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ON EXTENSION OF VECTOR POLYMEASURES, II

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ABSTRACT. We prove a necessary and sufficient condition for extension of a vector polymeasure from Cartesian product of rings to the Cartesian product of generated σ-rings.

In this addition to [2], we give a necessary and sufficient condition for the existence of a unique separately countably additive extension $\gamma\colon \sigma(R_1) \times \dots \times \sigma(R_d) \to Y$ of a separately countably additive $\gamma_0\colon R_1 \times \dots \times R_d \to Y$. Here R_i is a ring of subsets of a non empty set T_i, $\sigma(R_i)$ is the generated σ-ring, for $i = 1, \dots, d$, and Y is a Banach space.

Since for any sequence $A_n \in \sigma(R)$, $n = 1, 2, \dots$, there is a countable subring $R' \subset R$ such that $A_n \in \sigma(R')$ for each $n = 1, 2, \dots$, see [6; §5, Theorems C and D], the uniqueness of the extension of a vector polymeasure, see [1; Corollary of Lemma 4], implies the following:

LEMMA. *A separately countably additive $\gamma_0\colon R_1 \times \dots \times R_d \to Y$ has a unique separately countably additive extension $\gamma\colon \sigma(R_1) \times \dots \times \sigma(R_d) \to Y$ if and only if $\gamma_0\colon R'_1 \times \dots \times R'_d \to Y$ has a separately countably additive extension $\gamma\colon \sigma(R'_1) \times \dots \times \sigma(R'_D) \to Y$ for any countable subrings $R'_i \subset R_i$, $i = 1, \dots, d$.*

Note that [2; Corollary of Theorem 5] gives a necessary and sufficient condition for the extension in the case of countable rings R_i, $1, \dots, d$. The theorem below is not limited, but only reducible, to this case. In a sense, the theorem is a counterpart of [4; Theorem 9] (with iterated limits there) and [5; Theorem 2], which give similar double limit characterizations of L_1-representability of bounded multilinear operators on $\times C_0(T_i)$ and on $\times C_0(T_i, X_i)$ respectively.

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THEOREM. *A separately countably additive* $\gamma_0\colon R_1 \times \cdots \times R_d \to Y$ *has a unique separately countably additive extension* $\gamma\colon \sigma(R_1) \times \cdots \times \sigma(R_d) \to Y$ *if and only if the limits below exist in* Y *and*

$$\lim_{n_1\to\infty} \lim_{k_1\to\infty} \lim_{n\to\infty} \lim_{k\to\infty} \gamma_0(A_{1,n_1,k_1}, A_{2,n,k}, \dots, A_{d,n,k}) = \lim_{n\to\infty} \lim_{k\to\infty} \lim_{n_1\to\infty} \lim_{k_1\to\infty} \gamma_0(A_{1,n_1,k_1}, A_{2,n,k}, \dots, A_{d,n,k}),$$

$$\lim_{n_2\to\infty} \lim_{k_2\to\infty} \lim_{n\to\infty} \lim_{k\to\infty} \gamma_0(A_{1,n,k}, A_{2,n_2,k_2}, A_{3,n,k}, \dots, A_{d,n,k}) = \lim_{n\to\infty} \lim_{k\to\infty} \lim_{n_2\to\infty} \lim_{k_2\to\infty} \gamma_0(A_{1,n,k}, A_{2,n_2,k_2}, A_{3,n,k}, \dots, A_{d,n,k}),$$

....

$$\lim_{n_d\to\infty} \lim_{k_d\to\infty} \lim_{n\to\infty} \lim_{k\to\infty} \gamma_0(A_{1,n,k}, \dots, A_{d-1,n,k}, A_{d,n_d,k_d}) = \lim_{n\to\infty} \lim_{k\to\infty} \lim_{n_d\to\infty} \lim_{k_d\to\infty} \gamma_0(A_{1,n,k}, \dots, A_{d-1,n,k}, A_{d,n_d,k_d})$$

whenever

$$(A_{1,n,k}, \dots, A_{d,n,k}) \in R_1 \times \cdots \times R_d, \qquad n, k = 1, 2, \dots$$

and $\lim_{n\to\infty} \lim_{k\to\infty} \chi_{A_{i,n,k}}(t_i)$ *exists for each* $t_i \in T_i$ *and each* $i = 1, \dots, d$.

P r o o f. The necessity of the conditions follows immediately from [1; Theorem 1].

Conversely, assume the conditions of the theorem hold. By Lemma, we may and will suppose that each R_i, $i = 1, \dots, d$, is a countable ring. Having this reduction we obtain the existence of the extension γ by induction in the dimension d. For $d = 1$ it follows from K l u v á n e k 's extension theorem, see [7]. Suppose the assertion holds for $d - 1$, $d > 1$, and let $R_i \in R_i$, $i = 1, \dots, d$. Then, by the inductive hypothesis, there are uniquely determined separately countably additive extensions $\gamma_1(R_1, \dots)\colon \sigma(R_2) \times \cdots \times \sigma(R_d) \to Y$ and $\gamma_2(\cdot, R_2, \dots, R_d)\colon \sigma(R_1) \to Y$. Since $R_1, \dots, R_d$ are countable rings, by [1; Theorem 11], there are countably additive measures $\lambda_i\colon \sigma(R_i) \to [0, 1]$, $i = 1, \dots, d$, such that $M_i \in \sigma(R_i)$ and $\lambda_i(M_i) = 0$, $i = 1, \dots, d$, imply that $\gamma_1(R_1', M_2, \dots, M_d) = 0$ for each $R_1' \in R_1$, and $\gamma_2(M_1, R_2', \dots R_d') = 0$ for each $(R_2', \dots, R_d') \in R_2 \times \cdots \times R_d$.

Let $(E_1, \dots, E_d) \in \sigma(R_1) \times \cdots \times \sigma(R_d)$. For each $i = 1, \dots, d$ take $A_i \in (R_i)_{\sigma\delta}$ so that $E_i \subset A_i$ and $\lambda_i(A_i - E_i) = 0$, and let $A_{i,n,k} \in R_i$, $n, k = 1, 2, \dots$ be such that $A_{i,n,k} \nearrow A_{i,n} \searrow A_i$, see [3; Lemma C in the proof of Theorem 18]. Then

$$\gamma_1(R_1, E_2, \dots, E_d) = \gamma_1(R_1, A_2, \dots, A_d) = \lim_{n\to\infty} \lim_{k\to\infty} \gamma_0(R_1, A_{2,n,k}, \dots, A_{d,n,k})$$

for each $R_1 \in \mathcal{R}_1$, and

$$\gamma_2(E_1, R_2, \ldots, R_d) = \gamma_2(A_1, R_2, \ldots, R_d) = \lim_{n\to\infty} \lim_{k\to\infty} \gamma_0(A_{1,n,k}, R_2, \ldots, R_d)$$

for each $(R_2, \ldots, R_d) \in \mathcal{R}_2 \times \cdots \times \mathcal{R}_d$, by [1; Theorem 1].

Suppose $R_{1,n_1} \in \mathcal{R}_1$, $n_1 = 1, 2, \ldots$ are pairwise disjoint, and put $A_{1,2k-1} = R_{1,k}$ and $A_{1,2k} = \emptyset$ for $k = 1, 2, \ldots$. Then $\lim_{n_1\to\infty} \gamma_1(A_{1,n_1}, E_2, \ldots, E_d) = 0$. Hence, by Kluvánek's extension theorem, see [7], there is a unique countably additive extension $\gamma(\cdot, E_2, \ldots, E_d)\colon \sigma(\mathcal{R}_1) \in Y$ of $\gamma_1(\cdot, E_2, \ldots, E_d)\colon \mathcal{R}_1 \to Y$. Further we have the equalities:

$$\begin{aligned}\gamma(A_1, E_2, \ldots, E_d) &= \lim_{n_1\to\infty} \lim_{k_1\to\infty} \gamma_1(A_{1,n_1,k_1}, E_2, \ldots, E_d)\\ &= \lim_{n_1\to\infty} \lim_{k_1\to\infty} \lim_{n\to\infty} \lim_{k\to\infty} \gamma_0(A_{1,n_1,k_1}, A_{2,n,k}, \ldots A_{d,n,k})\\ &= \lim_{n\to\infty} \lim_{k\to\infty} \lim_{n_1\to\infty} \lim_{k_1\to\infty} \gamma_0(A_{1,n_1,k_1}, A_{2,n,k}, \ldots, A_{d,n,k})\\ &= \lim_{n\to\infty} \lim_{k\to\infty} \gamma_2(E_1, A_{2,n,k}, \ldots, A_{d,n,k})\end{aligned}$$

by the assumption of the theorem. Since analogous equations hold for any $A_1' \in (\mathcal{R}_1)_{\sigma\delta}$ such that $A_1' \supset E_1$ and $\lambda_1(A_1' - E_1) = 0$, we may uniquely define $\gamma(E_1, E_2, \ldots, E_d) = \gamma(A_1, E_2, \ldots, E_d)$. By the assumption, $\gamma_1(A_{1,n_1,k_1}, \ldots)\colon \sigma(\mathcal{R}_2) \times \cdots \times \sigma(\mathcal{R}_d) \to Y$ is separately countably additive for each $n_1, k_1 = 1, 2, \ldots$, hence $\gamma(E_1, \ldots)\colon \sigma(\mathcal{R}_2) \times \cdots \times \sigma(\mathcal{R}_d) \to Y$ being their set wise iterated limit is also separately countably additive by the (VHSN)-theorem for polymeasures, see the beginning of [1]. The theorem is proved. □

It will be of interest to solve the following:

Problem. Let $\gamma_0\colon \mathcal{R}_1 \times \cdots \times \mathcal{R}_d \to Y$ be separately countably additive, and suppose that there is a separately countably additive extension $\gamma_1\colon \sigma(\mathcal{R}_1) \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_d \to Y$ of γ_0, for each $A_1 \in \sigma(\mathcal{R}_1)$ there is a separately countably additive extension $\gamma_2(A_1, \ldots)\colon \sigma(\mathcal{R}_2) \times \mathcal{R}_3 \times \cdots \times \mathcal{R}_d \to Y$ of $\gamma_1(A_1, \ldots)\colon \mathcal{R}_2 \times \cdots \times \mathcal{R}_d \to Y$, ..., for each $(A_1, \ldots, A_{d-1}) \in \sigma(\mathcal{R}_1) \times \ldots \times \sigma(\mathcal{R}_{d-1})$ there is a countably additive extension $\gamma_d(A_1, \ldots, A_{d-1}, \cdot)\colon \sigma(\mathcal{R}_d) \to Y$ of $\gamma_{d-1}(A_1, \ldots, A_{d-1})\colon \mathcal{R}_d \to Y$. Assume analogous subsequent extensions exist when we start from $\sigma(\mathcal{R}_2), \ldots, \sigma(\mathcal{R}_d)$ and end on $\sigma(\mathcal{R}_1), \ldots, \sigma(\mathcal{R}_{d-1})$ respectively. Are then all the d final set functions mutually equal on $\sigma(\mathcal{R}_1) \times \cdots \times \sigma(\mathcal{R}_d)$?

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