Ivan Dobrakov; Jana Farková
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ON SUBMEASURES II

IVAN DOBRAKOV—JANA FARKOVÁ

Introduction

In the present paper we investigate connections between uniform exhaustivity, equi-absolute continuity, common or equi-subadditive continuity and sequential compactness in the topology of pointwise convergence for families of submeasures. (For the terminology see section 1 and Definitions 2 and 3).

The concept of subadditive continuity of μ is linked with absolute continuity in the following obvious way: μ is subadditively continuous if and only if the set functions ν_1^A, ν_2^A:

$$\nu_1^A(B) = \mu(A \cup B) - \mu(A)$$

and

$$\nu_2^A(B) = \mu(A) - \mu(A - B),$$

are absolutely — μ-continuous.

For such considerations of a family ν_i, $i \in I$ of set functions, the behaviour of the set function ν_I, $\nu_I(E) = \sup_{i \in I} \nu_i(E)$ is dominant. As the example following Corollary 2 of Theorem 7 shows, ν_I need not be a submeasure even if ν_i, $i = 1, 2, \ldots$ are uniformly exhaustive uniform submeasures on a σ-algebra.

It is mainly for this reason that we introduce and investigate a concept of a semimeasure, see Definition 1, which on a σ-ring is more general then the concept of a submeasure. Namely Theorems 7 and 11 are true only within the framework of semimeasures but not within that of submeasures.

Investigation of absolute continuity of subadditive set functions was initiated by W. Orlicz in [15] and [16] and was succesfully continued in [1], [2], [5], [8], [9] and [11].

Although most of our results are generalizations of the subadditive case, we prove results which have no meaning in the subadditive case, see Theorems 1, 2, 6, 8, 9.

In § 1 we introduce basic notations and terminology. In § 2 we consider subsequently set functions on a ring, on a σ-ring and on a generated σ-ring.

For a solution of Problem 1 on page 14 of part I (In the following [4] will be cited as part I.) and for other results on submeasures see the recent paper of L. Drewnowski: *On the continuity of certain non-additive set functions,* Colloquium Math. 38 (1978), 243—253.

§ 1. Notations and preliminaries

In the following $R_+ = \langle 0, +\infty)$ and $\bar{R}_+ = \langle 0, +\infty\rangle$. T will denote a non empty set, $\mathcal{R}$ a ring and $\mathcal{S}$ a σ-ring of subsets of T. If $\mathcal{E} \subset 2^T$, then $\sigma(\mathcal{E})$ denotes the smallest σ-ring containing $\mathcal{E}$. I will be a non empty set of indices.

All the considered set functions are supposed to be monotone and equal to zero on the empty set (we deal as in part I only with set functions with values in $\bar{R}_+$). If $\mathcal{E} \subset 2^T$ and $v_i: \mathcal{E} \to \bar{R}_+$, $i \in I$, are given, then $v_I: \mathcal{E} \to \bar{R}_+$ denotes the set function defined by the equality

$$v_I(E) = \sup_{i \in I} v_i(E), \quad E \in \mathcal{E}.$$

Let $\mathcal{E} \subset 2^T$ and let $v: \mathcal{E} \to \bar{R}_+$. We say that v is exhaustive, if $v(E_n) \to 0$ for any sequence of pairwise disjoint sets $E_n \in \mathcal{E}$, $n = 1, 2, \ldots$. We shall need the following two well-known facts about exhaustive set functions defined on a ring, see [5, 4.1 and 4.6].

Lemma 1. *A set function $v: \mathcal{R} \to \bar{R}_+$ is exhaustive if and only if every monotone sequence $E_n \in \mathcal{R}$, $n = 1, 2, \ldots$ is v-Cauchy, i.e., $v(E_n \Delta E_m) \to 0$ if $n \wedge m \to \infty$. ($a \vee b$, resp. $a \wedge b$, means the maximum, resp. the minimum, of the real numbers a and b.)*

Lemma 2. *Let $v: \mathcal{R} \to \bar{R}_+$ be exhaustive and let $E_n \in \mathcal{R}$, $n = 1, 2, \ldots$ Then for each $\varepsilon > 0$ there is an n_0 such that*

$$v\left(E_n - \bigcup_{k=1}^{n_0} E_k\right) < \varepsilon$$

for $n > n_0$.

We say that the family $v_i: \mathcal{E} \to \bar{R}_+$, $i \in I$, is uniformly exhaustive if v_I is exhaustive.

Let $v: \mathcal{R} \to \bar{R}_+$. We say that v is continuous at $\emptyset$, shortly continuous if $v(E_n) \to 0$ for any sequence $E_n \in \mathcal{R}$, $n = 1, 2, \ldots$ such that $E_n \searrow \emptyset$. If $v_i: \mathcal{R} \to \bar{R}_+$, $i \in I$, and if v_I is continuous, then we say that the family v_i, $i \in I$, is uniformly continuous.

We say that $v: \mathcal{R} \to \bar{R}_+$ has the Fatou property, briefly the (F.p.) if $E_n \in \mathcal{R}$, $n = 1, 2, \ldots$ and $E_n \nearrow E \in \mathcal{R} \Rightarrow v(E_n) \to v(E)$. If $v_i: \mathcal{R} \to \bar{R}_+$, $i \in I$ have the (F.p.), then clearly v_I has also the (F.p.).

If $\nu:\mathscr{R}\to\bar{R}_+$ is exhaustive and has the (F.p.), then it is clearly continuous. If $\nu:\mathscr{S}\to\bar{R}_+$ is continuous, then it is exhaustive.

Let ν, $\mu:\mathscr{R}\to\bar{R}_+$. We say that ν is absolutely μ-continuous, briefly $\nu\ll\mu$ if for each $\varepsilon>0$ there is an $\delta>0$ such that $A\in\mathscr{R}$, $\mu(A)<\delta\Rightarrow\nu(A)<\varepsilon$. If $\nu\ll\mu$ and also $\mu\ll\nu$, then we say that ν and μ are equivalent and write $\nu\sim\mu$. If μ, $\nu_i:\mathscr{R}\to\bar{R}_+$, $i\in I$ and if $\nu_I\ll\mu$, then we say that the family ν_i, $i\in I$, is equi-μ-continuous.

We say that $\nu:\mathscr{R}\to R_+$ is pseudometric generating if there is a subadditive $\lambda:\mathscr{R}\to R_+$ such that $\nu\sim\lambda$.

This terminology is clear, since then the function $\varrho(E, F) = \lambda(E\Delta F)$, $E, F\in\mathscr{R}$ is really a pseudometric on $\mathscr{R}$.

The following result is due to L. Drewnowski.

Theorem 1. *Let $\nu:\mathscr{R}\to R_+$. Then ν is pseudometric generating if and only if it has the following property: for each $\varepsilon>0$ there is a $\delta>0$ such that*

$$A, B\in\mathscr{R},\quad \nu(A)\vee\nu(B)<\delta\Rightarrow\nu(A\cup B)<\varepsilon.$$

(The property stated in this theorem will be called the pseudometric generating property, briefly the (p.g.p.).)

Proof. Necessity is immediate. Sufficiency: Monotonicity of ν and the (p.g.p.) imply that the families $\mathscr{V}_n=\{A\in\mathscr{R}:\nu(A)<n^{-1}\}$, $n=1, 2, \ldots$, form a base at $\emptyset$ for a unique Frechet—Nikodym topology $\Gamma(\nu)$ on $\mathscr{R}$, see [5, 1.5]. Since this base is countable, the topology $\Gamma(\nu)$ is pseudometrizable by an invariant pseudometric d on $\mathscr{R}$, see [3, chap. 9, § 3]. Now it is enough to put

$$\lambda(E)=\sup\{d(F,\emptyset):F\in\mathscr{R}, F\subset E\}.$$

Lemma 3. *Let $\mu:\mathscr{R}\to\bar{R}_+$ have the (p.g.p.). Then there is a sequence $\delta_k\in R_+$, $k=1, 2, \ldots$, $\delta_k\searrow 0$, such that $A_k\in\mathscr{R}$, $\mu(A_k)<\delta_k$ imply $\mu\left(\bigcup_{i=k+1}^{k+p}A_i\right)<\delta_k$ for each $k, p=1, 2, \ldots$*

Proof. Take arbitrary $\delta_1\in R_+$ and put subsequently $\delta_k=1/2[\delta_{k-1}\wedge\delta(\delta_{k-1})]$ for $k=2, 3, \ldots$, where $\delta(\delta_{k-1})$ is a δ from the (p.g.p.) corresponding to $\varepsilon=\delta_{k-1}$.

One of our basic concepts is introduced by the next

Definition 1. *We say that $\nu:\mathscr{R}\to\bar{R}_+$ is a semimeasure if it has the following properties:*

(i) the (p.g.p.),

(ii) the (F.p.),

(iii) $N\in\mathscr{R}$, $\nu(N)=0\Rightarrow\nu(A\cup N)=\nu(A)$ for each $A\in\mathscr{R}$, and

(iv) ν is exhaustive on $\mathscr{R}$.

Let us remind, see Definition 1 in part I, that $\mu:\mathscr{R}\to R_+$ is a submeasure if it is 1) monotone, 2) continuous and 3) subadditively continuous: for every $A\in\mathscr{R}$ and $\varepsilon>0$ there is a $\delta>0$ such that $B\in\mathscr{R}$, $\mu(B)<\delta$ implies: a) $\mu(A\cup B)\leqslant\mu(A)+\varepsilon$, and b) $\mu(A) \leqslant \mu(A-B)+\varepsilon$. If the δ in condition 3) is uniform with respect to $A \in \mathcal{R}$, then we say that μ is a uniform submeasure.

By Theorems 1 and 3 from part I each submeasure on a σ-ring is a semimeasure (on a ring this is not true even for countably additive measures, since they are not necessarily exhaustive).

The converse is not true as the following simple example demonstrates: Let $T=\langle 0, 1\rangle$. Let $\mathcal{S}$ be the Borel σ-algebra of T and let $\lambda: \mathcal{S} \to \langle 0, 1\rangle$ be the Lebesgue measure. Put $v(A)=\lambda(A)$ if $\lambda(A) \leqslant 1/2$ and $v(A)=1$ if $\lambda(A)>1/2$. Then obviously $v: \mathcal{S} \to \langle 0, 1\rangle$ is a semimeasure which is not a submeasure. As the Corollary 1 of Theorems 5 will show a semimeasure $v: \mathcal{S} \to R_+$ is a submeasure if and only if $A_n \in \mathcal{S}$, $n=1, 2, \ldots$ and $A_n \searrow A$ imply $v(A_n) \to v(A)$.

It is easy to verify that the analogs of Theorem 4—9, 11, 12, 14, 15 and Corollaries 1 and 2 of Theorem 15 from part I are valid for semimeasures. See also Theorem 10 below. On the other hand, as the example above shows, Theorem 10 from part I is in general not valid for semimeasures. Note also that in Theorems 3a) and 13 in part I the subadditive continuity can be replaced by the (p.g.p.).

Concerning the notion of the submeasure, let us note that the subadditive continuity may be replaced by the following one

3)*: *If* $A, A_n \in \mathcal{R}$, $n=1, 2, \ldots$ *and* $\mu(A \Delta A_n) \to 0$, *then* $\mu(A_n) \to \mu(A)$.

Proof: 3)$\Rightarrow$3)*. Suppose that $\mu(A_n) \not\to \mu(A)$. Then we can assume that for some $\varepsilon>0$ either $\mu(A_n)>\mu(A)+\varepsilon$ for each n, or $\mu(A_n)<\mu(A)-\varepsilon$ for each n. In the first case we get that $\mu(A \cup (A \triangle A_n)) \geqslant \mu(A \triangle (A \triangle A_n)) > \mu(A)+\varepsilon$, which contradicts 3a). Similarly the second case is inconsistent with 3b).

3)*$\Rightarrow$3). Let $\mu(B_n) \to 0$. Then $\mu(A \cup B_n) = \mu(A \triangle (B_n - A)) \to \mu(A)$ and $\mu(A - B_n) = \mu(A \triangle (A \cap B_n)) \to \mu(A)$.

Similarly, the uniform subadditive continuity is equivalent with the following one

3u)*: *for each* $\varepsilon>0$ *there is a* $\delta>0$ *such that* $A, B \in \mathcal{R}$ *and* $\mu(A \triangle B)<\delta \Rightarrow |\mu(A)-\mu(B)|<\varepsilon$.

Using these facts, Theorem 1, and Theorem 3b) from part I., we immediately obtain the following characterization of submeasures defined on a σ-ring:

Theorem 2. *A set function* $\mu: \mathcal{S} \to R_+$ *is a submeasure if and only if there is an equivalent subadditive submeasure* $\lambda: \mathcal{S} \to R_+$ *such that* μ *is a continuous function on the pseudometric space* $(\mathcal{S}, \lambda)$.

§ 2. Uniform exhaustivity and absolute continuity of set functions

1. On a ring

The following theorem is a generalization of Theorem 6.1 (a) from [5]. On the other hand it follows immediately from this result if we use Theorem 1. We give, however, a direct proof and thus the metrization result of Theorem 1 is not needed.

Theorem 3. *Let μ, $\nu:\mathscr{R}\to\bar{R}_+$ have both the (p.g.p.), let ν be exhaustive and suppose that $B_k\in\mathscr{R}$, $k=1, 2, \dots$, $B_k\searrow$ and $\mu(B_k)\to 0$ imply $\nu(B_k)\to 0$.*

Then $\nu\ll\mu$.

Proof. Suppose the contrary. According to Lemma 3 take a sequence $\{\delta_k\}$ with stated properties. Then there is an $\varepsilon_0>0$ and a sequence $E_k\in\mathscr{R}$, $k=1, 2, \dots$ such that $\mu(E_k)<\delta_k$ and $\nu(E_k)>\varepsilon_0$ for each $k=1, 2, \dots$

Since ν has the (p.g.p.), there is an $\varepsilon>0$ such that

$$(1)\qquad A, B\in\mathscr{R},\quad \nu(A)\vee\nu(B)<\varepsilon\Rightarrow\nu(A\cup B)<\varepsilon_0.$$

Further, by Lemma 3 we choose a sequence $\varepsilon_k\in R_+$, $k=1, 2, \dots$ such that $\varepsilon>\varepsilon_1$, $\varepsilon_k\searrow 0$ and $A_k\in\mathscr{R}$, $\nu(A_k)<\varepsilon_k$, $k=1, 2, \dots$ imply $\nu\left(\bigcup_{i=1}^{k}A_i\right)<\varepsilon$ for each $k=1, 2, \dots$

Since ν is exhaustive, applying Lemma 2 to the sequence E_n, $n=1, 2, \dots$ and to ε_2 we find an n_1 such that

$$\nu\left(E_n-\bigcup_{i=1}^{n_1}E_i\right)<\varepsilon_2\quad\text{for}\quad n>n_1.$$

Put $B_1=\bigcup_{i=1}^{n_1}E_i$ and apply Lemma 2 to the sequence $B_1\cap E_n$, $n=n_1+1, n_1+2, \dots$ and to ε_3. Then there is an $n_2>n_1$ such that

$$\nu\left(B_1\cap E_n-B_1\cap\left(\bigcup_{i=n_1+1}^{n_2}E_i\right)\right)<\varepsilon_3\quad\text{for}\quad n>n_2.$$

Define $B_2=B_1\cap\left(\bigcup_{i=n_1+1}^{n_2}E_i\right)$ and apply Lemma 2 to the sequence $B_2\cap E_n$, $n=n_2+1$, $n_2+2, \dots$ and to ε_4. Continuing in this way we obtain a required sequence $B_k\in\mathscr{R}$, $k=1, 2, \dots$ In fact, $B_k\searrow$, and

$$\mu(B_k)\leq\mu\left(\bigcup_{i=n_{k-1}+1}^{n_k}E_i\right)<\delta_{n_{k-1}}\searrow 0$$

as $k\to\infty$. Clearly

$$(2)\qquad \begin{aligned}E_n=(E_n\cap B_0-B_1)\cup(E_n\cap B_1-B_2)\cup\dots\cup\\ \cup(E_n\cap B_{k-1}-B_k)\cup E_n\cap B_k\end{aligned}$$

for each n, $k=1, 2, \dots$, where $B_0=T$.

Since $\nu(E_n\cap B_{k-1}-B_k)<\varepsilon_{k+1}$ for each $k=1, 2, \dots$ and each $n>n_k$, we have

$$\nu\left(\bigcup_{i=1}^{k}(E_n\cap B_{i-1}-B_i)\right)<\varepsilon_1<\varepsilon$$

for each $k=1, 2, \dots$ and each $n>n_k$. But then $\nu(B_k)\geq\nu(E_n\cap B_k)>\varepsilon$ for each $k=1, 2, \dots$ and each $n>n_k$, because otherwise by (1) and (2) the inequality $\nu(E_n)>\varepsilon_0$ cannot hold for $n>n_k$. Since $\varepsilon>0$, we have a contradiction.

The next theorem generalizes Theorem 1 in § 2 in [11].

Theorem 4. *Let μ, $\nu_i: \mathcal{R} \to \bar{R}_+$ $i \in I$, let $\nu_i \ll \mu$ for each $i \in I$ and let each ν_i, $i \in I$, have the following property (the property 3b) of a submeasure):*

For each $A \in \mathcal{R}$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that $B \in \mathcal{R}$, $\nu_i(B) < \delta \Rightarrow \nu_i(A) \leq \nu_i(A - B) + \varepsilon$.

Suppose further that both μ and ν_I have the (p.g.p.) and that ν_I is exhaustive. Then $\nu_I \ll \mu$.

Proof. Suppose the contrary. Then by Theorem 3 there is an $\varepsilon > 0$ and a sequence $B_k \in \mathcal{R}$, $k = 1, 2, \ldots$ such that $B_k \searrow$, $\mu(B_k) \to 0$ and $\nu_I(B_k) > \varepsilon$ for each $k = 1, 2, \ldots$. For each $k = 1, 2, \ldots$ take $i_k \in I$ so that $\nu_{i_k}(B_j) > \varepsilon$.

Put $k_1 = 1$. Since $\nu_{i_{k_1}}$ has the property 3b) of a submeasure, there is an $\eta > 0$ such that $B \in \mathcal{R}$, $\nu_{i_{k_1}}(B) < \eta \Rightarrow \nu_{i_{k_1}}(B_{k_1} - B) \geq \nu_{i_{k_1}}(B_{k_1}) - \varepsilon/2 \geq \varepsilon/2$. But $\nu_{i_{k_1}} \ll \mu$, hence there is a $\delta > 0$ such that $B \in \mathcal{R}$, $\mu(B) < \delta \Rightarrow \nu_{i_{k_1}}(B) < \eta$. Since $\mu(B_k) \to 0$, there is a $k_2 > k_1$ such that $\mu(B_{k_2}) < \delta$. In this way we have found a $k_2 > k_1$ such that $\nu_I(B_{k_1} - B_{k_2}) \geq \nu_{i_{k_1}}(B_{k_1} - B_{k_2}) \geq \varepsilon/2$. Repeating this consideration subsequently for $k_2, k_3, \ldots$, we obtain a subsequence B_{k_n}, $n = 1, 2, \ldots$ such that $\nu_I(B_{k_n} - B_{k_{n+1}}) \geq \varepsilon/2$ for each $n = 1, 2, \ldots$ But this contradicts the exhaustivity of ν_I, since $B_k \searrow$ and therefore the sets $B_{k_n} - B_{k_{n+1}}$, $n = 1, 2, \ldots$ are pairwise disjoint.

2. On a σ-ring

The next lemma immediately follows from the monotonicity of the considered set functions.

Lemma 4. *Let $\nu_n: \mathcal{R} \to \bar{R}_+$, $n = 1, 2, \ldots$ and let $\lim_{n\to\infty} \nu_n(A) = \nu(A)$ exist for each $A \in \mathcal{R}$ Then ν_n, $n = 1, 2, \ldots$ are uniformly continuous if and only if ν is continuous.*

The following simple theorem is the key to the most of our results which will follow.

Theorem 5. *Let μ, $\nu_i: \mathcal{S} \to \bar{R}_+$, $i \in I$ have the (F.p.) and let $N \in \mathcal{S}$, $\mu(N) = 0 \Rightarrow \nu_i(A \cup N) = \nu_i(A)$ for each $i \in I$ and each $A \in \mathcal{S}$. Let further μ have the (p.g.p.) and let ν_I be exhaustive. Then $\nu_I \ll \mu$.*

Proof. Suppose the contrary. Take a sequence δ_k, $k = 1, 2, \ldots$ for μ according to Lemma 3. Then there is an $\varepsilon > 0$ and a sequence $A_k \in \mathcal{S}$, $k = 1, 2, \ldots$ such that $\mu(A_k) < \delta_k$ and $\nu_I(A_k) > \varepsilon$ for each $k = 1, 2, \ldots$. But then $\mu\left(\bigcup_{i=k+1}^{\infty} A_i\right) \leq \delta_k$ for each $k = 1, 2, \ldots$ by Lemma 3 and the (F.p.) of μ.

Put $N = \bigcap_{k=1}^{\infty} \bigcup_{i=k+1}^{\infty} A_i$. Then $\mu(N) = 0$ by the monotonicity of μ, hence

$\nu_I\left(\bigcup_{i=k+1}^{\infty} A_i - N\right) = \nu_I\left(\bigcup_{i=k+1}^{\infty} A_i\right) > \nu_I(A_{k+1}) > \varepsilon$ for each $k = 1, 2, \ldots$ Since ν_I has the (F.p.) and is exhaustive, it is continuous. Clearly $\bigcup_{i=k+1}^{\infty} A_i - N \searrow \emptyset$ as $k \to \infty$, hence $\nu_I\left(\bigcup_{i=k+1}^{\infty} A_i - N\right) \to 0$ by the continuity of ν_i, a contradiction.

In connection with the next corollary see also Theorem 2 in part I.

Corollary 1. *For a set function $\mu: \mathcal{S} \to R_+$ the following conditions are equivalent:*

1) *μ is a submeasure*

2) *μ has the (p.g.p.), is monotonely continuous, i.e. $A_n \nearrow (\searrow) A \Rightarrow \mu(A_n) \to \mu(A)$, and $\mu(N) = 0 \Rightarrow \mu(A \cup N) = \mu(A)$ for each $A \in \mathcal{S}$.*

Particularly a semimeasure $\mu: \mathcal{S} \to R_+$ is a submeasure if and only if $A_n \searrow A \Rightarrow \mu(A_n) \to \mu(A)$.

Proof. 1)$\Rightarrow$2) by Theorem 3b), Theorem 1a) from part I and the subadditive continuity of μ.

2)$\Rightarrow$1). We have to show that μ is subadditively continuous. Let $A \in \mathcal{S}$ and put $\nu_1(B) = \mu(A \cup B) - \mu(A)$ and $\nu_2(B) = \mu(A) - \mu(A - B)$, $B \in \mathcal{S}$. Then it is easy to see that 2) implies that μ, ν_1 and ν_2 satisfy all assumptions of the theorem. Thus $(\nu_1 \vee \nu_2) \ll \mu$, what we wanted to show.

Using Lemma 4 we immediately have the following version of the Vitali—Hahn—Saks theorem.

Corollary 2. *Let $\mu, \nu_n: \mathcal{S} \to \bar{R}_+$, $n = 1, 2, \ldots$ have the (F.p.), let μ have the (p.g.p.) and let $N \in \mathcal{S}$, $\mu(N) = 0 \Rightarrow \nu_n(A \cup N) = \nu_n(A)$ for each $n = 1, 2, \ldots$ and each $A \in \mathcal{S}$. Let further $\nu_0: \mathcal{S} \to \bar{R}_+$ be continuous and let $\nu_n(A) \to \nu_0(A)$ for each $A \in \mathcal{S}$. Then the sequence ν_n, $n = 0, 1, 2, \ldots$ is equi-μ-continuous.*

From this we obtain the necessity of conditions II and III in Theorem 18 and of condition II in Theorem 23, part I, as we promised there. Namely we have

Corollary 3. *Let $\mu: \mathcal{S} \to R_+$ be a submeasure and let $A_n \in \mathcal{S}$, $n = 1, 2, \ldots$ be a monotone sequence with the limit A_0. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that $B \in \mathcal{S}$, $\mu(B) < \delta \Rightarrow \mu(A_n \cup B) \leqslant \mu(A_n) + \varepsilon$ and $\mu(A_n - B) \geqslant \mu(A_n) - \varepsilon$ for each $n = 0, 1, 2, \ldots$*

Proof. For $n = 0, 1, 2, \ldots$ put $\nu_n(B) = [\mu(A_n \cup B) - \mu(A_n)] \vee [\mu(A_n) - \mu(A_n - B)]$, $B \in \mathcal{S}$. Then by Theorem 1a), Theorem 3b), part I and the subadditive continuity of μ clearly all assumptions of Corollary 2 are satisfied.

Note that the last corollary is generalized by Theorem 6.

For the next theorem we need two lemmas. The first is immediate.

Lemma 5. *Let $\nu_{n,k}: \mathcal{R} \to R_+$, $n, k = 1, 2, \ldots$ and suppose that:*

1) *for each $n = 1, 2, \ldots$ the sequence $\nu_{n,k}$, $k = 1, 2, \ldots$ is uniformly exhaustive,*

2) *for each* $k=1, 2, \ldots$ *the sequence* $\nu_{n,k}$, $n=1, 2, \ldots$ *is uniformly exhaustive, and*

3) *for each subsequences* $n_i \to \infty$, $k_i \to \infty$ *as* $i \to \infty$ *the sequence* ν_{n_i, k_i}, $i=1, 2, \ldots$ *is uniformly exhaustive.*

Then the family $\nu_{n,k}$, n, $k=1, 2, \ldots$ *is uniformly exhaustive.*

Lemma 6. *Let* $\mu_n: \mathscr{S} \to R_+$, $n=1, 2, \ldots$ *be semimeasures or submeasures and put*

$$\mu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\mu_n(A)}{1+\mu_n(T)}, \quad A \in \mathscr{S}.$$

Then μ *is a semimeasure or a submeasure, respectively.*

Proof: We prove the lemma for semimeasures. The case of submeasures may be proved similarly. First we note that for each $n=1, 2, \ldots$, $\mu_n(T) = \sup_{A \in \mathscr{S}} \mu_n(A) < +\infty$ (if $A_k \in \mathscr{S}$, $k=1, 2, \ldots$ and $\mu_n(A_k) \nearrow \mu_n(T)$, then $\mu_n(T) = \lim_{k \to \infty} \mu_n(A_k) \leqslant \mu_n\left(\bigcup_{k=1}^{\infty} A_k\right) < +\infty$ by the monotonicity of μ_n).

Now only the (p.g.p.) is not immediate. Let $\varepsilon > 0$. Take n_0 so that $\sum_{n=n_0+1}^{\infty} \frac{1}{2^n} < \varepsilon/2$, and for $n=1, 2, \ldots, n_0$ take δ_n by the (p.g.p.) of μ_n so that $\mu_n(A) \vee \mu_n(B) < \delta_n \Rightarrow \mu_n(A \cup B) < \varepsilon/2$. Put $\delta = \frac{1}{2^{n_0}} \frac{a}{1+b}$, where $a = \min_{1 \leqslant n \leqslant n_0} \delta_n$ and $b = \max_{1 \leqslant n \leqslant n_0} \mu_n(T)$. Then clearly $\mu(A) \vee \mu(B) < \delta \Rightarrow \mu(A \cup B) < \varepsilon$, what we wanted to show.

We shall need also the following

Definition 2. *We say that the family of set functions* $\nu_i: \mathscr{R} \to R_+$, $i \in I$ *is commonly subadditively continuous if for each* $A \in \mathscr{R}$ *and each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $B \in \mathscr{R}$, $\nu_i(B) < \delta$ *imply* $\nu_i(A \cup B) \leqslant \nu_i(A) + \varepsilon$ *and* $\nu_i(A - B) \geqslant \nu_i(A) - \varepsilon$ *for each* $i \in I$.

Note that if $\nu_i: \mathscr{R} \to R_+$, $i \in I$ are commonly subadditively continuous, then clearly $\nu_I: \mathscr{R} \to \bar{R}_+$ is subadditively continuous.

Theorem 6. *Let* μ_0, $\mu_n: \mathscr{S} \to R_+$, $n=1, 2, \ldots$ *be submeasures and let* $\mu_n(A) \to \mu_0(A)$ *for each* $A \in \mathscr{S}$. *Let further* $A_k \in \mathscr{S}$, $k=1, 2, \ldots$ *and let* $A_k \to A_0$, *i.e.* $\limsup_k A_k = \liminf_k A_k = A_0$. *Then for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $B \in \mathscr{S}$, $\mu_n(B) < \delta$ *for each* $n=1, 2, \ldots$ *imply* $\mu_n(A_k \cup B) \leqslant \mu_n(A_k) + \varepsilon$ *and* $\mu_n(A_k - B) \geqslant \mu_n(A_k) - \varepsilon$ *for each* n, $k=1, 2, \ldots$

Proof. Put

$$\mu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu_n(A)}{1+\mu_n(T)}, \quad A \in \mathscr{S}.$$

Then $\mu:\mathcal{S}\to R_+$ is a submeasure by Lemma 6. For n, $k=0, 1, 2, \ldots$ define $\nu^+_{n,k}$, $\nu^-_{n,k}:\mathcal{S}\to R_+$ by the equalities: $\nu^+_{n,k}(B) = \mu_n(A_k\cup B) - \mu_n(A_k)$ and $\nu^-_{n,k}(B) = \mu_n(A_k) - \mu_n(A_k - B)$, $B\in\mathcal{S}$. Since $\mu(B) \leqslant \sup_n \nu_n(B)$ for each $B\in\mathcal{S}$, to prove the theorem it suffices to show that the family $\{\nu^+_{n,k}, \nu^-_{n,k}, n, k=1, 2, \ldots\}$ is equi-μ-continuous. To show this it is enough to check that all assumptions of Theorem 5 are satisfied. Since μ is a submeasure by Corollary 1 of Theorem 5, it has the required properties. Similarly, since each μ_n, $n=1, 2, \ldots$ is monotonely continuous, each $\nu^+_{n,k}$ and $\nu^-_{n,k}$, n, $k=1, 2, \ldots$ is continuous and has the (F.p.). The property: $N\in\mathcal{S}$, $\mu(N)=0 \Rightarrow \nu^+_{n,k}(A\cup N) = \nu^+_{n,k}(A)$ and $\nu^-_{n,k}(A\cup N) = \nu^-_{n,k}(A)$ for each $A\in\mathcal{S}$ is immediate. Theorem 1b) in part I implies that $\nu^+_{n,k}(B) \to \nu^+_{n,0}(B)$ and $\nu^-_{n,k}(B) \to \nu^-_{n,0}(B)$ for each $B\in\mathcal{S}$ and each $n=1, 2, \ldots$. Thus according to Lemma 4 the sequence $\nu^+_{n,k}\vee\nu^-_{n,k}$ is uniformly exhaustive for each $n=1, 2, \ldots$. Similarly, since $\mu_n(B)\to\mu_0(B)$ for each $B\in\mathcal{S}$, the sequence $\nu^+_{n,k}\vee\nu^-_{n,k}$ $n=1, 2, \ldots$ is uniformly exhaustive for each $k=1, 2, \ldots$. If now $n_i\wedge k_i\to\infty$, then it is easy to see that

$$(\nu^+_{n_i,k_i}\vee\nu^-_{n_i,k_i})(B)\to(\nu^+_{0,0}\vee\nu^-_{0,0})(B)$$

for each $B\in\mathcal{S}$, hence again by Lemma 4 the sequence $\nu^+_{n_i,k_i}\vee\nu^-_{n_i,k_i}$, $i=1, 2, \ldots$ is uniformly exhaustive. Thus by Lemma 5 the family $\{\nu^+_{n,k}, \nu^-_{n,k}, n, k=1, 2, \ldots\}$ is uniformly exhaustive, what we wanted to show.

Corollary. *Let the family of submeasures $\nu_i:\mathcal{S}\to R_+$, $i\in I$ be sequentially compact in the topology of pointwise convergence on $\mathcal{S}$. Then $\nu_I(A)<+\infty$ for each $A\in\mathcal{S}$, $\nu_I:\mathcal{S}\to R_+$ is a submeasure and the family ν_i, $i\in I$ is commonly subadditively continuous.*

The idea of the proof of assertion 2) of the next theorem is taken from [10, Theorem 3.10], see also [1, Theorem 1] and [5, 10.5].

Theorem 7. *Let $\nu_i:\mathcal{S}\to R_+$, $i\in I$ be semimeasures and let ν_I be exhaustive. Then:*

1) *$\nu_I:\mathcal{S}\to\langle 0, +\infty\rangle$ is a semimeasure, and*

2) *there exists a sequence $i_n\in I$, $n=1, 2, \ldots$, such that $\nu_I\ll\mu$, where*

$$\mu(A)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\nu_{i_n}(A)}{1+\nu_{i_n}(T)},\quad A\in\mathcal{S}.$$

Proof. 1) Only the (p.g.p.) of ν_I is not immediate. Suppose ν_I has not got it. Then there is an $\varepsilon>0$ and for each $n=1, 2, \ldots$ sets A_n, $B_n\in\mathcal{S}$ and $i_n\in I$, $n=1, 2, \ldots$ such that $\nu_I(A_n)\vee\nu_I(B_n)<1/n$ and $\nu_{i_n}(A_n\cup B_n)>\varepsilon$. Thus if $J=\{i_n, n=1, 2, \ldots\}$, then ν_J has not the (p.g.p.) either. Hence we reduced the case of general I to the case when $I=\{1, 2, \ldots\}$. Let $I=\{1, 2, \ldots\}$ and for $A\in\mathcal{S}$ put

$$\mu(A)=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{\nu_i(A)}{1+\nu_i(T)}.$$

Then $\mu : \mathcal{S} \to R_+$ is a semimeasure by Lemma 6, hence $\nu_I \ll \mu$ by Theorem 5. Let $\varepsilon > 0$. Since $\nu_I \ll \mu$, there is a $\delta_0 > 0$ such that $\mu(E) < \delta_0 \Rightarrow \nu_I(E) < \varepsilon$. Since μ has the (p.g.p.), there is a $\delta > 0$ such that $\mu(A) \vee \mu(B) < \delta \Rightarrow \mu(A \cup B) < \delta_0$. Since $\mu(A) \leq \nu_I(A)$ for each $A \in \mathcal{S}$, $\nu_I(A) \vee \nu_I(B) < \delta \Rightarrow \nu_I(A \cup B) < \varepsilon$, what we wanted to show.

2) First we show that for each $\varepsilon > 0$ there exists a finite subset $J_\varepsilon \subset I$ such that $A \in \mathcal{S}$, $\nu_{J_\varepsilon}(A) = 0 \Rightarrow \nu_I(A) \leq \varepsilon$. Suppose the contrary. Then there is an $\varepsilon_0 > 0$ such that for any finite subset $J \subset I$ there is a set $A \in \mathcal{S}$ and $i \in I - J$ such that $\nu_J(A) = 0$ and $\nu_i(A) > \varepsilon_0$. Take arbitrary $i_1 \in I$. Then there is an $A_1 \in \mathcal{S}$ and $i_2 \in I$ such that $\nu_{i_2}(A_1) = 0$ and $\nu_{i_2}(A_1) > \varepsilon_0$. Similarly there is an $A_2 \in \mathcal{S}$ and $i_3 \in I$ such that $\nu_{i_1}(A_2) \vee \nu_{i_2}(A_2) = 0$ and $\nu_{i_3}(A_2) > \varepsilon_0$. Continuing in this way we obtain a sequence $A_n \in \mathcal{S}$, $n = 1, 2, \ldots$ and a subsequence $i_n \in I$, $n = 1, 2, \ldots$ such that $\nu_{i_{n+1}}(A_n) > \varepsilon_0$ and $\nu_{i_n}(A_k) = 0$ for $k \geq n$, $n = 1, 2, \ldots$ By the (F.p.) of each ν_i we have $\nu_{i_n}\left(\bigcup_{k=n}^{\infty} A_k\right) = 0$ for each $n = 1, 2, \ldots$, hence $\nu_{i_{n+1}}\left(A_n - \bigcup_{k=n+1}^{\infty} A_k\right) > \varepsilon_0$ for each n. But this contradicts the exhaustivity of ν_I, since the sets $A_n - \bigcup_{k=n+1}^{\infty} A_k$, $n = 1, 2, \ldots$ are pairwise disjoint. In this way we have shown that for each $\varepsilon > 0$ there is a finite subset $J_\varepsilon \subset I$ such that $A \in \mathcal{S}$, $\nu_{J_\varepsilon}(A) = 0 \Rightarrow \nu_I(A) \leq \varepsilon$. Putting subsequently $\varepsilon = 1/k$, $k = 1, 2, \ldots$ we obtain a sequence $i_n \in I$, $n = 1, 2, \ldots$ such that $A \in \mathcal{S}$,

$$\mu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\nu_{i_n}(A)}{1 + \nu_{i_n}(T)} = 0 \Rightarrow \nu_I(A) = 0.$$

Now clearly all assumption of Theorem 5 are satisfied, hence we have the desired result $\nu_I \ll \mu$.

From 1) and the Corollary 1 of Theorem 5 we immediately have

Corollary 1. *Let $\nu_i : \mathcal{S} \to R_+$, $i \in I$ be semimeasures and let $\nu_I(A) < +\infty$ for each $A \in \mathcal{S}$. Then ν_I is a submeasure if and only if $A_n \in \mathcal{S}$, $n = 1, 2, \ldots$ and $A_n \searrow A$ implies $\nu_I(A_n) \to \nu_I(A)$.*

From assertion 2) of the theorem we easily have

Corollary 2. *Under the assumptions of the theorem suppose that each pseudometrizable uniform space $(\mathcal{S}, \mathcal{U}_{\nu_i})$, $i \in I$ is separable or that each ν_i, $i \in I$ is a regular Borel semimeasure on $\sigma(\mathcal{B})$, or that each ν_i, $i \in L$ has the property (p), see Definition 4, part I. Then the semimeasure ν_I also has the corresponding property.*

The next simple example shows that in Theorem 7 ν_I need not be a submeasure even if each ν_i, $i \in I$ is a uniform submeasure.

Example. Let $T = \langle 0, 1 \rangle$, let $\mathcal{B}$ be the Borel σ-algebra of T and let $\mu : \mathcal{B} \to \langle 0, 1 \rangle$ be the Lebesgue measure. For $n = 1, 2, \ldots$ and $A \in \mathcal{B}$ put

$$\nu_n(A) = \mu(A) \wedge 1/2 + [n(\mu(A) - 1/2) \wedge 1/2] \vee 0.$$

Then each $\nu_n : \mathscr{B} \to \langle 0, 1 \rangle$ is a uniform submeasure. Let $A_k = \langle 0, 1/2 + 1/(k+1))$, $k = 1, 2, \ldots$ Then $A_k \searrow \langle 0, 1/2 \rangle = A$, $\nu_I(A_k) = 1$ for each $k = 1, 2, \ldots$, but $\nu_I(A) = 1/2$. Thus ν_I is not a submeasure by Corollary 1 of Theorem 7.

Theorem 8. *Let $\nu_i : \mathscr{S} \to R_+$, $i \in I$ be atomless semimeasures, see Definition 2, part I, let ν_I be exhaustive and let $A, B \in \mathscr{S}$ and $\nu_I(A) \vee \nu_I(B) < +\infty$ imply $\nu_I(A \cup B) < +\infty$. Then $\nu_I(A) < +\infty$ for each $A \in \mathscr{S}$.*

Proof. Suppose $\nu_I(A) = +\infty$ for some $A \in \mathscr{S}$. Then there is a countable set $J \subset I$ such that $\nu_J(A) = +\infty$. In this way we may suppose that $I = \{1, 2, \ldots\}$.

Let $I = \{1, 2, \ldots\}$ and put

$$\mu(A) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\nu_i(A)}{1 + \nu_i(T)}, \quad A \in \mathscr{S}.$$

Then $\mu : \mathscr{S} \to R_+$ is a semimeasure by Lemma 6. Now it is easy to check that all assumptions of Theorem 5 are satisfied, hence $\nu_I \ll \mu$. It remains to apply the Saks decomposition of μ, see Theorem 8, part I, and the assumed property

$$\nu_I(A) \vee \nu_I(B) < +\infty \Rightarrow \nu_I(A \cup B) < +\infty.$$

Theorem 9. *Let $\nu_i : \mathscr{S} \to R_+$, $i \in I$ be semimeasures and let $\nu_I(A) < +\infty$ for each $A \in \mathscr{S}$. Then $\nu_I : \mathscr{S} \to R_+$, is a uniform submeasure if and only if the set function*

$\nu : \mathscr{S} \to R_+$, $\nu(B) = \sup_{A \in \mathscr{S}} [\nu_I(A \cup B) - \nu_I(A)]$, $B \in \mathscr{S}$ is exhaustive.

Proof: Let $\nu_I : \mathscr{S} \to R_+$ be a uniform submeasure. Since ν_I is then continuous, it is exhaustive. Now the exhaustivity of $\nu_I : \mathscr{S} \to R_+$ and its subadditive continuity imply the exhaustivity of ν.

Conversely, suppose that $\nu : \mathscr{S} \to R_+$ is exhaustive. Taking $A = \emptyset$ we obtain that $\nu_I : \mathscr{S} \to R_+$ is exhaustive. Since $\nu_I : \mathscr{S} \to R_+$ has also the (F.p.), it is continuous. Thus it remains to prove its uniform subadditive continuity. In fact we have to show that $\nu \ll \nu_I$. By Theorem 5 it is enough to check that with $\mu = \nu_I$ its assumptions are satisfied. Since each ν_i, $i \in I$ has the (F.p.), ν_I and ν also have the (F.p.). Since ν_I is exhaustive, it has the (p.g.p.) by Theorem 7. The implication $\nu_I(N) = 0 \Rightarrow \nu_I(A \cup N) = \nu(A)$ for each $A \in \mathscr{S}$ is immediate. Finally the exhaustivity of ν is assumed.

3. On a generated σ-ring and sequential compactness in the topology of pointwise convergence

For submeasures the next result is contained in the lemmas of the proof of Theorem 18, part I.

(By $\mathscr{R}_\sigma(\mathscr{R}_\delta)$ as usually we denote the class of limits of increasing (decreasing) sequences of sets of $\mathscr{R}$.)

Theorem 10. *Let $v:\sigma(\mathscr{R})\to\bar{R}_+$ be a semimeasure. Then:*

1) *for each $A\in\sigma(\mathscr{R})$ and each $\varepsilon>0$ there are $E\in\mathscr{R}_\sigma$ and $F\in\mathscr{R}_\delta$ such that $F\subset A\subset E$ and $v(E-F)<\varepsilon$.*

2) *for each $A\in\sigma(\mathscr{R})$ there are $F\in\mathscr{R}_{\delta\sigma}$ and $E\in\mathscr{R}_{\sigma\delta}$ such that $F\subset A\subset E$ and $v(E-F)=0$, and*

3) *$v(A)=\sup\{v(F), F\subset A, F\in\mathscr{R}_\delta\}$ for each $A\in\sigma(\mathscr{R})$.*

Proof. 1) Denote by $\mathscr{S}$ the class of all sets $A\in\sigma(\mathscr{R})$ for which 1) is valid. Then clearly $\mathscr{R}\subset\mathscr{S}$ and $\mathscr{S}$ is a ring by the (p.g.p.) of v. Let $A_n\in\mathscr{S}$, $n=1, 2, \ldots$ and let $A_n\nearrow A$. According to Lemma 3 and the (F.p.) of v there is a sequence $\delta_k\searrow 0$ such that $B_k\in\sigma(\mathscr{R})$, $v(B_k)<\delta_k$, $k=1, 2, \ldots$ imply $v\left(\bigcup_{k=1}^{\infty}B_k\right)<\varepsilon$. Since v is exhaustive, by Lemma 2 and the (F.p.) of v there is an n_0 such that $v(A-A_{n_0})<\delta_1$. Take $F\in\mathscr{R}_\delta$ so that $F\subset A_{n_0}$ and $v(A_{n_0}-F)<\delta_2$, for each $n=n_0+k$, $k=1, 2, \ldots$ take $E_n\in\mathscr{R}_\sigma$ such that $E_n\supset A_n$ and $v(E_{n_0+k}-A_{n_0+k})<\delta_{2+k}$, and put $E=\bigcup_{k=1}^{\infty}E_{n_0+k}$. Then $E\in\mathscr{R}_\sigma$, $F\subset A\subset E$ and $v(E-F)<\varepsilon$.

Thus $A\in\mathscr{S}$, hence $\mathscr{S}=\sigma(\mathscr{R})$.

2) follows immediately from 1) by the monotonicity of v.

3) Let $A\in\sigma(\mathscr{R})$. By 2) take $F\in\mathscr{R}_{\delta\sigma}$ so that $F\subset A$ and $v(A-F)=0$. Then $v(A)=v(F)$ and $v(F)=\sup\{v(G), G\in\mathscr{R}_\delta, G\subset F\}$ by the (F.p) of v.

The implication 1)$\Rightarrow$3) of the next theorem in the case when each v_i, $i\in I$ is additive was proved in [17, Theorem 2.1] and for subadditive v_i it follows from Theorem 7.2 in [5], see also Theorem 2.1 in [9].

Theorem 11. *Let $v_i:\sigma(\mathscr{R})\to R_+$, $i\in I$ be semimeasures. Then the following conditions are equivalent.*

1) *$v_I:\mathscr{R}\to\bar{R}_+$ is a semimeasure*

2) *$v_I:\sigma(\mathscr{R})\to\bar{R}_+$ is exhaustive*

3) *$v_I:\sigma(\mathscr{R})\to\bar{R}_+$ is a semimeasure.*

Proof. 2)$\Rightarrow$3) by Theorem 7.1) and obviously 3)$\Rightarrow$1).

1)$\Rightarrow$2). Suppose the contrary. Then there is an $\varepsilon_0>0$ and a sequence $A_k\in\sigma(\mathscr{R})$, $k=1, 2, \ldots$ of pairwise disjoint sets such that $v_I(A_k)>\varepsilon$ for each $k=1, 2, \ldots$. According to Theorem 10.3) there are $F_k\in\mathscr{R}_\delta$, $k=1, 2, \ldots$ such that $F_k\subset A_k$ and $v_I(F_k)>\varepsilon_0$ for each $k=1, 2, \ldots$ For each $k=1, 2, \ldots$ take $R_j^k\in\mathscr{R}$, $j=1, 2, \ldots$ so that $R_j^k\searrow F_k$. Let $k\in\{1, 2, \ldots\}$ be fixed. Since $v_I:\mathscr{R}\to\bar{R}_+$ is exhaustive, by Lemma 1 there is an j_0 such that $v_I(R_{j_0}^k-R_j^k)<\varepsilon_0$ for each $j\geq j_0$. But then $v_I(R_{j_0}^k-F_k)\leq\varepsilon_0$ by the (F.p.) of $v_I:\sigma(\mathscr{R})\to\bar{R}_+$, hence $v_I(R_j^k-F_k)\to 0$ as $j\to\infty$ for each $k=1, 2, \ldots$. By the (p.g.p.) of $v_I:\mathscr{R}\to\bar{R}_+$ take $\delta_0>0$ so that $A, B\in\mathscr{R}$, $v_i(A)\vee v_I(B)<\delta_0 \Rightarrow v_I(A\cup B)<\varepsilon_0$. According to Lemma 3 take a sequence

$\delta_k \searrow 0$, $k=1, 2, \ldots$ such that $A_k \in \mathscr{R}$ and $\nu_I(A_k) < \delta_k$, $k=1, 2, \ldots$ imply $\nu_I\left(\bigcup_{i=1}^{k} A_i\right) < \delta_0$ for each $k=1, 2, \ldots$. Further for each $k=1, 2, \ldots$ choose j_k so that $\nu_I(R_{j_k} - F_k) < \delta_k$, put $R_1 = R^1_{j_1}$ and $R_k = R^k_{j_k} - \bigcup_{i=1}^{k-1} R_i$ for $k \geqslant 2, 2, \ldots$. Then R_k, $k=1, 2, \ldots$ are pairwise disjoint elements of $\mathscr{R}$ and $R^k_{j_k} = R_k \cup \left(R^k_{j_k} \cap \left(\bigcup_{i=1}^{k-1} R_i\right)\right)$ for each $k=1, 2, \ldots$

Since F_k, $k=1, 2, \ldots$ are pairwise disjoint, it is easy to see that

$$S_k = R^k_{j_k} \cap \left(\bigcup_{i=1}^{k-1} R_i\right) \subset \bigcup_{i=1}^{k} (R^i_{j_i} - F_i)$$

for each $k=1, 2, \ldots$. Hence $\nu_I(S_k) < \delta_0$ for each $k=1, 2, \ldots$. Since $R^k_{j_k} = R_k \cup S_k$ and since $\nu_I(R^k_{j_k}) \geqslant \nu_I(F_k) > \varepsilon_0$ for each $k=1, 2, \ldots$, we have obtained that $\nu_I(R_k) > \delta_0$ for each $k=1, 2, \ldots$, a contradiction with the exhaustivity of $\nu_I : \mathscr{R} \to \bar{R}_+$.

The theorem is proved.

Theorem 12. *Let μ, $\nu_i : \sigma(\mathscr{R}) \to \bar{R}_+$, $i \in I$ be semimeasures, let $\nu_I : \sigma(\mathscr{R}) \to \bar{R}_+$ be exhaustive and let $\nu_i \ll \mu$ on $\mathscr{R}$ for each $i \in I$. Then $\nu_I \ll \mu$ on $\sigma(\mathscr{R})$.*

Proof. According to Theorem 5 it is enough to show that $N \in \sigma(\mathscr{R})$, $\mu(N)=0 \Rightarrow \nu_i(N)=0$ for each $i \in I$. Let us have fixed $i \in I$, $N \in \sigma(\mathscr{R})$ with $\mu(N)=0$ and $\varepsilon > 0$. Since $\nu_i \ll \mu$ on $\mathscr{R}$, there is a $\delta > 0$ such that $A \in \mathscr{R}$, $\mu(A) < \delta \Rightarrow \nu_i(A) < \varepsilon$. By Theorem 10.1) there is an $E \in \mathscr{R}_\sigma$ such that $N \subset E$ and $\mu(E) < \delta$. Choose $A_n \in \mathscr{R}$, $n=1, 2, \ldots$ so that $A_n \nearrow E$. Then $\mu(A_n) < \delta$, hence $\nu_i(A_n) < \varepsilon$ for each $n=1, 2, \ldots$. But then $\nu_i(E) \leqslant \varepsilon$ by the (F.p.) of ν_i. Since $\varepsilon > 0$ was arbitrary, $\nu_i(N)=0$, what we wanted to show.

Definition 3. *We say that $\nu_i : \mathscr{R} \to R_+$, $i \in I$ are subadditively equicontinuous if for each $A \in \mathscr{R}$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that $i \in I$, $B \in \mathscr{R}$ and $\nu_i(B) < \delta$ imply:*

$$\nu_i(A \cup B) \leqslant \nu_i(A) + \varepsilon \quad \textit{and} \quad \nu_i(A - B) \geqslant \nu_i(A) - \varepsilon.$$

If such a $\delta > 0$ exists commonly for all $A \in \mathscr{R}$, then we say that $\nu_i : \mathscr{R} \to R_+$, $i \in I$ are subadditively equicontinuous uniformly on $\mathscr{R}$.

Clearly, if $\nu_i : \mathscr{R} \to R_+$, $i \in I$ are subadditively equicontinuous and $\nu_I(A) < +\infty$ for each $A \in \mathscr{R}$, then $\nu_I : \mathscr{R} \to R_+$ is subadditively continuous. Further, if $\nu_n : \mathscr{R} \to R_+$, $n=1, 2, \ldots$ are subadditively equicontinuous and if $\nu_n(A) \to \nu(A) \in R_+$ for each $A \in \mathscr{R}$, then obviously $\nu : \mathscr{R} \to R_+$ is subadditively continuous.

Subadditive equicontinuity clearly implies common subadditive continuity. The following simple example shows that the converse is not true even if we have uniform submeasures on a σ-algebra.

Let $T = \{0, 1, 2, \ldots\}$ and let $\mathscr{S} = 2^T$. For $n=1, 2, \ldots$ define $\nu_n : \mathscr{S} \to R_+$ as follows: $\nu_n(\emptyset)=0$, $\nu_n(\{n\})=1/n$, $\nu_n(A)=2$ if $\{0\}\cup\{n\}\subset A$ and $\nu_n(A)=1$ if $\{0\}\cup\{n\}\not\subset A$ and A contains some $k<n$. Finally we define $\nu_n(A)=0$ if $\inf\{k: k\in A\}>n$.

Theorem 13. *Let the semimeasures $\nu_n: \sigma(\mathcal{R})\to R_+$, $n=1, 2, \ldots$ be commonly subadditively continuous, let them be uniformly exhaustive on $\mathcal{R}$ and let $\lim_{n\to\infty}\nu_n(A)\in R_+$ exist for each $A\in\mathcal{R}$. Then $\lim_{n\to\infty}\nu_n(E)\in R_+$ exists for each $E\in\sigma(\mathcal{R})$ and $\nu(E)=\lim_{n\to\infty}\nu_n(E)$, $E\in\sigma(\mathcal{R})$, is monotone and continuous on $\sigma(\mathcal{R})$.*

Proof. Let $E\in\sigma(\mathcal{R})$ and let $\varepsilon>0$. By assumption there is an $\varepsilon_1>0$ such that $B\in\sigma(\mathcal{R})$, $\nu_n(B)<\varepsilon_1$ for each $n=1, 2, \ldots$ implies $\nu_n(E\cup B)\leqslant\nu_n(B)+\varepsilon$ and $\nu_n(E-B)\geqslant\nu_n(E)-\varepsilon$ for each $n=1, 2, \ldots$.

Put

$$\mu(A)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\nu_n(A)}{1+\nu_n(T)},\quad A\in\sigma(\mathcal{R}).$$

Then $\mu: \sigma(\mathcal{R})\to R_+$ is a semimeasure by Lemma 6, and $N\in\sigma(\mathcal{R})$, $\mu(N)=0\Rightarrow\nu_n(A\cup N)=\nu_n(A)$ for each $A\in\sigma(\mathcal{R})$ and each $n=1, 2, \ldots$. Further, by Theorem 11 the sequence ν_n, $n=1, 2, \ldots$ is uniformly continuous on $\sigma(\mathcal{R})$. Thus by Theorem 5 the sequence ν_n, $n=1, 2, \ldots$ is equi-μ-continuous on $\sigma(\mathcal{R})$. Hence there is a $\delta>0$ such that $B\in\sigma(\mathcal{R})$, $\mu(B)<\delta\Rightarrow\nu_n(B)<\varepsilon_1$ for each $n=1, 2, \ldots$. Applying Theorem 15 from part I to μ we find an $A\in\mathcal{R}$ such that $\mu(E\triangle A)<\delta$. Since $E-(A\triangle E)\subset A\subset E\cup(A\triangle E)$, we have the inequality $\nu_n(E)-\varepsilon\leqslant\nu_n(A)\leqslant\nu_n(E)+\varepsilon$, i.e. $|\nu_n(E)-\nu_n(A)|\leqslant\varepsilon$ for each $n=1, 2, \ldots$. Since $\lim_{n\to\infty}\nu_n(A)\in R_+$ exists for each $A\in\mathcal{R}$ by the assumption, there is an n_0 such that $|\nu_n(E)-\nu_m(E)|\leqslant 3\varepsilon$ for each $n, m\geqslant n_0$. Since $\varepsilon>0$ and $E\in\sigma(\mathcal{R})$ were arbitrary, $\lim_{n\to\infty}\nu_n(E)\in R_+$ exists for each $E\in\sigma(\mathcal{R})$. Since also the sequence ν_n, $n=1, 2, \ldots$ is uniformly continuous on $\sigma(\mathcal{R})$, the set function $\nu(E)=\lim_{n\to\infty}\nu_n(E)$, $E\in\sigma(\mathcal{R})$ is continuous on $\sigma(\mathcal{R})$. Its monotonicity is obvious and thus the theorem is proved.

From here and from Theorem 6 we immediately have

Corollary 1. *Let $\nu, \nu_n: \sigma(\mathcal{R})\to R_+$, $n=1, 2, \ldots$ be submeasures and let $\nu_n(A)\to\nu(A)$ for each $A\in\mathcal{R}$. Then the following conditions are equivalent:*

1) *the sequence ν_n, $n=1, 2, \ldots$ is commonly subadditively continuous on $\sigma(\mathcal{R})$, and*

2) *$\nu_n(A)\to\nu(A)$ for each $A\in\sigma(\mathcal{R})$.*

Further, we have

Corollary 2. *Let $\nu_n: \sigma(\mathcal{R})\to R_+$, $n=1, 2, \ldots$ be subadditively equicontinuous* *submeasures and let* $\lim_{n\to\infty} \nu_n(A) \in R_+$ *exist for each* $A \in \mathcal{R}$. *Then the following conditions are equivalent:*

1) ν_n, $n = 1, 2, \ldots$ *are uniformly exhaustive on* $\mathcal{R}$, *and*

2) $\lim_{n\to\infty} \nu_n(A) \in R_+$, *exists for each* $A \in \sigma(\mathcal{R})$ *and* $\nu(A) = \lim_{n\to\infty} \nu_n(A)$, $A \in \sigma(\mathcal{R})$ *is a submeasure.*

Where from using the Cantor diagonal process and the Corollary of Theorem 6 we immediately have

Corollary 3. *Let* $\mathcal{R}$ *be a countable family, see Theorem C, § 5 in* [12], *and let* $\nu_i : \sigma(\mathcal{R}) \to R_+$, $i \in I$ *be subadditively equicontinuous submeasures. Then the following conditions are equivalent:*

1) $\nu_i : \sigma(\mathcal{R}) \to R_+$, $i \in I$ *is a relatively sequentially compact family in the topology of pointwise convergence on* $\sigma(\mathcal{R})$, *and*

2) $\nu_I(A) < +\infty$ *for each* $A \in \mathcal{R}$ *and* $\nu_I : \mathcal{R} \to R_+$ *is exhaustive.*

This Corollary 3 generalizes Theorem 2, § 3 in [2], while the next theorem generalizes Theorem 1, § 3 in [2].

Theorem 14. *Let* $\nu_i : \mathcal{S} \to R_+$, $i \in I$ *be subadditively equicontinuous submeasures and let for each* $i \in I$ *the pseudometrizable uniform space* $(\mathcal{S}, \mathcal{U}_{\nu_i})$ *be separable. (For the definition of* $(\mathcal{S}, \mathcal{U}_{\nu_i})$ *see the paragraph preceding Theorem 14, part I.). Then the following conditions are equivalent:*

1) $\nu_i : \mathcal{S} \to R_+$, $i \in I$ *is a relatively sequentially compact family in the topology of pointwise convergence on* $\mathcal{S}$, *and*

2) $\nu_I(A) < +\infty$ *for each* $A \in \mathcal{S}$ *and* $\nu_I : \mathcal{S} \to R_+$ *is exhaustive.*

Proof. 1)⇒2) by the Corollary of Theorem 6.

2)⇒1). By assertion 2) of Theorem 7 there is a sequence $i_n \in I$, $n = 1, 2, \ldots$ such that $\nu_I \ll \mu$, where

$$\mu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\nu_{i_n}(A)}{1 + \nu_{i_n}(T)}, \quad A \in \mathcal{S}.$$

Clearly, the pseudometrizable uniform space $(\mathcal{S}, \mathcal{U}_\mu)$ is also separable. Hence there is a sequence $E_n \in \mathcal{S}$, $n = 1, 2, \ldots$ which is dense in $(\mathcal{S}, \mathcal{U}_\mu)$. Let $\mathcal{R}$ be the ring generated by this sequence. Then $\mathcal{R}$ is countable, see Th. C, § 5 in [12] and in the same way as in Lemma 3. 1 in [2] we can show that to each $E \in \mathcal{S}$ there is a set $A \in \sigma(\mathcal{R})$ such that $\mu(E \triangle A) = 0$. But then $\nu_I(E \triangle A) = 0$, hence it is enough to prove 1) on $\sigma(\mathcal{R})$. But this is the implication 2)⇒1) of Corollary 3 of Theorem 13.

Lemma 7. *Let* T *be a locally compact Hausdorff topological space and let* $\nu_n : \sigma(\mathcal{B}_\wedge) \to R_+$, $n = 1, 2, \ldots$ *be regular Borel (Baire) semimeasures. Then the following conditions are equivalent:*

1) $\nu_I: \mathscr{C}_\wedge \to R_+$ *is exhaustive, and*

2) $\nu_I: \sigma(\mathscr{C}_\wedge) \to R_+$ *is exhaustive.*

(For notations see Definition 3, part I.)

Proof. 1)$\Rightarrow$2) by the regularity of ν_n, $n = 1, 2, \ldots$, while 2)$\Rightarrow$1) is immediate.

Now in the same way as in Theorem 13 we can prove the following

Theorem 15. *Let T be a locally compact Hausdorff topological space, let $\nu_n: \sigma(\mathscr{B}_\wedge) \to R_+$, $n = 1, 2, \ldots$ be regular Borel (Baire) semimeasures and let $\lim_{n\to\infty} \nu_n(C) \in R_+$ exist for each $C \in \mathscr{C}_\wedge$. Let further $\nu_n: \mathscr{C}_\wedge \to R_+$, $n = 1, 2, \ldots$ be uniformly exhaustive and let their extensions $\nu_n: \sigma(\mathscr{B}_\wedge) \to R_+$ be commonly subadditively continuous. Then $\lim_{n\to\infty} \nu_n(E) \in R_+$ exists for each $E \in \sigma(\mathscr{B}_\wedge)$ and $\nu(E) = \lim_{n\to\infty} \nu_n(E)$, $E \in \sigma(\mathscr{B}_\wedge)$, is monotone and continuous on $\sigma(\mathscr{B}_\wedge)$.*

We omit the obvious formulations of the analogs of Corollaries 1, 2, and 3 of Theorem 13.

We finish this section with the following version of the Vitali—Hahn—Saks theorem.

Theorem 16. *Let the submeasures ν, $\nu_n: \mathscr{R} \to R_+$, $n = 1, 2$, be exhaustive and subadditively equicontinuous uniformly on $\mathscr{R}$. Let further $\mu: \sigma(\mathscr{R}) \to R_+$ be a semimeasure, let $\nu_n \ll \mu$ on $\mathscr{R}$ for each $n = 1, 2, \ldots$, and let $\nu_n(A) \to \nu(A)$ for each $A \in \mathscr{R}$. Then*

1) *the extended submeasures $\nu_n: \sigma(\mathscr{R}) \to R_+$, $n = 1, 2, \ldots$, are subadditively equicontinuous uniformly on $\sigma(\mathscr{R})$,*

2) *$\nu_n(E) \to \nu(E)$ for each $E \in \sigma(\mathscr{R})$, and*

3) *the extended submeasures $\nu_n: \sigma(\mathscr{R}) \to R_+$, $n = 1, 2, \ldots$ are equi-μ-continuous on $\sigma(\mathscr{R})$.*

Proof. 1) follows immediately from the extension procedure for submeasures given in the proof of Theorem 18, part I.

2) follows immediately from 1) and Corollary 1 of Theorem 13, and

3) follows immediately from 2), from Lemma 4 and from Theorem 12.

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Matematický ústav SAV
Obrancov mieru 49
886 25 Bratislava

О СУБМЕРАХ II

Иван Добраков–Яна Фаркова

Резюме

Пусть $\mathscr{R}$ кольцо подмножеств непустого множества T. Функция $\mu:\mathscr{R}\to\langle 0, +\infty)$ называется субмерой, если она монотонна, непрерывна ($A_n\searrow\emptyset \Rightarrow \mu(A_n)\to 0$), и полуаддитивно непрерывна ($\forall A\in\mathscr{R}$ и $\forall\varepsilon>0$ $\exists\delta>0$; $B\in\mathscr{R}$, $\mu(B)<\delta \Rightarrow \mu(A\cup B) \leqq \mu(A)+\varepsilon$ и $\mu(A-B)\geqq\mu(A)-\varepsilon$). В первой части, смотри [4], было показано, что почти все результаты об отдельных мерах имеют обобщения для субмер. В настоящей части исследуются отдельные связи между равномерным отсутствием ускользяющей нагрузки, равностепенной абсолютной непрерывностью, совместной или равностепенной полуаддитивной непрерывностью и слабой компактностью для некоторых семейств функций множеств, в частности для субмер. После вводных замечаний данных в §1, в §2 исследуются упомянутые связи вначале на кольце, после того на σ-кольце, и наконец на σ-кольце порожденном кольцом. Решающую роль в этих исследованиях семейства v_i, $i\in I$, играет поведение функции v_I, $v_I(E) = \sup_{i\in I} v_i(E)$, и эквивалентность полуаддитивной непрерывности μ с абсолютной μ-непрерывностью функций v_1^{A} и v_2^{A}, где $v_1^{A}(B) = \mu(A\cup B)-\mu(A)$, и $v_2^{A}(A) = \mu(A)-\mu(A-B)$.