equation defining M is $y\mathrm{e}^{kv} = \text{constant}$.

I.2. Let $\cos\psi = 0$. If $\partial f_2/\partial x = 0$ one comes back to the previous case and if $\partial f_2/\partial x \neq 0$ one considers $f_2 = f_2(x,y,v) = x - G(y,v)$. We obtain the PDE

$$G_v - kyG_y - kG = 0 \quad (\text{along } M)$$

having the solution

$$G(y,v) = \mathrm{e}^{kv}\mathbf{q}(y\mathrm{e}^{kv})$$

where $\mathbf{q}$ is an arbitrary smooth function depending on one variable. Hence the second equation defining M is

$$x = \mathrm{e}^{kv}\mathbf{q}(y\mathrm{e}^{kv}). \tag{3.9}$$

Let us analyze the first equation.

- If $\partial f_1/\partial w = 0$, we get $\cos\psi = 0$ and as previously $x = \mathrm{e}^{kv}\tilde{\mathbf{q}}(y\mathrm{e}^{kv})$. This and (3.9) yield $y\mathrm{e}^{kv} = \text{constant}$ and consequently M is defined by

$$M\colon \begin{cases} x\mathrm{e}^{-kv} = \text{constant}, \\ y\mathrm{e}^{kv} = \text{constant}. \end{cases}$$

- If $\partial f_1/\partial w \neq 0$, consider $f_1 = w - F(x,y,v)$. The two equations in (3.8) corresponding to $i=1$ yield

$$\begin{cases} F_v - kyF_y + kxF_x = 0, \\ \cos\psi\,F_x - \sin\psi = 0. \end{cases}$$

Obviously, $\cos\psi \neq 0$ and hence

$$\begin{cases} F_x = \tan\psi, \\ F_v = -kx\tan\psi + kyF_y. \end{cases} \tag{3.10}$$

The involutivity condition (3.7) becomes

$$kx\varrho_x - ky\varrho_y + \varrho_v = -k$$

where $\varrho = \ln\tan\psi$, and has the solution

$$\varrho = \boldsymbol{\varrho}(y\mathrm{e}^{kv}, x\mathrm{e}^{-kv}) - kv$$

with $\boldsymbol{\varrho}$ an arbitrary smooth function of two variables. The solution of the PDE's system (3.10) is

$$F(x,y,v) = \mathbf{q}(y\mathrm{e}^{kv}, x\mathrm{e}^{-kv})$$

with $\mathbf{q}$ a differentiable function. Recall that, since $\cos\psi \neq 0$, the second equation defining M is $y\mathrm{e}^{kv}$ = constant; hence we will take $F(x,y,v) = \mathbf{q}(x\mathrm{e}^{-kv})$, $\mathbf{q} \in C^\infty$. Thus M is given by

$$M =: \begin{cases} y\mathrm{e}^{kv} = \text{constant}, \\ w = \mathbf{q}(x\mathrm{e}^{-kv}) \ \text{ or, equivalently, } \ w = \mathbf{q}(xy). \end{cases}$$

Subcase II. Suppose $\partial f_i/\partial w \neq 0$ for $i = 1, 2$. In this situation we can assume that M is given by

$$\begin{cases} w = F(x,y,v), \\ w = G(x,y,v) \end{cases}$$

or, equivalently, by

$$\begin{cases} w = F(x,y,v), \\ \varphi(x,y,v) = 0 \end{cases}$$

which yields the first subcase.

We can state:

Theorem 7. *Let M be a 2-dimensional totally real submanifold in $(\mathbb{R}^4, g, J)$ tangent to the Lee vector field B. Then M is given by either*

A:

$$\begin{cases} x\mathrm{e}^{-kv} = \text{constant}, \\ y\mathrm{e}^{kv} = \text{constant}, \end{cases}$$

i.e., $M = \gamma \times \mathbb{R}$, where $\gamma(t) = (c_1\mathrm{e}^{kt}, c_2\mathrm{e}^{-kt}, t) \subset \mathbb{R}^3$ with c_1 and c_2 real constants; or

B:

$$\begin{cases} y\mathrm{e}^{kv} = \text{constant}, \\ w = \mathbf{q}(xy) \end{cases}$$

with $\mathbf{q} \in C^\infty$.

Moreover, all submanifolds belonging to the case A *are minimal.*

P r o o f. We have to sketch the proof only for the second part of the statement. The vector fields

$$E_1 = kx\partial_x - ky\partial_y + \partial_v \quad \text{and} \quad E_2 = \partial_w$$

are unitary, orthogonal and tangent to M. Then, the normal bundle of M is spanned by

$$V_1 = \partial_y - kx\partial_w \quad \text{and} \quad V_2 = \partial_x.$$

One has $\nabla_{E_1} E_1 = \nabla_{E_2} E_2 = 0$, which implies that the mean curvature vanishes. □

Let us turn our attention to the case B. We have

Theorem 8. *Let M be a minimal, totally real 2-dimensional submanifold in $(\mathbb{R}^4, g, J)$ belonging to the case* B. *Then M is given by*

$$M\colon \begin{cases} y\mathrm{e}^{kv} = \text{constant}, \\ w = \text{constant}. \end{cases}$$

($M = \mathbb{R}\times\gamma$ where $\gamma(t) = (t, c\mathrm{e}^{-kt}) \subset \mathbb{R}^2$ with c a real constant; hence $M \subset \mathbb{R}^3 \subset \mathbb{R}^4$.)

P r o o f. The normal bundle of M is spanned by

$$V_1 = \partial_y - kx\partial_w \quad \text{and} \quad V_2 = -\sin\psi\,\partial_x + \cos\psi\,\partial_w.$$

We have $\nabla_{E_1} E_1 = 0$ and

$$\nabla_{E_2} E_2 = -k\cos^2\psi\, E_1 + k\sin\psi\cos\psi\, V_1 + \cos\psi\, V_2.$$

One obtains that the mean curvature is

$$H = \tfrac{1}{2}(k\sin\psi\cos\psi\, V_1 + \cos\psi\psi_x\, V_2)$$

and taking into account the fact that $\cos\psi \neq 0$ it follows that $\sin\psi = 0$ and $\psi_x = 0$. Thus $\psi = \text{constant}$.

(As a remark, the involutivity condition is fulfilled.)

After some computations, one gets $F = F(y\mathrm{e}^{kv}) = \text{constant}$ since $y\mathrm{e}^{kv} = \text{constant}$; hence the conclusion. □

A submanifold M in a Hermitian manifold $(\tilde{M}, J, \tilde{g})$ is a *CR-submanifold* if it is endowed with a holomorphic distribution $\mathcal{D}$ (i.e. $J_x\mathcal{D}_x = \mathcal{D}_x$ for all $x \in M$) such that its orthogonal complement $\mathcal{D}^\perp$ (with respect to $g = \jmath^*\tilde{g}$) in $T(M)$ is anti-invariant, namely $J_x\mathcal{D}^\perp_x \subseteq T(M)^\perp_x$, $\forall x \in M$. (Here $T(M)^\perp$ is the normal bundle of the immersion $\jmath\colon\ M \hookrightarrow \tilde{M}$.) Let us consider only proper CR-submanifolds, i.e. $\dim\mathcal{D} = 2s \geqslant 2$, $\dim\mathcal{D}^\perp = q \geqslant 1$. Since the ambient manifold is $\mathbb{R}^4$ it follows that $s = 1$, $q = 1$ and hence M is a generic (i.e. $J_x\mathcal{D}^\perp_x = T(M)^\perp_x$) 3-dimensional CR submanifold in $(\mathbb{R}^4, g, J)$.

We are interested in finding all proper CR submanifolds M (of dimension 3) for which the Lee vector field B is tangent or normal to M. (The case B oblique yields (more) complicated relations.)

Case a) B is tangent to M.

a.1) B belongs to the distribution $\mathcal{D}$; it follows that the anti-Lee vector field A lies in $\mathcal{D}$, too. Since $[A,B] = -k^2A$, it results that the holomorphic distribution is integrable. Consider M given by $f(x,y,v,w) \equiv 0$ with $\operatorname{grad} f \neq 0$ on M. One gets the following PDE's system:

$$(4.11)\qquad \begin{cases} f_y - kxf_w = 0,\\ kxf_x - kyf_y + f_v = 0 \quad \text{(along } M). \end{cases}$$

a.1.1) If $f_w = 0$, then $f_y = 0$ and $f_v = -kxf_x$. Thus, $f_x \neq 0$ and by using the same argument as in the previous section we can write the function defining the manifold M as $f = x - F(v)$. One obtains the ODE $kF = F'$ and consequently M is given by $x\mathrm{e}^{-kv} =$ constant.

a.1.2) If $f_w \neq 0$, then we may assume $f = w - F(x,y,v)$ and thus $f_y = kx$. It follows from $(4.11)_1$ that $F = -kxy + G(x,v)$ with G a smooth function satisfying $G_v + kxG_x = 0$ having the general solution $G(x,v) = \mathbf{q}(x\mathrm{e}^{-kv})$, $\mathbf{q} \in C^\infty$. Thus, M is given by $f(x,y,v,w) = w + kxy - \mathbf{q}(x\mathrm{e}^{-kv}) \equiv 0$.

We can state now

Theorem 9. *Let M be a proper CR submanifold such that the Lee vector field belongs to the holomorphic distribution. Then either*

1. *$M = M^T \times \mathbb{R}$ (M^T is described above) or*
2. *M is given by $f = w + kxy - \mathbf{q}(x\mathrm{e}^{-kv}) \equiv 0$ where $\mathbf{q} \in C^\infty$.*

Moreover, every hypersurface in case 1 is minimal.

P r o o f. As we have already mentioned, the holomorphic distribution $\mathcal{D}$ is involutive and M^T is the integral submanifold of the distribution $\mathcal{D}$ (generated e.g. by

$kx\,\partial/\partial x + \partial/\partial v - k^2 xy\,\partial/\partial w$ and $\partial/\partial y - kx\,\partial/\partial w$). Similarly, the anti-holomorphic distribution $\mathcal{D}^\perp$ is involutive, too. Moreover, the leaves of both distributions are totally geodesic in M.

We have only to study the minimality of the submanifolds in case a.1.1). Consider the orthonormal frame on M: $\{E_1, E_2, E_3\}$ where $E_1 = k^{-1}B$, $E_2 = k^{-1}A$ and $E_3 = \partial/\partial w$. We have

$\dot{\nabla}_{E_1} E_1 = 0$ and $h(E_1, E_1) = 0$,
$\dot{\nabla}_{E_2} E_2 = kE_1$ and $h(E_2, E_2) = 0$,
$\dot{\nabla}_{E_3} E_3 = 0$ and $h(E_3, E_3) = 0$

and hence the mean curvature H vanishes. (We have denoted by $\dot{\nabla}$ and h the Levi-Civita connection on M and the second fundamental form, respectively.) □

Let us study now the minimality of the submanifolds in case a.1.2): Consider the orthonormal frame $\{E_1, E_2, E_3\}$ on M, where $E_1 = k^{-1}B$, $E_2 = k^{-1}A$ and $E_3 = (\partial/\partial x + T\partial/\partial w)/\sqrt{1+T^2}$. We have denoted $T = \mathrm{e}^{-kv}\mathbf{q}'(x\mathrm{e}^{-kv}) - ky$.

The unitary normal to M is $V = (-T\partial/\partial x + \partial/\partial w)/\sqrt{1+T^2}$. We have

$$\dot{\nabla}_{E_3} E_3 = -\frac{k}{1+T^2} E_1 + \frac{kT}{1+T^2} E_2 \quad \text{and} \quad h(E_3, E_3) = \frac{\mathrm{e}^{-2kv}\mathbf{q}''}{(1+T^2)^{3/2}} V.$$

Proposition 10. *Let M be as in case 2 in the previous theorem. Suppose M is minimal in $(\mathbb{R}^4, g)$. Then M is given by*

$$w + kxy - cx\mathrm{e}^{-kv} = \text{constant}$$

where c is a real constant.

P r o o f. The statement follows from the relation

$$H = \frac{\mathrm{e}^{-2kv}\mathbf{q}''}{3(1+T^2)^{3/2}} V.$$

□

If X is a vector field on M we denote by PX the tangent part of JX (see [7] for details).

Proposition 11. *Let M be as in case 1 in Theorem 9. Then P is parallel (with respect to the Levi-Civita connection $\dot{\nabla}$ on M).*

Remark 12.

- The previous result is not surprising (cf. Theorem 1, [1]).
- Let M be as in case 2 in Theorem 9. It is easy to verify that the leaves of $\mathcal{D}^\perp$ are not totally geodesic in M (for example, the condition (5) in [1, page 7], is not satisfied). Moreover, $\dot{\nabla}P \neq 0$. Consequently, there are no product CR submanifolds in case 2.

a.2) B belongs to the distribution $\mathcal{D}^\perp$; it follows that the anti-Lee vector field A is normal to M. If M is given by $f(x,y,v,w)=0$, then A and the gradient of f are collinear. From this and the relation $B(f)=0$ it follows that M has the equation $y\mathrm{e}^{kv} = \text{constant}$. The holomorphic distribution $\mathcal{D}$ is generated by $E_1 = \partial/\partial x$ and $E_2 = \partial/\partial w$, which means that it is involutive.

Theorem 13. *Let M be a proper CR submanifold in $(\mathbb{R}^4, J, g)$ such that the Lee vector field belongs to the anti-holomorphic distribution $\mathcal{D}^\perp$. Then M is the hypersurface in $\mathbb{R}^4$ given by the equation $y\mathrm{e}^{kv} = \text{constant}$. Moreover, M is minimal.*

P r o o f. The three vector fields E_1, E_2 and $E_3 = k^{-1}B$ form an orthonormal basis in $\chi(M)$. We have

$$\begin{aligned}
\dot\nabla_{E_1}E_1 &= -kE_3 \quad \text{and} \quad h(E_1,E_1)=0,\\
\dot\nabla_{E_2}E_2 &= 0 \quad \text{and} \quad h(E_2,E_2)=0,\\
\dot\nabla_{E_3}E_3 &= 0 \quad \text{and} \quad h(E_3,E_3)=0.
\end{aligned}$$

Hence the conclusion. □

a.3) B has components both in $\mathcal{D}$ and in $\mathcal{D}^\perp$; let us say $B = B_1 + B_2$ with $B_1 \in \mathcal{D}$ and $B_2 \in \mathcal{D}^\perp$. Then A is oblique and $\tan(A) = -JB_1$ and $\mathrm{nor}(A) = -JB_2$, where $\tan(A)$ and $\mathrm{nor}(A)$ denote the tangent and the normal part of A, respectively. Moreover, the tangent part of A belongs to the holomorphic distribution $\mathcal{D}$. Consider $\tan(A) = a\,\partial/\partial x + b\,\partial/\partial y + p\,\partial/\partial v + q\,\partial/\partial w$, with a, b, p, q smooth functions on $\mathbb{R}^4$. The orthogonality between B and $\mathrm{nor}(A)$ yields $p=0$. Thus we have

$$\begin{cases}
\tan(A) = a\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y} + q\dfrac{\partial}{\partial w},\\[2ex]
B_1 = -q\dfrac{\partial}{\partial x} - bky\dfrac{\partial}{\partial y} + b\dfrac{\partial}{\partial v} + a\dfrac{\partial}{\partial w},\\[2ex]
B_2 = (q+k^2x)\dfrac{\partial}{\partial x} - (k-b)ky\dfrac{\partial}{\partial y} + (k-b)\dfrac{\partial}{\partial v} - a\dfrac{\partial}{\partial w}
\end{cases}$$

which generate $\chi(M)$ and $\mathrm{nor}(A) = -a\,\partial/\partial x + (k-b)\partial/\partial y - (k^2x+q)\partial/\partial w$, which is normal to M. Then, since $\tan(A) \perp \mathrm{nor}(A)$, one gets

$$a^2 = b(k-b) - (bkx+q)^2$$

with $a^2+b^2+q^2 \neq 0$ and $a^2 + (k-b)^2 + (k^2x+q)^2 \neq 0$. It is easy to see that we have $\mathrm{nor}(A) \perp B_{1,2}$ and $B_1 \perp B_2$.

If the submanifold M is given by $f(x, y, v, w) = 0$, one obtains, from the tangency conditions, the following PDE's system (on M):

$$
\begin{cases} af_x + bf_y + qf_w = 0, \\ -qf_x - bkyf_y + bf_v + af_w = 0, \\ kxf_x - kyf_y + f_v = 0. \end{cases} \tag{4.12}
$$

Moreover, $\mathrm{grad}(f)$ is parallel to the normal part of A.

From $(4.12)_{2,3}$ we get $af_w = (bkx + q)f_x$.

a.3.1) $a = 0$: It is not difficult to prove that $bkx + q$ cannot be 0 (otherwise $\mathrm{nor}(A) = 0$). It follows that $f_x = 0$ and hence, by substituting in (4.12), we have $f_v = kyf_y$.

a.3.1.i) $f_y = 0$: Then $f_v = 0$ and M is given by $w =$ constant. Moreover, $q = 0$ and $b = k/\alpha$.

a.3.1.ii) $f_y \neq 0$: Then $f = y - F(v, w)$ and, together with $f_v = kyf_y$, we obtain that $F(v, w) = \mathbf{q}(w)\mathrm{e}^{-kv}$, $\mathbf{q} \in C^\infty(I)$, $I \subset \mathbb{R}$. After some computations, one arrives at

$$
b = \frac{k(\mathbf{q}'(w)\mathrm{e}^{-kv})^2}{1 + (kx + \mathbf{q}'(w)\mathrm{e}^{-kv})^2}, \quad q = \frac{k\mathbf{q}'(w)\mathrm{e}^{-kv}}{1 + (kx + \mathbf{q}'(w)\mathrm{e}^{-kv})^2}
$$

and hence $\mathbf{q}$ cannot be constant (otherwise $\tan(A) = 0$). In this case, M is given by $y\mathrm{e}^{kv} = \mathbf{q}(w)$.

a.3.2) $a \neq 0$: It follows that $f_w = (q + bkx)/af_x$. The case $f_x = 0$ yields a contradiction (namely $b = 0$ and $a^2 = -q^2$). Consequently $f_x \neq 0$ and we can consider $f = x - F(y, v, w)$ with F verifying $-F_w = (q + bkF)/a$. From (4.12) we also obtain a PDE, namely $F_v = kyF_y + kF$ with the general solution $F(y, v, w) = \mathrm{e}^{kv}\mathbf{q}(y\mathrm{e}^{kv}, w)$ ($\mathbf{q}$ is a smooth function depending on two variables). After some computations one has

$$
b = \frac{k}{1 + (t_1 + kx)^2 + t_2^2}, \quad q = t_1 b, \ \ a = t_2 b
$$

where

$$
t_1 = -\frac{kF + F_y F_w}{1 + F_w^2} \quad \text{and} \quad t_2 = \frac{F_y - kFF_w}{1 + F_w^2}.
$$

M is defined by the equation $x\mathrm{e}^{-kv} = \mathbf{q}(y\mathrm{e}^{kv}, w)$.

Theorem 14. *Let M be a proper CR-submanifold in $(\mathbb{R}^4, J, g)$ such that the Lee vector field B is tangent to M and has components either in $\mathcal{D}$ or in $\mathcal{D}^{\perp}$. Then we have one of the following three situations:*

1. *M is the hyperplane $w = \text{constant}$. In this case M is minimal.*
2. *M is given by $y\mathrm{e}^{kv} = \mathbf{q}(w)$.*
3. *M is defined by the equation $x\mathrm{e}^{-kv} = \mathbf{q}(y\mathrm{e}^{kv}, w)$.*

If we require M to be minimal, we get:

in case 2: $\mathbf{q}'(w) = c$, a nonzero real constant and hence M is given by

$$cw - y\mathrm{e}^{kv} = \text{constant};$$

in case 3: $\mathbf{q}(y\mathrm{e}^{kv}, w) = cy\mathrm{e}^{kv} + \text{constant}$, where c is a nonzero real number; consequently, M is defined by

$$x\mathrm{e}^{-kv} - cy\mathrm{e}^{kv} = \text{constant}.$$

Proof. We will sketch the proof only for the second part of the statement.

Let M be given by the equation $y\mathrm{e}^{kv} = \mathbf{q}(w)$ with q a nonconstant smooth function. Consider in $\chi(M)$ the following orthonormal frame: $E_1 = \partial/\partial x$, $E_2 = \frac{1}{k}B$, $E_3 = 1/\sqrt{T}(\mathbf{q}'(w)\partial/\partial y + \mathrm{e}^{kv}\partial/\partial w)$ where $T = (\mathbf{q}'(w)kx + \mathrm{e}^{kv})^2 + \mathbf{q}'(w)^2$. Then $V = (\varrho\,\partial/\partial y + \mu\,\partial/\partial w)/\|\varrho\,\partial/\partial y + \mu\,\partial/\partial w\|$ is a normal and unitary vector field on M, where $\varrho = \mathrm{e}^{kv} + kx\mathbf{q}'(w)$ and $\mu = -(kx\mathrm{e}^{kv} + \alpha\mathbf{q}'(w))$.

The Gauss and Weingarten formulas yield

$$\begin{aligned}
&\dot{\nabla}_{E_1}E_1 = -kE_2, \quad h(E_1, E_1) = 0,\\
&\dot{\nabla}_{E_2}E_2 = 0, \quad h(E_2, E_2) = 0,\\
&\dot{\nabla}_{E_3}E_3 = -\frac{k\varrho\mathbf{q}'(w)}{T}E_1 + \frac{k\mathbf{q}'(w)^2}{T}E_2 + \frac{\mathrm{e}^{kv}\mu\mathbf{q}''(w)}{T}\Big(\lambda + \frac{2}{\sqrt{T}}\Big)E_3,\\
&\qquad\qquad\qquad\qquad\qquad\qquad h(E_3, E_3) = \frac{\lambda\mathbf{q}''(w)}{T}V
\end{aligned}$$

with λ a certain nonzero function.

Thus M is minimal if and only if $\mathbf{q}''(w) = 0$. Hence we get the conclusion.

In this case

$$T = (ckx + \mathrm{e}^{kv})^2 + c^2,$$
$$E_3 = \frac{1}{\sqrt{T}}\Big(c\frac{\partial}{\partial y} + \mathrm{e}^{kv}\frac{\partial}{\partial w}\Big)$$

and

$$\dot{\nabla}_{E_3}E_3 = -\frac{ck}{T}(ckx + \mathrm{e}^{kv})E_1 + \frac{c^2k}{T}E_2.$$

Let now M be given by $x\mathrm{e}^{-kv} = \mathbf{q}(y\mathrm{e}^{kv}, w)$.

First let us introduce some notations $P = \mathrm{e}^{kv}\mathbf{q}_2'$, $Q = \mathrm{e}^{kv}\mathbf{q}_1' - kx\mathbf{q}_2'$ where by $\mathbf{q}_1'$ and $\mathbf{q}_2'$ we have denoted the (first) partial derivatives with respect to the first and to the second coordinate, respectively. Put also $T = 1/\sqrt{1+P^2}$ and $S = 1/\sqrt{1+P^2+\mathrm{e}^{2kv}Q^2}$. Consider now the following orthonormal frame in $\chi(M)$:

$$\begin{aligned}
E_1 &= \frac{1}{k}B,\\
E_2 &= T\Big(P\frac{\partial}{\partial x} + \frac{\partial}{\partial w}\Big),\\
E_3 &= TS\Big(\mathrm{e}^{kv}Q\frac{\partial}{\partial x} + (1+P^2)\frac{\partial}{\partial y} - (\mathrm{e}^{kv}PQ + kx(1+P^2))\frac{\partial}{\partial w}\Big).
\end{aligned}$$

Then, a unitary vector field normal to M is

$$V = S\Big(\frac{\partial}{\partial x} - \mathrm{e}^{kv}Q\frac{\partial}{\partial y} - (P - \mathrm{e}^{kv}kxQ)\frac{\partial}{\partial w}\Big).$$

The Gauss and Weingarten formulas yield

$$\dot{\nabla}_{E_1}E_1 = 0, \quad h(E_1,E_1) = 0,$$

$$\dot{\nabla}_{E_2}E_2 = -kP^2T^2E_1 + TS(kP + \mathrm{e}^{2kv}QT^2\mathbf{q}_{22}'')E_3,$$
$$h(E_2,E_2) = \mathrm{e}^{kv}T^2S(-kPQ + \mathbf{q}_{22}'')V,$$

$$\begin{aligned}
\dot{\nabla}_{E_3}E_3 = \frac{E_3(TS)}{TS}E_3 + T^2S^2\{k[(1+P^2)^2 - \mathrm{e}^{2kv}Q^2]E_1 + k\mathrm{e}^{kv}P^2(1+P^2)QTE_2\\
-k\mathrm{e}^{2kv}PQ^2\frac{S}{T}E_3 + e^{2kv}T[\mathrm{e}^{kv}P^3\mathbf{q}_{11}'' - \mathrm{e}^{kv}(1+P^2)Q\mathbf{q}_{12}''\\
-Q(\mathrm{e}^{kv}PQ + kx(1+P^2))\mathbf{q}_{22}'']\\
+\mathrm{e}^{2kv}S[\mathrm{e}^{2kv}(1+P^2)^2QT\mathbf{q}_{11}'' + \frac{2}{T}(1+P^2)(P - kx\mathrm{e}^{kv}Q)\mathbf{q}_{12}''\\
+(\mathrm{e}^{2kv}P^2Q^2 - k^2x^2(1+P^2)^2)QT\mathbf{q}_{22}'']E_3\}h(E_3,E_3) =\\
= T^2S^2\mathrm{e}^{kv}\{kPQ^3\mathrm{e}^{2kv} + \mathrm{e}^{2kv}(1+P^2)^2\mathbf{q}_{11}''\\
-2\mathrm{e}^{kv}(1+P^2)(\mathrm{e}^{kv}PQ + kx(1+P^2))\mathbf{q}_{12}'' + (\mathrm{e}^{kv}PQ + kx(1+P^2))^2\mathbf{q}_{22}''\}V.
\end{aligned}$$

Consequently, if the mean curvature of M vanishes, then

$$\begin{aligned}
-kPQ(1+P^2) + \mathrm{e}^{2kv}(1+P^2)^2\mathbf{q}_{11}''\\
-2\mathrm{e}^{kv}(1+P^2)(\mathrm{e}^{kv}PQ + kx(1+P^2))\mathbf{q}_{12}'' + (1+P^2+\mathrm{e}^{2kv}Q^2)\mathbf{q}_{22}'' = 0.
\end{aligned} \tag{4.13}$$

The above expression is a polynomial of second order in x (since P does not depend on x and Q is affine in x). So, if we look at the coefficient of x^2 we get $k^2(1+P^2)\mathbf{q}''_{22} = 0$ and hence $\mathbf{q}'_2$ is constant with respect to the second variable. Thus,

$$\mathbf{q}(y\mathrm{e}^{kv}, w) = \sigma(y\mathrm{e}^{kv})w + \tau(y\mathrm{e}^{kv})$$

for some smooth functions σ and τ. We return to (4.13) and find the coefficient of x. It is

$$(1+P^2)(k\mathrm{e}^{kv}(k\sigma^2 - 2\sigma') + 2k\mathrm{e}^{3kv}\sigma^2(1-\sigma'))$$

and must vanish on M. It follows that $\sigma = 0$. Finally, the remaining term is $\mathrm{e}^{2kv}\tau'' = 0$. Thus $\tau(y\mathrm{e}^{kv}) = cy\mathrm{e}^{kv} + \text{constant}$ and from this we get the conclusion.

In this case we have

$$P = 0, \quad Q = c\mathrm{e}^{kv}, \quad T = 1, \quad S = \frac{1}{\sqrt{1+c^2\mathrm{e}^{4kv}}}$$

and

$$E_1 = \frac{1}{k}B, \quad E_2 = \frac{\partial}{\partial w}, \quad E_3 = \frac{1}{\sqrt{1+c^2\mathrm{e}^{4kv}}}\Big(\frac{\partial}{\partial x} - c\mathrm{e}^{2kv}\frac{\partial}{\partial y} + ckx\mathrm{e}^{2kv}\frac{\partial}{\partial w}\Big).$$

Moreover, $h(E_1, E_1) = h(E_2, E_2) = h(E_3, E_3) = 0$. □

Case b) B is normal to M. It follows that A belongs to the anti-holomorphic distribution $\mathcal{D}^\perp$. If M is given by $f(x, y, v, w) = 0$ (with $\operatorname{grad} f \neq 0$), we have $A(f) = 0$ and $B \| \operatorname{grad} f$. Consequently, one gets $f_w = 0$, $f_y = 0$ and $f_x = 0$. Hence $f = f(v) = 0$, which means that M is the hyperplane $v = \text{constant}$. We easily obtain that the holomorphic distribution $\mathcal{D}$ is generated by $E_1 = \partial/\partial x$ and $E_2 = \partial/\partial w$. Thus $\mathcal{D}$ is involutive.

Theorem 15. *Let M be a proper CR submanifold in $(\mathbf{R}^4, J, g)$ normal to the Lee vector field B. Then M is a hyperplane in $\mathbf{R}^4$. Moreover, M is minimal but not totally geodesic.*

Proof. The three vector fields E_1, E_2 and $E_3 = k^{-1}A$ form an orthonormal basis in $\chi(M)$. We have

$$\begin{aligned}
\dot\nabla_{E_1}E_1 = 0 \quad &\text{and} \quad h(E_1, E_1) = -kV, \\
\dot\nabla_{E_2}E_2 = 0 \quad &\text{and} \quad h(E_2, E_2) = 0, \\
\dot\nabla_{E_3}E_3 = 0 \quad &\text{and} \quad h(E_3, E_3) = kV
\end{aligned}$$

where $V = \frac{1}{k}B$ is the unitary normal to M. Hence the conclusion. □

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