Mathematica Slovaca

Zbigniew Grande

On Borsík's problem concerning quasiuniform limits of Darboux quasicontinuous functions

Mathematica Slovaca, Vol. 44 (1994), No. 3, 297--301

Persistent URL: http://dml.cz/dmlcz/128668

Math. Slovaca, **44** (1994), No. 3, 297–301

ON BORSÍK'S PROBLEM CONCERNING QUASIUNIFORM LIMITS OF DARBOUX QUASICONTINUOUS FUNCTIONS

ZBIGNIEW GRANDE[1]

(Communicated by Ladislav Mišík)

ABSTRACT. It is proved that every cliquish function $f: \mathbb{R} \to \mathbb{R}$ is a quasiuniform limit of a sequence of Darboux quasicontinuous functions.

Let $\mathbb{R}$ be the set of all reals. A function $f: \mathbb{R} \to \mathbb{R}$ is said to be *quasicontinuous* (*cliquish*) at a point $x \in \mathbb{R}$ if for every $\varepsilon > 0$ and every neighbourhood U of x there is a nonempty open set $V \subset U$ such that $|f(t) - f(x)| < \varepsilon$ for each $t \in V$ ($\operatorname{osc} f < \varepsilon$ on V).

A function f is *quasicontinuous* (*cliquish*) if it is such at each point of its domain [2]. A sequence (f_n), $f_n: \mathbb{R} \to \mathbb{R}$, quasiuniformly converges to $f: \mathbb{R} \to \mathbb{R}$ ([3]) if (f_n) pointwise converges to f and

$$\forall \varepsilon > 0\ \forall m\ \ \exists p\ \forall x \in \mathbb{R}: \ \min\{|f_{m+1}(x) - f(x)|, \dots, |f_{m+p}(x) - f(x)|\} < \varepsilon .$$

In the article [1], B o r s í k proved that every cliquish function $f: \mathbb{R} \to \mathbb{R}$ is a quasiuniform limit of a sequence of quasicontinuous functions and he puts the following problem:

PROBLEM. ([1]) Let $f: \mathbb{R} \to \mathbb{R}$ be a cliquish function. Is the function f a quasiuniform limit of a sequence of Darboux quasicontinuous functions?

In this article, I prove that the answer to the above Borsík's question is affirmative.

AMS Subject Classification (1991): Primary: 26A15.
Key words: Cliquishness, Quasicontinuity, Darboux property, Quasiuniform convergence.
[1] Supported by KBN grant 2 1144 91 01, 1992-94.

THEOREM. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a cliquish function. There is a sequence of Darboux quasicontinuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ which quasiuniformly converges to f.*

In the proof of this theorem, we use the following lemmata:

LEMMA. *If a continuous function $f\colon [a,b] \to \mathbb{R}$ and a closed interval $[c,d]$ are such that $f([a,b]) \subset [c,d]$, then there is a continuous function $g\colon [a,b] \to \mathbb{R}$ such that $g(a) = f(a)$, $g(b) = f(b)$, and $g\big([a,b]\big) = [c,d]$.*

The proof of this lemma is obvious.

LEMMA. *Let $\varepsilon > 0$ and let $f\colon (a,b) \to \mathbb{R}$ be a function such that for every $x \in (a,b)$ we have $\operatorname{osc} f(x) < \varepsilon$. There is a continuous function $g\colon (a,b) \to \mathbb{R}$ such that $|f(x) - g(x)| < 2\varepsilon$ for each $x \in (a,b)$.*

P r o o f. It suffices to prove that for every closed interval $[c,d] \subset (a,b)$ there is a continuous function $h\colon [c,d] \to \mathbb{R}$ such that $h(c) = f(c)$, $h(d) = f(d)$ and $|h(x) - f(x)| < 2\varepsilon$ for every $x \in [c,d]$. Let $[c,d] \subset (a,b)$ be a closed interval. Since $\operatorname{osc} f(x) < \varepsilon$ for every $x \in [c,d]$, there are open intervals $J_i = (a_i, b_i)$, $i = 1,\dots,k$, such that $a_1 < c < a_2 < b_1 < a_3 < b_2 < \dots < a_k < b_{k-1} < d < b_k$, and $\operatorname{osc} f < \varepsilon$ on every J_i, $i = 1,\dots,k$. In every interval (a_{i+1}, b_i), $i = 1,\dots,k-1$, we find a point x_i. Let $x_0 = c$, $x_k = d$. Put $h(x_i) = f(x_i)$ for $i = 0,1,\dots,k$ and let h be linear in every interval $[x_i, x_{i+1}]$, $i = 0,1,\dots,k-1$. Obviously h is continuous and $h(c) = f(c)$ and $h(d) = f(d)$. Let $x \in (c,d)$. Then $x \in (x_i, x_{i+1})$ for some $i < k$. Since $[x_i, x_{i+1}] \subset (a_{i+1}, b_{i+1})$, we have $\operatorname{osc} f < \varepsilon$ on $[x_i, x_{i+1}]$. Consequently, $|f(x_i) - f(x_{i+1})| < \varepsilon$, $|f(x) - f(x_i)| < \varepsilon$, and $|h(x) - f(x)| \le |h(x) - h(x_i)| + |h(x_i) - f(x)| \le |h(x_{i+1}) - h(x_i)| + |f(x_i) - f(x)| = |f(x_{i+1}) - f(x_i)| + |f(x_i) - f(x)| < \varepsilon + \varepsilon = 2\varepsilon$. Thus the proof is completed.

P r o o f o f T h e o r e m. Put $A_n = \{x \in \mathbb{R};\ \operatorname{osc} f(x) \ge 1/n\}$, $n = 1,2,\dots$. Then all sets A_n, $n = 1,2,\dots$, are closed and nowhere dense. Fix a positive integer n. For every component (a,b) of the set $\mathbb{R} - A_n$ we have $\operatorname{osc} f(x) < 1/n$ for every $x \in (a,b)$. So, by Lemma 2, there is a continuous function $g_{(a,b)}\colon (a,b) \to \mathbb{R}$ such that $|f(x) - g_{(a,b)}(x)| < 2/n$ for every $x \in (a,b)$. Let

$$g_n(x) = f(x)\,, \qquad \text{for} \quad x \in A_n\,,$$

and

$$g_n(x) = g_{(a,b)}(x)$$

if x belongs to some component (a,b) of the set $\mathbb{R} - A_n$. If $a > -\infty$, $i \le n$, and $\operatorname{dist}(a, A_i) = \inf\{|a - x|;\ x \in A_i\} < 1/n$, then there is a sequence $(I_{i,k})$

of closed intervals (which depends on (a,b)) such that:

- $I_{i,k} = [a_{i,k}, b_{i,k}] \subset \big(a, a + \min(1/n, (b-a)/2)\big)\big) \cap \{x \in (a,b);\ \operatorname{dist}(x, A_i) < 1/n\}$ for $k = 1, 2, \dots$;
- $\lim\limits_{k\to\infty} b_{i,k} = a$;
- $I_{i,k} \cap I_{j,l} = \emptyset$ if $(i,k) \neq (j,l)$, $i, j \le n$, $k, l = 1, 2, \dots$;
- $\operatorname{osc} g_n < 1/n$ on every $I_{i,k}$, $k = 1, 2, \dots$.

Similarly, if $b < \infty$, $i \le n$, and $\operatorname{dist}(b, A_i) < 1/n$, then we find a sequence of closed intervals $J_{i,k} = [c_{i,k}, d_{i,k}]$, $k = 1, 2, \dots$, such that:

- $J_{i,k} \subset \big(b - \min(1/n, (b-a)/2), b\big) \cap \{x \in (a,b);\ \operatorname{dist}(x, A_i) < 1/n\}$ for $k = 1, 2, \dots$;
- $\lim\limits_{k\to\infty} c_{i,k} = b$;
- $J_{i,k} \cap J_{j,l} = \emptyset$ if $(i,k) \neq (j,l)$, $i, j \le n$, $k, l = 1, 2, \dots$;
- $\operatorname{osc} g_n < 1/n$ on every $J_{i,k}$, $k = 1, 2, \dots$.

For every $k = 1, 2, \dots$ and $i \le n$ there are closed intervals $K_{i,k} \supset g_n(I_{i,k})$ and $L_{i,k} \supset g_n(J_{i,k})$ such that $K_{i,k}$ has the same center as $g_n(I_{i,k})$, $L_{i,k}$ has the same center as $g_n(J_{i,k})$, and the diameters $d(K_{i,k})$, $d(L_{i,k})$ are equal to $25/(i-1)$ for $i > 1$ and $K_{1,k} \cap L_{1,k} \supset [-k, k]$ for $k = 1, 2, \dots$. By Lemma 1, for every $k = 1, 2, \dots$ and $i \le n$ there are continuous functions $s_{n,i,k}\colon I_{i,k} \to K_{i,k}$, and $t_{n,i,k}\colon J_{i,k} \to L_{i,k}$ such that $s_{n,i,k}(I_{i,k}) = K_{i,k}$, $t_{n,i,k}(J_{i,k}) = L_{i,k}$, $s_{n,i,k}(a_{i,k}) = g_n(a_{i,k})$, $s_{n,i,k}(b_{i,k}) = g_n(b_{i,k})$, $t_{n,i,k}(c_{i,k}) = g_n(c_{i,k})$, $t_{n,i,k}(d_{i,k}) = g_n(d_{i,k})$. If $x \in (a,b)$, then let $f_{2n-1}(x) = s_{n,i,2k-1}(x)$ for $x \in I_{i,2k-1}$, $f_{2n-1}(x) = t_{n,i,2k-1}(x)$ for $x \in J_{i,2k-1}$, $i \le n$, $k = 1, 2, \dots$ and let $f_{2n-1}(x) = g_n(x)$ at other points of (a,b). Moreover, let $f_{2n-1}(x) = f(x)$ for $x \in A_n$. Similarly, let $f_{2n}(x) = s_{n,i,2k}(x)$ for $x \in I_{i,2k}$, $f_{2n}(x) = t_{n,i,2k}(x)$ for $x \in J_{i,2k}$, $i \le n$, $k = 1, 2, \dots$, $f_{2n}(x) = g_n(x)$ otherwise in (a,b), and $f_{2n}(x) = f(x)$ for $x \in A_n$. Now, we shall prove that the sequence (f_n) pointwise converges to f. If $x \in A_n$ for some $n = 1, 2, \dots$, then $f_k(x) = f(x)$ for every $k > 2n-1$ and $\lim\limits_{k\to\infty} f_k(x) = f(x)$. Suppose that x is not in any A_n, $n = 1, 2, \dots$. Fix a positive ε. There is a positive integer n such that $15/n < \varepsilon$, and a positive integer $m > n$ such that $\operatorname{dist}(x, A_n) > 1/m$. Then for $k > m$ we have $\operatorname{dist}(x, A_n) > 1/k$, and if $x \in I_{i,2p-1} \cup J_{i,2p-1}$ for some i and p, then $i > n$. Since for $k > m$ and $i > n$ we have $f_{2k-1}(I_{i,2p-1}) = K_{i,2p-1}$, $f_{2k-1}(J_{i,2p-1}) = L_{i,2p-1}$, $K_{i,2p-1}(L_{i,2p-1})$ has the same center as $g_k(I_{i,2p-1})\big(g_k(J_{i,2p-1})\big)$, $d(K_{i,2p-1}) = d(L_{i,2p-1}) = 25/(i-1) \le 25/n$, and $d\big(g_k(I_{i,2p-1})\big) < 1/k < 1/n$, $d\big(g_k(J_{i,2p-1})\big) < 1/k < 1/n$, we may observe that $|f_{2k-1}(x) - g_k(x)| < 13/n$. Consequently, for $k > m$ we have $|f_{2k-1}(x) - f(x)| \le |f_{2k-1}(x) - g_k(x)| + |g_k(x) - f(x)| < 13/n + 2/k < 15/n < \varepsilon$, and similarly, $|f_{2k}(x) - f(x)| < \varepsilon$. So, the sequence (f_n) pointwise converges to f. Since $\min\{|f_{2n-1}(x) - f(x)|, |f_{2n}(x) - f(x)|\} \le |g_n(x) - f(x)| < 2/n$ for every $x \in \mathbb{R}$ and $n = 1, 2, \dots$, the above convergence of the sequence (f_n) is quasiuniform. We will show that every function f_{2n}, $n = 1, 2, \dots$, is quasicontinuous. Fix a positive integer n. Since f_{2n} is continuous at every point $x \in \mathbb{R} - A_n$, it suffices to prove that it is quasicontinuous at each point $x \in A_n$. Fix $x \in A_n$ and $\varepsilon > 0$. If $x \in A_1$, then there is an interval $I_{1,2k} \subset (x - \varepsilon, x + \varepsilon)$ such that $f_{2n}(x) \in (-k, k)$. Consequently, there is an open interval $I \subset I_{1,2k}$ such that $f_{2n}(I) \subset \big(f_{2n}(x) - \varepsilon,\, f_{2n}(x) + \varepsilon\big)$. So, in this case, f_{2n} is quasicontinuous at x. If $x \in A_i - A_{i-1}$, $1 < i \le n$, then there is a positive number $\delta < \varepsilon$ such that $\operatorname{osc} f < 1/(i-1)$ on $(x - \delta, x + \delta) \subset (x - \varepsilon, x + \varepsilon)$. There is an interval $I_{i,2k} \subset (x - \delta, x + \delta)$. Let $z \in I_{i,2k}$ be a point. Then $|f(x) - f(z)| < 1/(i-1)$, and $|g_n(z) - f(z)| < 2/n < 2/(i-1)$. Consequently, $f(x) \in \big(g_n(z) - 3/(i-1),\, g_n(z) + 3/(i-1)\big)$, and there is a point $u \in I_{i,2k}$ such that $f_{2n}(u) = f(x)$. Since the function f_{2n} is continuous at u, there is an open interval $I \subset I_{i,2k}$ such that $f_{2n}(I) \subset \big(f(x) - \varepsilon,\, f(x) + \varepsilon\big) = \big(f_{2n}(x) - \varepsilon,\, f_{2n}(x) + \varepsilon\big)$. So f_{2n} is quasicontinuous at x. The proof of the quasicontinuity of the function f_{2n-1} is analogous. Now we shall prove that f_{2n} has the Darboux property. Let $K \subset \mathbb{R}$ be a closed interval. If $K \subset \mathbb{R} - A_n$, then f_{2n} is continuous on K, and $f_{2n}(K)$ is a connected set in $\mathbb{R}$. If $A_1 \cap K \ne \emptyset$, then $f_{2n}(K) = \mathbb{R}$. Assume that the set $f_{2n}(K)$ is not connected. Let $c \in \mathbb{R}$ be such that

$$A = \{x \in K;\ f_{2n}(x) < c\} \ne \emptyset, \qquad B = \{x \in K;\ f_{2n}(x) > c\} \ne \emptyset.$$

and $f_{2n}(x) \ne c$ for every $x \in K$. Find a point $z \in K \cap \operatorname{cl} A \cap \operatorname{cl} B$ (cl denotes the closure operation). Evidently, $z \in A_n$. Since z is not in A_1, there is $i \le n$, $i > 1$, such that $z \in A_i - A_{i-1}$. Assume that $f_{2n}(z) = f(z) > c$. Since $\operatorname{osc} f(z) < 1(i-1)$ and $|g_n(u) - g_n(v)| \le |g_n(u) - f(u)| + |f(u) - f(v)| + |f(v) - g_n(v)| < 2/n + |f(u) - f(v)| + 2/n = |f(u) - f(v)| + 4/n$ for all points $u, v \in \mathbb{R}$, we may observe that $\operatorname{osc} g_n(z) < 1/(i-1) + 4/n < 5/(i-1)$. Let U be an open set containing z such that $\operatorname{osc} g_n < 5/(i-1)$ on U and $\operatorname{osc} f < 1/(i-1)$ on U. Assume that $g_n(u) < c$ at a point $u \in U$. Then for every $x \in U$ we have $|g_n(x) - c| \le |g_n(x) - g_n(u)| + |g_n(u) - c| < 5/(i-1) + \big(c - g_n(u)\big) < 5/(i-1) + \big(f(z) - g_n(u)\big) = 5/(i-1) + |g_n(z) - g_n(u)| < 5/(i-1) + 5/(i-1) = 10/(i-1)$. There is $I_{i,2k} \subset U \cap \operatorname{int} K$ (or $J_{i,2k} \subset U \cap \operatorname{int} K$). If $I_{i,2k} \subset U \cap \operatorname{int} K$, then $g_n(I_{i,2k}) \subset \big(c - 10/(i-1), c + 10/(i-1)\big)$, and consequently $c \in K_{i,2k} = f_{2n}(I_{i,2k})$. Similarly, if $J_{i,2k} \subset U \cap \operatorname{int} K$, then also $c \in f_{2n}(J_{i,2k})$. This contradiction proves that $g_n(x) > c$ on the set U. Now we find $I_{i,2k} \subset U \cap \operatorname{int} K$ (or $J_{i,2k} \subset U \cap \operatorname{int} K$). Since c is not in $K_{i,2k}$ (or in $L_{i,2k}$), we obtain that $g_n(x) > c + 10/(i-1)$ for $x \in I_{i,2k}$ (or for $x \in J_{i,2k}$). But $\operatorname{osc} g_n < 5/(i-1)$ on U, so $g_n(x) \ge c + 5/(i-1)$ for $x \in U$. In particular, $f(z) = g_n(z) \ge c + 5/(i-1)$

Since $z \in \operatorname{cl} A$, there is a point $w \in A \cap U$. Then $f(w) < c$, in a contradiction with the facts $f(z) \geq 5/(i-1) + c$ and $\operatorname{osc} f < 1/(i-1)$ on U. In the case where $f(z) < c$, the proof is analogous.

REFERENCES

[1] BORSÍK, J.: *Quasiuniform limits of quasicontinuous functions*, Math. Slovaca **42** (1992), 269–274.

[2] NEUBRUNN, T.: *Quasi-continuity*, Real Anal. Exchange **14** (1988-89), 259–306.

[3] SIKORSKI, R.: *Real Functions I* (Polish), PWN, Warszawa, 1959.

Received December 21, 1992

Revised April 10, 1993

Department of Mathematics
Pedagogical University
Plac Slowianski 9
PL-65-069 Zielona Gora
Poland