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VARIETIES WITH MODULAR AND DISTRIBUTIVE LATTICES OF SYMMETRIC OR REFLEXIVE RELATIONS

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A binary relation R on an algebra (A, F) is called *compatible* if R satisfies the *Substitution Property* with respect to F, i.e. if for each n-ary $f \in F$, $\langle a_i, b_i \rangle \in R$ for $i = 1, \ldots, n$ imply $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in R$. It was shown in [1] that for any subcollection C of the properties: *reflexivity*, *symmetry*, *transitivity*, the set of all compatible relations on (A, F) satisfying C forms an algebraic lattice (with respect to set inclusion). The modularity or distributivity of such lattices were characterized by some authors, especially for varieties of algebras. For *congruences* (i.e. reflexive, symmetric and transitive compatible relations), it was done by A. Day [5] and B. Jónsson [6]. For *tolerances* (i.e. reflexive and symmetric compatible relations), it was solved in [2]. For *quasiorders* (i.e. reflexive and transitive compatible relations), the distributivity was characterized in [4]. For *weak congruences* (symmetric and transitive compatible relations), the answer has been given recently by G. Vojvodić and B. Šešelja in [8]. For general compatible relations, the solution is contained in [3].

The aim of this paper is to characterize varieties whose members have distributive or modular lattices of symmetric or reflexive compatible relations.

Notation. An algebra and its support will be denoted by the same letter. Let A be an algebra. Denote by $\mathrm{Sym}(A)$ the lattice of all symmetric compatible relations on A. Clearly, the empty relation is the least and A^2 is the greatest element of $\mathrm{Sym}(A)$. The operation $\wedge$ (meet) in $\mathrm{Sym}(A)$ coincides with set intersection. Denote by $\vee$ the join in $\mathrm{Sym}(A)$. For $a, b \in A$ denote by $S(a, b)$ the least element of $\mathrm{Sym}(A)$ containing the pair $\langle a, b \rangle$. If $x_1, \ldots, x_n$ are elements of A, denote by $\mathbf{x}$ the sequence $x_1, \ldots, x_n$.

Lemma 1. *Let a, b, c, d, x, y, a_i, b_i $(i = 1, \ldots, n)$ be elements of an algebra A and let $S_j \in \mathrm{Sym}(A)$ for $j \in J$. Then*

(a) $\langle c, d\rangle \in S(a, b)$ *if and only if* $c = t(a, b), d = t(b, a)$ *for some binary term* $t(x, y)$ *over* A;

(b) $\langle x, y\rangle \in \vee\{S_j\,;\, j \in J\}$ *if and only if there exist an* m*-ary term* p *and elements* x_k, y_k *of* A ($k = 1, \ldots, m$) *such that* $\langle x_k, y_k\rangle \in S_{j_k}$ *for some* $j_k \in J$ *and* $x = p(x_1, \ldots, x_m)$, $y = p(y_1, \ldots, y_m)$;

(c) $\langle x, y\rangle \in \bigvee\{S(a_i, b_i)\,;\, i = 1, \ldots, n\}$ *if and only if there exists a* $2n$*-ary term* q *with* $x = q(a_1, \ldots, a_n, b_1, \ldots, b_n)$, $y = q(b_1, \ldots, b_n, a_1, \ldots, a_n)$.

The proof is elementary, for details see e.g. [1].

Theorem 1. *For a variety* V, *the following conditions are equivalent:*

(1) $\mathrm{Sym}(A)$ *is distributive for each* $A \in V$;

(2) *For every* n*-ary term* p *there exist an* m*-ary term* q *and binary terms* r_j, s_j ($j = 1, \ldots, m$) *such that* $p(\mathbf{x}) = q(r_1(p(\mathbf{x}), p(\mathbf{y})), \ldots, r_m(p(\mathbf{x}), p(\mathbf{y})))$, *and for each* $j \in \{1, \ldots, m\}$ *there exists* $i \in \{1, \ldots, n\}$ *with* $r_j(p(\mathbf{x}), p(\mathbf{y})) = s_j(x_i, y_i)$.

Proof. (1) $\Rightarrow$ (2): Let p be an n-ary term and let $A = F_v(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a free algebra of V with $2n$ free generators $x_1, \ldots, x_n, y_1, \ldots, y_n$. Denote $x = p(\mathbf{x})$, $y = p(\mathbf{y})$. By Lemma 1 we have

$$\langle x, y\rangle \in S(x, y) \wedge \bigvee\{S(x_i, y_i)\,;\, i = 1, \ldots, n\}.$$

Distributivity of $\mathrm{Sym}(A)$ implies

$$\langle x, y\rangle \in \bigvee\{S(x, y) \wedge S(x_i, y_i)\,;\, i = 1, \ldots, n\},$$

thus, by Lemma 1, there exist an m-ary term q and elements $u_j, v_j \in A$ ($j = 1, \ldots, m$) such that $x = q(u_1, \ldots, u_m)$, $y = q(v_1, \ldots, v_m)$, where for each $j \in \{1, \ldots, m\}$,

$$\langle u_j, v_j\rangle \in S(x, y) \wedge S(x_i, y_i) \quad \text{for some} \quad i \in \{1, \cdots, n\}.$$

By Lemma 1, there exist binary terms r_j, s_j with

$$u_j = r_j(x, y) = s_j(x_i, y_i), \;\; v_j = r_j(y, x) = s_j(y_i, x_i),$$

whence (2) is evident.

(2) $\Rightarrow$ (1): Let $A \in V$ and $R, S, Q \in \mathrm{Sym}(A)$. Suppose $\langle a, b\rangle \in R \wedge (S \vee Q)$. By Lemma 1, there exist an n-ary term p and elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of A such that

$$a = p(a_1, \ldots, a_n), \;\; b = p(b_1, \ldots, b_n)$$

and $\langle a, b\rangle \in R$, thus $S(a,b) \subseteq R$, and either $\langle a_i, b_i\rangle \in S$ or $\langle a_i, b_i\rangle \in Q$ for $i = 1, \ldots, n$. By (2), there exist terms q, s_j, r_j such that

$$a = q(r_1(a,b), \ldots, r_m(a,b)), \qquad b = q(r_1(b,a), \ldots, r_m(b,a))$$

and, for each j,

$$r_j(a,b) = s_j(a_i, b_i) \quad \text{and} \quad r_j(b,a) = s_j(b_i, a_i)$$

for some $i \in \{1, \ldots, n\}$. Hence, if $\langle a_i, b_i\rangle \in S$, then $\langle r_j(a,b), r_j(b,a)\rangle \in R \wedge S$, and $\langle r_j(a,b), r_j(b,a)\rangle \in R \wedge Q$ provided $\langle a_i, b_i\rangle \in Q$. By Lemma 1, we conclude $\langle a, b\rangle \in (R \wedge S) \vee (R \wedge Q)$. □

Example 1. Every unary variety V has distributive $\mathrm{Sym}(A)$ for each $A \in V$.

Evidently, every n-ary term in a unary variety V is properly unary. Without loss of generality, suppose $p(x_1, \ldots, x_n) = p_0(x_1)$. We can put $m = 1$, $q(x) = x$, $r_1(x,y) = x$, $s_1(x,y) = p_0(x)$. Then (2) of Theorem 1 is satisfied:

$$p(\mathbf{x}) = p_0(x_1) = q(r_1(p(\mathbf{x}), p(\mathbf{y}))) \text{ and } r_1(p(\mathbf{x}), p(\mathbf{y})) = p(\mathbf{x}) = p_0(x_1) = s_1(x_1, y_1).$$

Now, we turn to the modularity of $\mathrm{Sym}(A)$.

Theorem 2. *For a variety V, the following conditions are equivalent:*

(1) $\mathrm{Sym}(A)$ *is modular for each* $A \in V$;

(2) *for every n-ary term p and each $k \in \{1, \ldots, n\}$ there exist an m-ary term q, $(2+2k)$-ary terms w_j, $(2n-2k)$-ary terms g_j and $2k$-ary terms t_j ($j = 1, \ldots, m$) such that $p(\mathbf{x}) = q(z_1, \ldots, z_m)$, where for each j either*

$$\begin{aligned} z_j &= w_j(p(\mathbf{x}), p(\mathbf{y}), x_1, \ldots, x_k, y_1, \ldots, y_k) \\ &= g_j(x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_n) \quad \textit{or} \\ z_j &= t_j(x_1, \ldots, x_k, y_1, \ldots, y_k). \end{aligned}$$

Proof. (1) ⇒ (2): Let p be an n-ary term over V, $k \in \{1, \ldots, n\}$, and let $A = F_v(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a free algebra of V. Denote $x = p(\mathbf{x})$, $y = p(\mathbf{y})$ and put

$$Q = \bigvee\{S(x_i, y_i);\, i = 1, \ldots, k\}, \quad T = \bigvee\{S(x_i, y_i);\, i = k+1, \ldots, n\},$$
$$R = S(x,y) \vee Q.$$

Then $\langle x, y\rangle \in T \vee Q$ and $\langle x, y\rangle \in R$, thus $\langle x, y\rangle \in R \wedge (T \vee Q)$. Since $Q \subseteq R$, the modularity of Sym (A) implies $\langle x, y\rangle \in (R \wedge T) \vee Q$. By Lemma 1, there exist an m-ary term q and elements $z_j, u_j \in A$ such that $x = q(z_1, \ldots, z_m)$, $y = q(u_1, \ldots, u_m)$, where for each $j \in \{1, \ldots, m\}$ either

$$\langle z_j, u_j\rangle \in R \wedge T \quad \text{or} \quad \langle z_j, u_j\rangle \in Q.$$

By an argument similar to that of the proof of Theorem 1, we obtain (2).

(2) $\Rightarrow$ (1): Let $A \in V$ and $R, Q, T \in \mathrm{Sym}(A)$. Let

$$\langle a, b\rangle \in R \wedge (T \vee (R \wedge Q)).$$

Then $\langle a, b\rangle \in R$ and, by Lemma 1, there exist an n-ary term p and elements a_i, b_i of A ($i = 1, \ldots, n$) such that $a = p(a_1, \ldots, a_n)$, $b = p(b_1, \ldots, b_n)$, where $\langle a_i, b_i\rangle \in R \wedge Q$ for $i \leqslant k$ and $\langle a_i, b_i\rangle \in T$ for $i > k$ for some $k \in \{1, \ldots, n\}$. By (2), we have

$$a = q(z_1, \ldots, z_m) \quad \text{and} \quad b = q(u_1, \ldots, u_m),$$

where either

$$\begin{aligned}
z_j &= w_j(a, b, a_1, \ldots, a_k, b_1, \ldots, b_k) = g_j(a_{k+1}, \ldots, a_n, b_{k+1}, \ldots, b_n),\\
u_j &= w_j(b, a, b_1, \ldots, b_k, a_1, \ldots, a_k) = g_j(b_{k+1}, \ldots, b_n, a_{k+1}, \ldots, a_n),\\
&\text{i.e. } \langle z_j, u_j\rangle \in (R \vee (R \wedge Q)) \wedge T = R \wedge T, \text{ or}\\
z_j &= t_j(a_1, \ldots, a_k, b_1, \ldots, b_k), u_j = t_j(b_1, \ldots, b_k, a_1, \ldots, a_k),\\
&\text{i.e. } \langle z_j, u_j\rangle \in R \wedge Q.
\end{aligned}$$

By Lemma 1, $\langle a, b\rangle \in (R \wedge T) \vee (R \wedge Q)$. □

Example 2. The variety $\mathscr{A}$ of all abelian groups has modular Sym(A) for each $A \in \mathscr{A}$.

Evidently, every n-ary term $p(x_1, \ldots, x_n)$ of A is of the form $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\alpha_1, \ldots, \alpha_n$ are integers. Put $m = 2$, $q(z, v) = v \circ z$,

$$\begin{aligned}
w_1(a, b, x_1, \ldots, x_k, y_1, \ldots, y_k) &= a \circ x_1^{-\alpha_1} \ldots x_k^{-\alpha_k},\\
g_1(x_{k+1}, \ldots, x_m, y_{k+1}, \ldots, y_n) &= x_{k+1}^{\alpha_{k+1}} \ldots x_n^{\alpha_n},\\
t_2(x_1, \ldots, x_k, y_1, \ldots, y_k) &= x_1^{\alpha_1} \ldots x_k^{\alpha_k}.
\end{aligned}$$

Then

$$\begin{aligned}
z_1 &= w_1(p(\mathbf{x}), p(\mathbf{y}), x_1, \ldots, x_k, y_1, \ldots, y_k) = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \circ x_1^{-\alpha_1} \ldots x_k^{-\alpha_k}\\
&= x_{k+1}^{\alpha_{k+1}} \ldots x_n^{\alpha_n} = g_1(x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_n),\\
z_2 &= t_2(x_1, \ldots, x_k, y_1, \ldots, y_k) = x_1^{\alpha_1} \ldots x_k^{\alpha_k},
\end{aligned}$$

and

$$q(z_1, z_2) = z_2 \circ z_1 = x_1^{\alpha_1} \dots x_k^{\alpha_k} \circ x_{k+1}^{\alpha_{k+1}} \dots x_n^{\alpha_n} = p(\mathbf{x}),$$

proving (2) of Theorem 2.

Similarly as in Example 2, we can show that $\mathrm{Sym}(B)$ is modular for every Boolean algebra B. In this case $p(\mathbf{x})$ is expressed in the form of the canonical disjunction and the proof is rather tedious.

Now we turn to *reflexive relations*. For an algebra A, denote by $\mathrm{Ref}(A)$ the lattice of all reflexive compatible relations on A; denote by $\vee$ or $\wedge$ the operation join or meet in $\mathrm{Ref}(A)$, respectively. Evidently, $\wedge$ coincides with set intersection and the identity relation ω is the least and A^2 is the greatest element of $\mathrm{Ref}(A)$. Denote by $R(a, b)$ the least relation of $\mathrm{Ref}(A)$ containing the given pair $\langle a, b\rangle$ of elements $a, b \in A$. The following elementary assertion has been proved in [1] (Theorems 4 and 6):

Lemma 2. *Let A be an algebra, let a, b, c, d, x, y, a_1, ..., a_n, b_1, ..., b_n be elements of A and $R_j \in \mathrm{Ref}(A)$ for $j \in J$.*

Then (a) *$\langle c, d\rangle \in R(a, b)$ iff there exists an $(n + 1)$-ary term t and elements e_1, ..., $e_n \in A$ such that $c = t(a, e_1, \dots, e_n)$, $d = t(b, e_1, \dots, e_n)$;*

(b) *$\langle x, y\rangle \in \bigvee\{R_j, j \in J\}$ iff there exist an m-ary term p and elements $x_k, y_k \in A$ ($k = 1$, ..., m) such that $\langle x_k, y_k\rangle \in R_{j_k}$ for some $j_k \in J$ and $x = q(x_1, \dots, x_m)$, $y = q(y_1, \dots, y_m)$;*

(c) *$\langle x, y\rangle \in \bigvee\{R(a_i, b_i);\ i = 1, \dots, n\}$ iff there exist an $(n + m)$-ary term q and elements e_1, ..., $e_m \in A$ such that*

$$x = q(a_1, \dots, a_n, e_1, \dots, e_m), \quad y = q(b_1, \dots, b_n, e_1, \dots, e_m).$$

Theorem 3. *For a variety V the following conditions are equivalent:*

(1) *$\mathrm{Ref}(A)$ is distributive for each $A \in V$;*

(2) *For every n-ary term p there exist an m-ary term q and $(2n + 1)$-ary terms r_j, s_j ($j = 1$, ..., m) such that*

$$p(\mathbf{x}) = q(r_1(p(\mathbf{x}), \mathbf{x}, \mathbf{y}), \dots, r_m(p(\mathbf{x}), \mathbf{x}, \mathbf{y})),$$
$$p(\mathbf{y}) = q(r_1(p(\mathbf{y}), \mathbf{x}, \mathbf{y}), \dots, r_m(p(\mathbf{y}), \mathbf{x}, \mathbf{y}))$$

and for each $j \in \{1, \dots, m\}$ there exists $i \in \{1, \dots, n\}$ with

$$r_j(p(\mathbf{x}), \mathbf{x}, \mathbf{y}) = s_j(x_i, \mathbf{x}, \mathbf{y}) \quad \text{and} \quad r_j(p(\mathbf{y}), \mathbf{x}, \mathbf{y}) = s_j(y_i, \mathbf{x}, \mathbf{y}).$$

The proof is word for word analogous to that of Theorem 1 only Lemma 2 is applied instead of Lemma 1.

Example 3. Every variety of unary algebras has distributive $\mathrm{Ref}(A)$ for each $A \in V$.

Without loss of generality, $p(\mathbf{x}) = p_0(x_1)$ for some unary term p_0 and $s_1(\mathbf{x}) = p_0(x)$.

Remark. If a variety V is congruence-permutable, then $\mathrm{Con}\, A = \mathrm{Ref}(A)$ for each $A \in V$, see [9] ($\mathrm{Con}\, A$ denotes the congruence lattice of A). Therefore, $\mathrm{Ref}(A)$ is distributive e.g. for every Boolean algebra A. However, we can give also the explicit boolean terms satisfying (2) of Theorem 3:

Let p be an n-ary boolean term. We can put $m = n$, $q = p$ and for every $j = 1, \ldots, n$, $s_j(z, \mathbf{x}, \mathbf{y}) = [(x \vee y) \wedge z] \vee (x \wedge y)$ and $r_j(z, \mathbf{x}, \mathbf{y}) = \{[(x \vee y) \wedge x_j] \vee (y \wedge z) \vee [x \wedge (x \oplus y \oplus z)]\} \wedge \{[(x \vee y) \wedge y_j] \vee [(y \vee z) \wedge (x \vee (x \oplus y \oplus z))]\}$, where $a \oplus b = (a \wedge b') \vee (a' \wedge b)$ and $x = p(\mathbf{x})$, $y = p(\mathbf{y})$. It is an easy exercise to verify (2) of Theorem 3.

However, we are able to give a more general example:

Example 4. For every distributive lattice L, $\mathrm{Ref}(L)$ is distributive.

Denote by D the variety of all distributive lattices. Since every n-ary term p over D arises by a finite number of lattice operations $\vee$ and $\wedge$, we can prove the existence of q, r_i, s_i satisfying (2) of Theorem 3 by induction over the rank of the term p. Hence, it suffices to show it for two cases, namely $p(x_1, x_2) = x_1 \vee x_2$ and $p(x_1, x_2) = x_1 \wedge x_2$.

(a) Let $p(x_1, x_2) = x_1 \vee x_2$. Put $n = 2$, $q = p$, $s_i(z, \mathbf{x}, \mathbf{y}) = [(x \vee y) \wedge z] \vee (x \wedge y)$, $r_i(z, \mathbf{x}, \mathbf{y}) = \{x \wedge z \wedge [((x \vee y) \wedge x_i) \vee (x \wedge y)]\} \vee \{y \wedge z \wedge [((x \vee y) \wedge y_i) \vee (x \wedge y)]\}$, where $x = p(x_1, x_2)$, $y = p(y_1, y_2)$. Hence $x_i \leqslant x, y_i \leqslant y$, thus also $(x \wedge x_i) \vee (x \wedge y) = ((x \vee y) \wedge x_i) \vee (x \wedge y)$, i.e. $r_i(x, \mathbf{x}, \mathbf{y}) = s_i(x_i, \mathbf{x}, \mathbf{y})$, analogously $r_i(y, \mathbf{x}, \mathbf{y}) = s_i(y_i, \mathbf{x}, \mathbf{y})$. It is easy to show that

$$x_1 \vee x_2 = s_1(x_1, \mathbf{x}, \mathbf{y}) \vee s_2(x_2, \mathbf{x}, \mathbf{y}),$$
$$y_1 \vee y_2 = s_1(y_1, \mathbf{x}, \mathbf{y}) \vee s_2(y_2, \mathbf{x}, \mathbf{y}),$$

thus (2) of Theorem 3 is satisfied.

(b) If $p(x_1, x_2) = x_1 \wedge x_2$, then we can choose r_i dually to the case (a); s_i is clearly self-dual. Moreover, $x = x_1 \wedge x_2$, $y = y_1 \wedge y_2$ gives $x \leqslant x_i$, $y \leqslant y_i$, thus (2) of Theorem 3 can be shown dually to (a). By induction over the rank of the term p, it can be generalized for any term p over D.

Corollary. *Let V be a non-trivial variety of lattices. The following conditions are equivalent:*

(1) Ref(L) *is distributive for each* $L \in V$;

(2) V *is a variety of all distributive lattices.*

Proof. (2) $\Rightarrow$ (1) By Example 4. Conversely, let V be a nontrivial lattice variety which is not a variety of distributive lattices. Then V contains at least one of the lattices N_5 or M_3, i.e. V contains at least one of the lattices L_1, L_2 in Fig. 1:

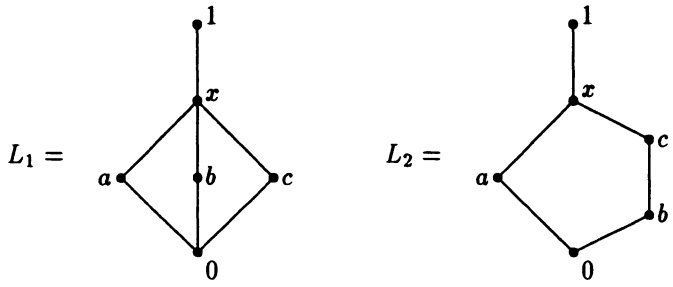

Fig. 1

Denote by B the subset $\{0, a, b, c, x\}$ and put

$$R_1 = \{1, x, a\}^2 \cup B^2,\ R_2 = \{1, x, b\}^2 \cup B^2,\ R_3 = \{1, x, c\}^2 \cup B^2.$$

It is an easy exercise to show that for L_1, $\{R_1 \wedge R_2 \wedge R_3, R_1, R_2, R_3, R_1 \vee R_2 \vee R_3\}$ forms a sublattice of Ref(L_1) isomorphic to M_3. In the case of L_2, $R_3 \subseteq R_2$ and $\{R_3 \wedge R_1, R_1, R_2, R_3, R_1 \vee R_2\}$ forms a sublattice of Ref(L_2) isomorphic to N_5. Hence neither Ref(L_1) nor Ref(L_2) are distributive. □

We can proceed to characterize the varieties with modular lattices of reflexive relations.

Applying Lemma 2 instead of Lemma 1, we can also prove similarly as in the case of Theorem 2:

Theorem 4. *For a variety V, the following conditions are equivalent:*

(1) Ref(A) *is modular for each* $A \in V$;

(2) *For every n-ary term p and each $k \in \{1, \ldots, n\}$ there exist an m-ary term q, $(2n+1+k)$-ary terms w_j, $(2n+k)$-ary t_j and $(3n-k)$-ary g_j ($j = 1, \ldots, m$) such that $p(\mathbf{x}) = q(u_1, \ldots, u_m)$, $p(\mathbf{y}) = q(v_1, \ldots, v_m)$ where for each j either*

$$\begin{aligned} u_j &= w_j(p(\mathbf{x}), x_1, \ldots, x_k, \mathbf{x}, \mathbf{y}) = g_j(x_{k+1}, \ldots, x_n, \mathbf{x}, \mathbf{y}), \\ v_j &= w_j(p(\mathbf{y}), y_1, \ldots, y_k, \mathbf{x}, \mathbf{y}) = g_j(y_{k+1}, \ldots, y_n, \mathbf{x}, \mathbf{y}) \\ \text{or} \quad u_j &= t_j(x_1, \ldots, x_k, \mathbf{x}, \mathbf{y}), \qquad v_j = t_j(y_1, \ldots, y_k, \mathbf{x}, \mathbf{y}). \end{aligned}$$

It can be shown that every variety of groups (or quasigroups) has modular $\mathrm{Ref}(A)$ for each $A \in V$. However, this is an easy corollary of [5] since groups are congruence-permutable, thus $\mathrm{Ref}(A) = \mathrm{Con}\, A$, see [9].

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