Czechoslovak Mathematical Journal

Jozef Džurina

Asymptotic properties of third order delay differential equations

Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 3, 443–448

Persistent URL: http://dml.cz/dmlcz/128546

Terms of use:

© Institute of Mathematics AS CR, 1995

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

ASYMPTOTIC PROPERTIES OF THIRD ORDER DELAY DIFFERENTIAL EQUATIONS

JOZEF DŽURINA, Košice

(Received July 12, 1993)

We consider the delay differential equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}\left(\frac{u(t)}{r_0(t)}\right)'\right)'\right)' - p(t)u(\tau(t)) = 0. \tag{1}$$

We always assume that

(i) $r_i(t)$, $0 \leqslant i \leqslant 2$, $\tau(t)$ and $p(t)$ are continuous on $[t_0, \infty)$, $r_i(t) > 0$, $p(t) > 0$, $\tau(t) < t$, $\tau(t) \to \infty$ as $t \to \infty$ and $\tau(t)$ is increasing;

(ii) $R_i(t) = \displaystyle\int_{t_0}^{t} \frac{\mathrm{d}s}{r_i(s)} \to \infty$ as $t \to \infty$ for $i = 1$ and 2.

For the sake of convenience we introduce the following functions:

$$\begin{aligned} L_0u(t) &= \frac{u(t)}{r_0(t)}, \\ L_iu(t) &= \frac{1}{r_i(t)}\frac{\mathrm{d}}{\mathrm{d}t}L_{i-1}u(t), \quad i = 1 \text{ and } 2, \\ L_3u(t) &= \frac{\mathrm{d}}{\mathrm{d}t}L_2u(t). \end{aligned}$$

By a solution of (1) we mean any function $u : [T_u, \infty) \to \mathbb{R}$ satisfying (1) on $[T_u, \infty)$ such that $L_iu(t)$, $0 \leqslant i \leqslant 3$, exist and are continuous on $[T_u, \infty)$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

The asymptotic behavior of the solutions of (1) is described in the following lemma which is a generalization of a lemma of Kiguradze [2, Lemma 3].

Lemma 1. *Let $u(t)$ be a nonoscillatory solution of* (1). *Then there exist an integer ℓ, $\ell \in \{1,3\}$ and $t_1 \geqslant t_0$ such that*

$$\begin{aligned} u(t)L_i u(t) > 0, &\quad 0 \leqslant i \leqslant \ell, \\ (-1)^{i-\ell} u(t) L_i u(t) > 0, &\quad \ell \leqslant i \leqslant 3 \end{aligned} \tag{2}$$

for all $t \geqslant t_1$.

A function $u(t)$ satisfying (2) is said to be a function of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then by Lemma 1

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3.$$

Following Kiguradze we say that equation (1) has property (B) if $\mathcal{N} = \mathcal{N}_3$.

In a recent paper [3] Kusano and Naito have presented a useful comparison principle which under conditions (i) and (ii) enables us to deduce property (B) of a delay equation of the form (1) from that of the ordinary differential equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}\left(\frac{u(t)}{r_0(t)}\right)'\right)'\right)' - \frac{p(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}u(t) = 0,$$

where $\tau^{-1}(t)$ is the inverse function to $\tau(t)$. The objective of this paper is to show that this comparison theorem may fail and then it is a good idea to compare equation (1) with the first order delay equation

$$y'(t) + q(t)y(w(t)) = 0, \tag{E}$$

where $w(t) \not\equiv \tau(t)$. We present the relationship between property (B) of equation (1) and the oscillation of equation (E).

In the sequel we shall consider the functions $g(t)$ and $w(t)$ satisfying

$$g(t) \in C([t_0, \infty)), \quad g(t) > t, \quad w(t) = \tau(g(t)) < t. \tag{3}$$

For the sake of convenience and further references we make use of the following notation:

$$q(t) = r_2(t) \int_t^{g(t)} p(s) r_0(\tau(s))(R_1(\tau(s)) - R_1(t_1))\,\mathrm{d}s \tag{4}$$

for sufficiently large t with $\tau(t) > t_1$.

Theorem 1. *Let* (3) *hold. Assume that the linear differential inequality*

$$(\widetilde{\mathrm{E}}) \qquad y'(t) + q(t)y(w(t)) \leqslant 0$$

has no eventually positive solutions. Then equation (1) *has property* (B).

Proof. By way of contradiction we assume that (1) has a nonoscillatory solution $u(t)$ which belongs to the class N_1. We may assume that $u(t)$ is positive. Then by Lemma 1, there exists a t_1 such that

$$L_0u(t) > 0, \quad L_1u(t) > 0, \quad L_2u(t) < 0 \quad \text{and} \quad L_3u(t) > 0$$

for $t \geqslant t_1$. Integration of the identity $L_1u(t) = L_1u(t)$ from t_1 to t leads to

$$L_0u(t) \geqslant \int_{t_1}^{t} r_1(s)L_1u(s)\,\mathrm{d}s, \quad t \geqslant t_1. \tag{5}$$

On the other hand, integrating the identity $L_3u(t) = L_3u(t)$ from t ($\geqslant t_1$) to ∞ one gets

$$-L_2u(t) \geqslant \int_{t}^{\infty} L_3u(s)\,\mathrm{d}s = \int_{t}^{\infty} p(s)u(\tau(s))\,\mathrm{d}s. \tag{6}$$

Combining (5) with (6) we get

$$\begin{aligned} -L_2u(t) &\geqslant \int_{t}^{\infty} p(s)r_0(\tau(s)) \int_{t_1}^{\tau(s)} r_1(x)L_1u(x)\,\mathrm{d}x\,\mathrm{d}s \\ &\geqslant \int_{t}^{g(t)} p(s)r_0(\tau(s)) \int_{t_1}^{\tau(s)} r_1(x)L_1u(x)\,\mathrm{d}x\,\mathrm{d}s, \quad t \geqslant t_2, \end{aligned}$$

where $t_2 \geqslant t_1$ is large enough. Since $L_1u(t)$ is decreasing we have in view of the last inequalities

$$-L_2u(t) \geqslant \int_{t}^{g(t)} p(s)r_0(\tau(s))L_1u(\tau(s))(R_1(\tau(s)) - R_1(t_1))\,\mathrm{d}s.$$

Hence, as $L_1u(t)$ is decreasing and $\tau(t)$ is increasing, the last inequalities imply

$$-L_2u(t) \geqslant L_1u(w(t)) \int_{t}^{g(t)} p(s)r_0(\tau(s))(R_1(\tau(s)) - R_1(t_1))\,\mathrm{d}s.$$

Put $z(t) = L_1u(t)$. Obviously, $z(t) > 0$ and $z(t)$ satisfies

$$-\frac{z'(t)}{r_2(t)} \geqslant z(w(t))\frac{q(t)}{r_2(t)}, \quad t \geqslant t_2,$$

which implies that $z(t)$ is a positive solution of the differential inequality

$$z'(t) + q(t)z(w(t)) \leqslant 0, \quad t \geqslant t_2,$$

which contradicts hypothesis. The proof is complete. □

Corollary 1. *Let* (3) *hold. Assume that either*

$$\liminf_{t\to\infty} \int_{w(t)}^{t} q(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}$$

or

$$\limsup_{t\to\infty} \int_{w(t)}^{t} q(s)\,\mathrm{d}s > 1.$$

Then equation (1) *has property* (B).

Proof. It is known (see [4]) that both conditions are sufficient for $(\widetilde{\mathrm{E}})$ to have no positive solutions. Our assertion follows from Theorem 1. □

Corollary 2. *Let* (3) *hold. Assume that the differential equation*

$$(E) \qquad y'(t) + q(t)y(w(t)) = 0$$

is oscillatory. Then equation (1) *has property* (B).

Proof. Corollary 2 follows from Theorem 1 and the fact that $(\widetilde{\mathrm{E}})$ has no positive solutions if and only if (E) is oscillatory (see [1]). □

Corollary 3. *Let* (3) *hold. For all large t define*

$$\tilde{q}(t) = r_2(t) \int_{t}^{g(t)} p(s) r_0(\tau(s)) R_1(\tau(s))\,\mathrm{d}s.$$

Assume that either

$$\liminf_{t\to\infty} \int_{w(t)}^{t} \tilde{q}(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{7}$$

or

$$\limsup_{t\to\infty} \int_{w(t)}^{t} \tilde{q}(s)\,\mathrm{d}s > 1. \tag{8}$$

Then equation (1) *has property* (B).

In the following illustrative example we show how to choose the function $g(t)$ satisfying $g(t) > t$ and $\tau(g(t)) < t$.

Example 1. Let us consider the third order differential equation

$$y'''(t) - \frac{a}{t^2\sqrt{t}} y(\sqrt{t}) = 0, \quad a > 0, \quad \text{and} \quad t \geqslant 1. \tag{9}$$

We have $\tau(t) = \sqrt{t}$. Let us put $g(t) = 2t$. Then $w(t) = \sqrt{2t}$ satisfies (3) and

$$\tilde{q}(t) = \int_t^{2t} \frac{a}{s^2\sqrt{s}} \sqrt{s}\, \mathrm{d}s = \frac{a}{2t}.$$

By Corollary 3 equation (9) has property (B) if

$$\liminf_{t\to\infty} \int_{\sqrt{2t}}^{t} \tilde{q}(s)\, \mathrm{d}s > \frac{1}{\mathrm{e}}. \tag{10}$$

Simple computation shows that (10) holds for any $a > 0$. Note that we obtain the same result if we choose $g(t) = \alpha^2 t^2$, $0 < \alpha < 1$. On the other hand, by the above-mentioned result of Kusano and Naito one gets that (9) has property (B) if the ordinary equation without delay

$$y'''(t) - \frac{2a}{t^4} y(t) = 0 \tag{11}$$

has property (B). However, as (11) has not property (B), the criterion of Kusano and Naito fails for (9).

For a special choice of the function $g(t)$ we have the following result:

Theorem 2. *Suppose that*

$$\limsup_{t\to\infty} \frac{\tau(t)}{t} < c < 1. \tag{12}$$

Assume that either

$$\liminf_{t\to\infty} \int_{ct}^{t} r_2(s) \int_s^{\tau^{-1}(cs)} p(x) r_0(\tau(x)) R_1(\tau(x))\, \mathrm{d}x\, \mathrm{d}s > \frac{1}{\mathrm{e}} \tag{13}$$

or

$$\limsup_{t\to\infty} \int_{ct}^{t} r_2(s) \int_s^{\tau^{-1}(cs)} p(x) r_0(\tau(x)) R_1(\tau(x))\, \mathrm{d}x\, \mathrm{d}s > 1. \tag{14}$$

Then equation (1) *has property* (B).

P r o o f. It is easy to verify that (12) is equivalent to

$$\limsup_{t\to\infty} \frac{t}{\tau^{-1}(t)} < c < 1,$$

which implies that for all large t

$$\tau^{-1}(t) > \frac{t}{c}. \tag{15}$$

Put $g(t) = \tau^{-1}(ct)$. Then $g(t)$ satisfies (3) as $w(t) = \tau(g(t)) = ct < t$ and in view of (15) we have $g(t) > t$. Noting that (7) and (8) are equivalent to (13) and (14), respectively, the assertion of this theorem follows from Corollary 3. The proof is complete. □

References

[1] *I. Győri, G. Ladas*: Oscillation theory of delay differential equations. Clarendon press, Oxford, 1991.

[2] *I. T. Kiguradze*: On the oscillation of solutions of the equation $\mathrm{d}^m u/\,\mathrm{d}t^m + a(t)|u|^n \times \operatorname{sign} u = 0$. Mat. Sb *65* (1964), 172–187. (In Russian.)

[3] *T. Kusano and M. Naito*: Comparison theorems for functional differential equations with deviating arguments. J. Math. Soc. Japan *3* (1981), 509–532.

[4] *G. S. Ladde, V. Lakshmikantham, B. G. Zhang*: Oscillation theory of differential equations with deviating arguments. Dekker, New York, 1987.

Author's address: Department of Mathematical Analysis, Šafárik University, Jesenná 5, 041 54 Košice, Slovakia.