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MORPHISMS OF A CERTAIN CLASS OF MONOUNARY ALGEBRAS

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1. Introduction

The power of monounary algebras was studied in [4]. The present paper is dealing with the so-called bipower (the carrier is the set of all bijective homomorphisms of the given monounary algebras), monopower (the carier is the set of injective homomorphisms) and epipower (the carier is the set of surjective homomorphisms) of algebras. Further, isomorphic embeddings and homomorphic images are studied, similarly as Birkhoff [2] did for ordered sets. We investigate their connection to operations of addition, multiplication and exponentiation for monounary algebras of a given class. Results are presented in the form of explicit formulas for the number and the type of the resulting algebras and in the form of rules which hold for the studied relations.

2. Basic notions

The cardinal number of a set M is denoted by the symbol $|M|$. The ordered pair $\boldsymbol{A} = (A, f)$, where A is a set and f a mapping of A into itself, is called a monounary algebra.

We study the class $\mathfrak{A}$ of monounary algebras consisting of a finite number of cyclic components. Compare [4]. The type $t(\boldsymbol{A})$ of any algebra $\boldsymbol{A}$ of the studied class $\mathfrak{A}$ can be expressed in the canonical form of a polynomial $a_1\boldsymbol{1} + a_2\boldsymbol{2} + \ldots + a_m\boldsymbol{m}$ meaning that the algebra $\boldsymbol{A}$ has exactly a_i i-element cycles for any i with $1 \leqq i \leqq m$ and no other elements.

The mapping $h\colon A \to B$ is called a homomorphism of $\boldsymbol{A} = (A, f)$ into $\boldsymbol{B} = (B, g)$ iff $h(f(x)) = g(h(x))$ for any $x \in A$. By $\operatorname{Hom}(\boldsymbol{A}, \boldsymbol{B})$ we denote the set of all homomorphisms of $\boldsymbol{A}$ into $\boldsymbol{B}$. An isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}$ is called an automorphism. If $\boldsymbol{A}$, $\boldsymbol{B}$ are isomorphic algebras, we write $\boldsymbol{A} \cong \boldsymbol{B}$. The set of all automorphisms on the algebra $\boldsymbol{A}$ is denoted by $\operatorname{Aut}\boldsymbol{A}$.

The sum $\boldsymbol{A} + \boldsymbol{B}$ of the algebras $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g) \in \mathfrak{A}$ where $A \cap B = \emptyset$ is defined to be the algebra $\boldsymbol{C} = (C, h)$ such that $C = A \cup B$, $h = f \cup g$.

By the product $\boldsymbol{A} \cdot \boldsymbol{B}$ of the algebras $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g) \in \mathfrak{A}$ we mean the algebra $\boldsymbol{C} = (C, h)$ such that $C = A \times B$ and $h(a, b) = (f(a), g(b))$ for any $(a, b) \in C$.

The power $\boldsymbol{A}^{\boldsymbol{B}}$ of the algebras $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g) \in \mathfrak{A}$ is defined to be the algebra $\boldsymbol{C} = (C, h)$ such that $C = \operatorname{Hom}(\boldsymbol{B}, \boldsymbol{A})$ and $h(\varphi) = \varphi \cdot g$ for any $\varphi \in C$.

By $\operatorname{Comp} \boldsymbol{A}$ we denote the set of all components of the algebra $\boldsymbol{A}$. Compare [4].

Let $(N, /)$ be the set of all positive integers ordered by divisibility, let $K \subset N$. Then we put $\min K = \{n \in K;$ for any $m \in K$, m/n implies $m = n\}$ which denotes the set of minimal elements of the ordered set $(K, /)$.

1. Let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathfrak{A}$. Then $\operatorname{Hom}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$ iff for any $m \in \min\{|T|;\ \boldsymbol{T} \in \operatorname{Comp} \boldsymbol{A}\}$ there is $n \in \min\{|T'|;\ \boldsymbol{T}' \in \operatorname{Comp} \boldsymbol{B}\}$ such that n/m.
Proof: $\operatorname{Hom}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$ iff for any $\boldsymbol{T} \in \operatorname{Comp} \boldsymbol{A}$ there is $\boldsymbol{T}' \in \operatorname{Comp} \boldsymbol{B}$ such that $|T'|/|T|$. This is equivalent to the following condition: For any $m \in \min\{|T|;\ \boldsymbol{T} \in \operatorname{Comp} \boldsymbol{A}\}$ there is $n \in \min\{|T'|;\ \boldsymbol{T}' \in \operatorname{Comp} \boldsymbol{B}\}$ such that n/m. □

2. Let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathfrak{A}$, $t(\boldsymbol{A}) = \boldsymbol{m}$, $t(\boldsymbol{B}) = \boldsymbol{n}$, $m, n > O$. Then $\boldsymbol{m} \cdot \boldsymbol{n} = \text{g.c.d.}\,(m, n)\ \textbf{l.c.m.}\,(\boldsymbol{m}, \boldsymbol{n})$, where g.c.d. means the greatest common divisor and l.c.m. the least common multiple.

For any positive integers $i_1, i_2, \ldots, i_k$ we have

$$\boldsymbol{i}_1 \cdot \boldsymbol{i}_2 \ldots \boldsymbol{i}_k = \frac{i_1 \cdot i_2 \ldots i_k}{[i_1, i_2, \ldots, i_k]} \qquad [\boldsymbol{i}_1, \boldsymbol{i}_2, \ldots, \boldsymbol{i}_k],$$

where $[i_1, i_2, \ldots, i_k]$ denotes the l.c.m. of $i_1, i_2, \ldots, i_k$.
Proof: The product $\boldsymbol{A} \cdot \boldsymbol{B}$ contains the elements (a_i, b_j), $1 \leqq i \leqq m$, $1 \leqq j \leqq n$. Each element (a_i, b_j) lies in a cycle whose number of elements is l.c.m. (m, n). Since $m \cdot n = \text{g.c.d.}\,(m, n) \cdot \text{l.c.m.}\,(m, n)$, the type of $\boldsymbol{A} \cdot \boldsymbol{B}$ is g.c.d. (m, n) **l.c.m.** $(\boldsymbol{m}, \boldsymbol{n})$. The following assertion can be proved by induction. □

3. Bimorphisms, monomorphisms and epimorphisms

A bijective homomorphism, (isomorphism in fact) of $\boldsymbol{A}$ into $\boldsymbol{B}$ is called a bimorphism. The symbol $\operatorname{Bi}(\boldsymbol{A}, \boldsymbol{B})$ denotes the set of all bimorphisms of $\boldsymbol{A}$ into $\boldsymbol{B}$.

1. Let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathfrak{A}$. Then, clearly, we have
(i) $\operatorname{Bi}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$ iff $\boldsymbol{A} \cong \boldsymbol{B}$,
(ii) $\operatorname{Bi}(\boldsymbol{A}, \boldsymbol{B}) \cong \operatorname{Aut} \boldsymbol{A}$.

2. Let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathfrak{A}$, $t(\boldsymbol{A}) = a_1\boldsymbol{1} + a_2\boldsymbol{2} + \ldots + a_m\boldsymbol{m} = t(\boldsymbol{B})$. Then

$$|\operatorname{Bi}(\boldsymbol{A}, \boldsymbol{B})| = \prod_{1 \leqq i \leqq m} a_i!\, i^{a_i}.$$

Proof: The assertion follows from 3.1 since the number of automorphisms is given by the same formula. See [4], 5.5. □

By the bipower **Bi** $(\boldsymbol{A}, \boldsymbol{B})$ of the algebras $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g) \in \mathfrak{A}$ we mean the algebra $\boldsymbol{C} = (C, h)$ such that $C = \mathrm{Bi}(\boldsymbol{A}, \boldsymbol{B})$ and $h(\varphi) = \varphi \cdot f$ for any $\varphi \in C$. Regarding 3.1 the algebra **Bi** $(\boldsymbol{A}, \boldsymbol{B})$ is isomorphic to the algebra **Aut** $\boldsymbol{A}$ which is defined analogously.

3. For the positive integers a, m, n, $m \neq n$ we have

(i) $\mathbf{Aut}\,\boldsymbol{m} \cong \boldsymbol{m}$,

(ii) $\mathbf{Aut}\,(\boldsymbol{m} + \boldsymbol{n}) \cong \text{g.c.d.}\,(m, n)\ \mathbf{l.c.m.}\,(\boldsymbol{m}, \boldsymbol{n})$,

(iii) $\mathbf{Aut}\,(a\boldsymbol{m}) \cong a!\, m^{a-1} \boldsymbol{m}$.

Proof: (i) is evident; (ii) follows from (i) and 2.2 taking into account that the number of automorphisms is $m \cdot n$; (iii) follows from the fact that automorphisms, whose number is $a!\, m^a$, turn out to be algebras of the type $\boldsymbol{m}$. □

4. Let $\boldsymbol{A} \in \mathfrak{A}$, $t(\boldsymbol{A}) = a_1\mathbf{1} + a_2\mathbf{2} + \ldots + a_m\boldsymbol{m}$. Then

$$\mathbf{Aut}\,\boldsymbol{A} \cong \frac{\prod\limits_{1 \leqq i \leqq m} a_i!\, i^{a_i}}{[I]} \qquad [\boldsymbol{I}],$$

where $[I]$ is the least common multiple of elements of the set $I = \{i;\ 1 \leqq i \leqq m,\ a_i > O\}$.

Proof: The resulting algebra is of the type which is the least common multiple of types of nonzero cycles of the given algebra (compare 3.3 (ii)). Now, the assertion follows from 3.2 and 3.1 (ii). □

An injective homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ is called a monomorphism. The symbol $\mathrm{Mon}(\boldsymbol{A}, \boldsymbol{B})$ denotes the set of all monomorphisms of $\boldsymbol{A}$ into $\boldsymbol{B}$.

5. Let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathfrak{A}$ be connected algebras (having just one component). Then, clearly, $\mathrm{Mon}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$ iff $\boldsymbol{A} \cong \boldsymbol{B}$.

6. Let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathfrak{A}$, $t(\boldsymbol{A}) = a_1\mathbf{1} + a_2\mathbf{2} + \ldots + a_m\boldsymbol{m}$, $t(\boldsymbol{B}) = b_1\mathbf{1} + b_2\mathbf{2} + \ldots + b_n\boldsymbol{n}$. Then, clearly, $\mathrm{Mon}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$ iff $m \leqq n$, $a_i \leqq b_i$, $1 \leqq i \leqq m$.

7. Let $\boldsymbol{A}$, $\boldsymbol{B} \in \mathfrak{A}$, $t(\boldsymbol{A}) = a_1\mathbf{1} + a_2\mathbf{2} + \ldots + a_m\boldsymbol{m}$, $t(\boldsymbol{B}) = b_1\mathbf{1} + b_2\mathbf{2} + \ldots + b_n\boldsymbol{n}$. If $\mathrm{Mon}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$, then

$$|\mathrm{Mon}(\boldsymbol{A}, \boldsymbol{B})| = \prod_{1 \leqq i \leqq m} V_{a_i}(b_i)\, i^{a_i},$$

where $V_a(b)$ is the number of a-tuples formed of different elements of a set with b-elements.

Proof: The number of injective mappings of an a-element set into a b-element set is equal to the number of a-tuples formed of different elements of a set with b-elements. From this and from 3.2 the assertion follows. □

By the monopower **Mon** $(\boldsymbol{A}, \boldsymbol{B})$ of the algebras $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g) \in \mathfrak{A}$ we mean the algebra $\boldsymbol{C} = (C, h)$ such that $C = \mathrm{Mon}(\boldsymbol{A}, \boldsymbol{B})$ and $h(\varphi) = \varphi \cdot f$ for any $\varphi \in C$.

8. For the positive integers a, b, c, d, m, n we have

(i) $\mathbf{Mon}(\boldsymbol{m}, \boldsymbol{n}) \cong \begin{cases} \boldsymbol{m} \text{ if } m = n, \\ \boldsymbol{0} \text{ otherwise;} \end{cases}$

(ii) $\mathbf{Mon}(\boldsymbol{m}, a\boldsymbol{n}) \cong \begin{cases} a\boldsymbol{m} \text{ if } m = n \\ \boldsymbol{0} \text{ otherwise;} \end{cases}$

(iii) $\mathbf{Mon}(a\boldsymbol{m}, b\boldsymbol{n}) \cong \begin{cases} V_a(b)\, m^{a-1}\boldsymbol{m} \text{ if } m = n,\ a \leqq b, \\ \boldsymbol{0} \text{ otherwise;} \end{cases}$

(iv) If $m \neq n$, then

$$\mathbf{Mon}(a\boldsymbol{m} + b\boldsymbol{n}, c\boldsymbol{m} + d\boldsymbol{n}) \cong \begin{cases} V_a(c) \cdot V_b(d)\, m^{a-1} n^{b-1} \\ \text{g.c.d. } (m, n)\ \textbf{l.c.m.}\ (\boldsymbol{m}, \boldsymbol{n}) \\ \text{if } a \leqq c,\ b \leqq d; \\ \boldsymbol{0} \text{ otherwise.} \end{cases}$$

Proof: (i) follows from 3.5; (ii) follows from (i); (iii) follows from (ii) and 3.7, since the type of the resulting algebra is $\boldsymbol{m}$; (iv) follows from (iii) and from the multiplicative rule (compare 2.2). □

9. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}$, $t(\boldsymbol{A}) = a_1\mathbf{1} + a_2\mathbf{2} + \ldots + a_m\boldsymbol{m}$, $t(\boldsymbol{B}) = b_1\mathbf{1} + b_2\mathbf{2} + \ldots + b_n\boldsymbol{n}$. If $\mathrm{Mon}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$, then

$$\mathbf{Mon}(\boldsymbol{A}, \boldsymbol{B}) \cong \frac{\prod_{1 \leqq i \leqq m} V_{a_i}(b_i)\, i^{a_i}}{[I]} \quad [\boldsymbol{I}],$$

where $[I]$ is the lowest common multiple of elements of the set $I = \{i;\ 1 \leqq i \leqq m,\ a_i > O\}$.

Proof: The assertion follows from 3.7 and 3.8. Compare the explanation in the proof of 3.4. □

A surjective homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$ is called an epimorphism. The symbol $\mathrm{Ep}(\boldsymbol{A}, \boldsymbol{B})$ denotes the set of all epimorphisms of $\boldsymbol{A}$ onto $\boldsymbol{B}$.

10. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}$ be connected algebras. Then, clearly $\mathrm{Ep}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$ iff $|B|/|A|$.

11. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}$, $t(\boldsymbol{A}) = a_1\mathbf{1} + a_2\mathbf{2} + \ldots + a_m\boldsymbol{m}$, $t(\boldsymbol{B}) = b_1\mathbf{1} + b_2\mathbf{2} + \ldots + b_n\boldsymbol{n}$. If $\mathrm{Ep}(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$, then

$$\sum_{1 \leqq i \leqq m} a_i \geqq \sum_{1 \leqq j \leqq n} b_j .$$

Proof: The assertion follows from the fact that any epimorphism maps components of $\boldsymbol{A}$ onto components of $\boldsymbol{B}$ satisfying the condition of divisibility (see 3.10). □

By the epipower $\mathbf{Ep}(\boldsymbol{A}, \boldsymbol{B})$ of the algebras $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g) \in \mathfrak{A}$ we mean the algebra $\boldsymbol{C} = (C, h)$ such that $C = \mathrm{Ep}(\boldsymbol{A}, \boldsymbol{B})$ and $h(\varphi) = \varphi \cdot f$ for any $\varphi \in C$.

12. For the positive integers a, m, n, p, q we have

$$\text{(i)} \quad \mathbf{Ep}(\boldsymbol{m}, \boldsymbol{n}) \cong \begin{cases} \boldsymbol{n} \text{ if } n/m; \\ \boldsymbol{O} \text{ otherwise.} \end{cases}$$

$$\text{(ii)} \quad \mathbf{Ep}(a\boldsymbol{m}, \boldsymbol{n}) \cong \begin{cases} n^{a-1}\boldsymbol{n} \text{ if } n/m; \\ \boldsymbol{O} \text{ otherwise.} \end{cases}$$

$$\text{(iii)} \quad \mathbf{Ep}(\boldsymbol{m} + \boldsymbol{n}, \boldsymbol{p} + \boldsymbol{q}) \cong \begin{cases} 2 \text{ g.c.d. } (p, q) \ \mathbf{l.c.m.}\ (\boldsymbol{p}, \boldsymbol{q}) \\ \text{if } p/m,\ p/n,\ q/m,\ q/n; \\ \text{g.c.d. } (p, q) \ \mathbf{l.c.m.}\ (\boldsymbol{p}, \boldsymbol{q}) \\ \text{if } p/m,\ p\not| n,\ q/n, \\ \text{or } p\not| m,\ p/n,\ q/m, \\ \text{or } p/m,\ p/n,\ q/m,\ q\not| n, \\ \text{or } p/m,\ p/n,\ q\not| m,\ q/n; \\ \boldsymbol{O} \text{ otherwise.} \end{cases}$$

Proof: (i) follows from 3.10; (ii) follows from (i) taking into account that the number of epimorphisms is n^a. If we consider, for instance, that $p/m, p/n, q/n$ we obtain $\boldsymbol{p} \cdot \boldsymbol{q}$ epimorphisms. Thus, by 2.2, this equals g.c.d. (p, q) **l.c.m.** $(\boldsymbol{p}, \boldsymbol{q})$. In the case when both p and q divide m and n, we obtain twice the number of epimorphisms. In the remaining cases there does not exist any epimorphism. Together, we have (iii). □

Let P be the set of p elements, Q the set of q elements. By the symbol $S(p, q)$ we denote the number of surjective mappings of the set P onto the set Q.

13. For the positive integers p, q we have

$$S(p, q) = \sum_{j=1}^{q} (-1)^{q-j} \binom{q}{j} j^p.$$

Compare [1] pp. 106 and 121 and [3] p. 109.
If $p = q$, the above mentioned formula gives $S(p, p) = p!$ and in the case of $p < q$ we obtain $S(p, q) = O$. Compare [3] p. 44.

14. For the positive integers $a, b, m, n, a > b$ we have

$$\mathbf{Ep}(a\boldsymbol{m}, b\boldsymbol{n}) \cong \begin{cases} n^{a-1} \sum_{i=1}^{b} (-1)^{b-i} \binom{b}{i} i^a \boldsymbol{n} & \text{if } n/m, \\ \boldsymbol{O} \text{ otherwise.} \end{cases}$$

Proof: The assertion follows from 3.12 (ii) and 3.13. □

4. Isomorphic embeddings and homomorphic images

Let us denote by the symbol $\boldsymbol{A} \Subset \boldsymbol{B}$ the fact that there is a monomorphism (injective homomorphism) of the algebra $\boldsymbol{A}$ into the algebra $\boldsymbol{B}$.

1. The relation $\Subset$ is a partial ordering on $\mathfrak{A}$. Formally, for any $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \in \mathfrak{A}$ we have

(1) $\boldsymbol{A} \Subset \boldsymbol{A}$,

(2) $\boldsymbol{A} \Subset \boldsymbol{B}$, $\boldsymbol{B} \Subset \boldsymbol{C}$ implies $\boldsymbol{A} \Subset \boldsymbol{C}$,

(3) $\boldsymbol{A} \Subset \boldsymbol{B}$, $\boldsymbol{B} \Subset \boldsymbol{A}$ implies $\boldsymbol{A} \cong \boldsymbol{B}$.

Proof. The reflexivity and transitivity are evident, let us prove the antisymmetry. Let $t(\boldsymbol{A}) = \sum_i a_i \boldsymbol{i}$, $t(\boldsymbol{B}) = \sum_j b_j \boldsymbol{j}$. If $\boldsymbol{A} \Subset \boldsymbol{B}$, then $a_i \leqq b_i$ for any i and if $\boldsymbol{B} \Subset \boldsymbol{A}$, then $b_j \leqq a_j$ for any j (see 3.6). From this we obtain $t(\boldsymbol{A}) = t(\boldsymbol{B})$. □

In general, it does not hold that either $\boldsymbol{A} \Subset \boldsymbol{B}$ or $\boldsymbol{B} \Subset \boldsymbol{A}$ (take, e.g., $t(\boldsymbol{A}) = = \boldsymbol{3}$, $t(\boldsymbol{B}) = \boldsymbol{4}$).

Let us denote by the symbol $\boldsymbol{A} \prec \boldsymbol{B}$ the fact that there is an epimorphism (surjective homomorphism) of the algebra $\boldsymbol{B}$ onto the algebra $\boldsymbol{A}$.

2. Analogously we have: The relation $\prec$ is a partial ordering on $\mathfrak{A}$. Formally, for any $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathfrak{A}$, we have

(4) $\boldsymbol{A} \prec \boldsymbol{A}$,

(5) $\boldsymbol{A} \prec \boldsymbol{B}$, $\boldsymbol{B} \prec \boldsymbol{C}$ implies $\boldsymbol{A} \prec \boldsymbol{C}$,

(6) $\boldsymbol{A} \prec \boldsymbol{B}$, $\boldsymbol{B} \prec \boldsymbol{A}$ implies $\boldsymbol{A} \cong \boldsymbol{B}$.

Since, in general, it does not hold that $\boldsymbol{A} \prec \boldsymbol{B}$ iff $\boldsymbol{A} \Subset \boldsymbol{B}$ (take, e.g., $t(\boldsymbol{A}) = = \boldsymbol{2}$, $t(\boldsymbol{B}) = \boldsymbol{4}$) we shall study the defined relations separately.

3. Both relations are clearly consistent and the one-element cycle is a homomorphic image of every nonempty algebra. Formally, for any $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in \mathfrak{A}$, we have

(7) Let $\boldsymbol{B} \cong \boldsymbol{C}$. Then $\boldsymbol{A} \Subset \boldsymbol{B}$ implies $\boldsymbol{A} \Subset \boldsymbol{C}$ and $\boldsymbol{B} \Subset \boldsymbol{D}$ implies $\boldsymbol{C} \Subset \boldsymbol{D}$.

(8) Let $\boldsymbol{B} \cong \boldsymbol{C}$. Then $\boldsymbol{A} \prec \boldsymbol{B}$ implies $\boldsymbol{A} \prec \boldsymbol{C}$ and $\boldsymbol{B} \prec \boldsymbol{D}$ implies $\boldsymbol{C} \prec \boldsymbol{D}$.

(9) $\boldsymbol{1} \prec \boldsymbol{A}$ for any $\boldsymbol{A} \neq \emptyset$.

The analogous relationship $\boldsymbol{1} \Subset \boldsymbol{A}$ does not hold in general. Take, e.g., $t(\boldsymbol{A}) = \boldsymbol{2}$.

4. The sum is isotone, formally, for any $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in \mathfrak{A}$, we have

(10) If $\boldsymbol{A} \Subset \boldsymbol{B}$, $\boldsymbol{C} \Subset \boldsymbol{D}$, then $\boldsymbol{A} + \boldsymbol{C} \Subset \boldsymbol{B} + \boldsymbol{D}$.

(11) If $\boldsymbol{A} \prec \boldsymbol{B}$, $\boldsymbol{C} \prec \boldsymbol{D}$, then $\boldsymbol{A} + \boldsymbol{C} \prec \boldsymbol{B} + \boldsymbol{D}$.

We have also the following assertions on decompositions.

(12) If $\boldsymbol{A} \Subset \boldsymbol{B} + \boldsymbol{C}$, then $\boldsymbol{A} \cong \boldsymbol{D} + \boldsymbol{E}$ where $\boldsymbol{D} \Subset \boldsymbol{B}$ and $\boldsymbol{E} \Subset \boldsymbol{C}$.

(13) If $\boldsymbol{B} + \boldsymbol{C} \prec \boldsymbol{A}$, then $\boldsymbol{A} \cong \boldsymbol{D} + \boldsymbol{E}$ where $\boldsymbol{B} \prec \boldsymbol{D}$ and $\boldsymbol{C} \prec \boldsymbol{E}$.

Further, for any $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}$, we have

(14) $\boldsymbol{A} \Subset \boldsymbol{A} + \boldsymbol{B}$, $\boldsymbol{B} \Subset \boldsymbol{A} + \boldsymbol{B}$.

The analogous relationship $\boldsymbol{A} \prec \boldsymbol{A} + \boldsymbol{B}$ does not hold in general since in $\boldsymbol{B}$ there might be a component which cannot be homomorphically mapped onto $\boldsymbol{A}$.

However, the following assertion is evident:

(15) $\boldsymbol{A} \Subset \boldsymbol{B}$ implies $\boldsymbol{A} \cong \boldsymbol{B}$ or $\boldsymbol{A} + \boldsymbol{X} \cong \boldsymbol{B}$ for some $\boldsymbol{X} \in \mathfrak{A}$.

5. The product is isotone, formally, for any $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C} \in \mathfrak{A}$, we have

(16) $\boldsymbol{A} \Subset \boldsymbol{B}$ implies $\boldsymbol{A} \cdot \boldsymbol{C} \Subset \boldsymbol{B} \cdot \boldsymbol{C}$,

(17) $\boldsymbol{A} \prec \boldsymbol{B}$ implies $\boldsymbol{A} \cdot \boldsymbol{C} \prec \boldsymbol{B} \cdot \boldsymbol{C}$.

Proof: Let $t(\boldsymbol{A}) = \sum_i a_i \boldsymbol{i}$, $t(\boldsymbol{B}) = \sum_j b_j \boldsymbol{j}$, $t(\boldsymbol{C}) = \sum_k c_k \boldsymbol{k}$. By 2.2 we have

$$t(\boldsymbol{A} \cdot \boldsymbol{C}) = \sum_i \sum_k a_i c_k \text{ g.c.d. } (i, k) \ \textbf{l.c.m.} \ (\boldsymbol{i}, \boldsymbol{k}),$$

$$t(\boldsymbol{B} \cdot \boldsymbol{C}) = \sum_i \sum_k b_j c_k \text{ g.c.d. } (j, k) \ \textbf{l.c.m.} \ (\boldsymbol{j}, \boldsymbol{k}).$$

Since by 3.6 $\boldsymbol{A} \Subset \boldsymbol{B}$ implies $a_i \leqq b_i$ for any i, we obtain the assertion (16).

If the mapping $b \mapsto a$ is a surjective homomorphism of $\boldsymbol{B}$ onto $\boldsymbol{A}$, then, clearly, $(b, c) \mapsto (a, c)$ is a surjective homomorphism of $\boldsymbol{B} \cdot \boldsymbol{C}$ onto $\boldsymbol{A} \cdot \boldsymbol{C}$ and thus we have (17). □

6. For any $\boldsymbol{A}$, $\boldsymbol{B} \in \mathfrak{A}$ we have

(18) $\boldsymbol{A} \prec \boldsymbol{A} \cdot \boldsymbol{B}$.

Proof: The mapping $(a, b) \mapsto a$ is clearly a surjective homomorphism of $\boldsymbol{A} \cdot \boldsymbol{B}$ onto $\boldsymbol{A}$. □

The analogous relationship $\boldsymbol{A} \Subset \boldsymbol{A} \cdot \boldsymbol{B}$, $\boldsymbol{B} \neq \emptyset$ does not hold in general, take, e.g., $t(\boldsymbol{A}) = \mathbf{2}$, $t(\boldsymbol{B}) = \mathbf{3}$.

7. For any $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C} \in \mathfrak{A}$ we have

(19) $\prec$ A $\Subset \boldsymbol{B}$ implies $\boldsymbol{A}^{\boldsymbol{C}} \Subset \boldsymbol{B}^{\boldsymbol{C}}$.

Proof: Let i be an injective homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. If we assign to any homomorphism $\varphi \in \boldsymbol{A}^{\boldsymbol{C}}$ the mapping $\psi = i \cdot \varphi$, the clearly $\psi \in \boldsymbol{B}^{\boldsymbol{C}}$. Let j be a mapping $\boldsymbol{A}^{\boldsymbol{C}}$ into $\boldsymbol{B}^{\boldsymbol{C}}$ such that $j(\varphi) = \psi$. We prove that j is an injective homomorphism. Let $\boldsymbol{C} = (C, h)$ and denote the operation in the algebra $\boldsymbol{A}^{\boldsymbol{C}}$ by the symbol f_{A^C} and similarly f_{B^C} denotes the operation in the algebra $\boldsymbol{B}^{\boldsymbol{C}}$.

For $\varphi \in \boldsymbol{A}^{\boldsymbol{C}}$ we have $j(f_{A^C}(\varphi)) = j(\varphi \cdot h) = i \cdot (\varphi \cdot h)$ and also $f_{B^C}(j(\varphi)) = = f_{B^C}(i \cdot \varphi) = (i \cdot \varphi) \cdot h$.

Further, to two different homomorphisms of $\boldsymbol{A}^{\boldsymbol{C}}$ there are assigned different homomorphisms of $\boldsymbol{B}^{\boldsymbol{C}}$. Thus, we have (19). □

The analogous implication $\boldsymbol{A} \prec \boldsymbol{B}$ implies that $\boldsymbol{A}^{\boldsymbol{C}} \prec \boldsymbol{B}^{\boldsymbol{C}}$ does not hold in general. Take, e.g., $t(\boldsymbol{A}) = \mathbf{4}$, $t(\boldsymbol{B}) = \mathbf{8}$, $t(\boldsymbol{C}) = \mathbf{12}$.

8. For any $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C} \in \mathfrak{A}$ we have

(20) $\boldsymbol{A} \prec \boldsymbol{B}$ implies $\boldsymbol{C}^{\boldsymbol{A}} \Subset \boldsymbol{C}^{\boldsymbol{B}}$.

Proof: Let Θ be a homomorphism of $\boldsymbol{B}$ onto $\boldsymbol{A}$. If we assign to any $\psi \in \boldsymbol{C}^{\boldsymbol{A}}$ the mapping $\psi = \varphi \cdot \Theta$, then, clearly, $\psi \in \boldsymbol{C}^{\boldsymbol{B}}$. Let i be a mapping of $\boldsymbol{C}^{\boldsymbol{A}}$ into $\boldsymbol{C}^{\boldsymbol{B}}$ such that $i(\varphi) = \psi$. We prove that i is an injective homomorphism. Let $\boldsymbol{A} =$

$= (A, f)$, $\boldsymbol{B} = (B, g)$ and denote the operation in the algebra $\boldsymbol{C}^{\boldsymbol{A}}$ by the symbol f_{C^A} and similary f_{C^B} denotes the operation in the algebra $\boldsymbol{C}^{\boldsymbol{B}}$. For $\varphi \in \boldsymbol{C}^{\boldsymbol{A}}$ we have $i(f_{C^A}(\varphi)) = i(\varphi \cdot f) = (\varphi \cdot f) \cdot \Theta$ and also $f_{C^B}(i(\varphi)) = f_{C^B}(\varphi \cdot \Theta) = (\varphi \cdot \Theta) \cdot g$. Since Θ is a homomorphism of $\boldsymbol{B}$ onto $\boldsymbol{A}$, we have $f \cdot \Theta = \Theta \cdot g$ and from this $i(f_{C^A}(\varphi)) = f_{C^B}(i(\varphi))$. Further, to two different homomorphism of $\boldsymbol{C}^{\boldsymbol{A}}$ there are assigned different homomorphisms of $\boldsymbol{C}^{\boldsymbol{B}}$. Thus, we obtain the assertion. $\square$

9. The relationship $\boldsymbol{A} \Subset \boldsymbol{A}^{\boldsymbol{B}}$ does not hold in general for all $\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{A}$. Take, e.g., $t(\boldsymbol{A}) = \boldsymbol{2}$, $t(\boldsymbol{B}) = \boldsymbol{3}$.

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МОРФИЗМЫ ОДНОГО КЛАССА МОНОУНАРНЫХ АЛГЕБР

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Резюме

В статье изуеаются биморфизмы, мономорфизмы и эпиморфизмы моноунарных алгебр данного класса. Далее изучаются свойства отношений изоморфного вложения и образования гомоморфных образов и их связь с операциями сложения, умножения и возведения в степень.