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Mathematica Slovaca, Vol. 54 (2004), No. 1, 13--21

Persistent URL: http://dml.cz/dmlcz/128934

Math. Slovaca, **54** (2004), No. 1, 13–21

Mathematica Slovaca
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Mathematical Institute
Slovak Academy of Sciences

Dedicated to Professor Sylvia Pulmannová on the occasion of her 65th birthday

ON THE PERMANENCE PROPERTIES OF INTERVAL HOMOGENEOUS ORTHOMODULAR LATTICES

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(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. An orthomodular lattice L is said to be *interval homogeneous* if it is σ-complete and satisfies the following property: Whenever L is isomorphic to an interval $[a, b]$ in L, then L is isomorphic to each interval $[c, d] \supseteq [a, b]$. This class was introduced in the effort to determine the orthomodular lattices which satisfy the Cantor-Bernstein theorem.

In this paper we carry on the investigation of this important class. We investigate permanence properties of this class with respect to the formation of substructures and σ-epimorphic images. We show that there are also fairly complex examples of interval homogeneous orthomodular lattices.

In fact, we show as a main result that every σ-complete orthomodular lattice (abbreviated σ-OML) can be embedded into an interval homogeneous orthomodular lattice. In a somewhat dual sense, we find that each σ-OML is a σ-epimorphic image of an interval homogeneous orthomodular lattice.

1. Introduction

The celebrated Cantor-Bernstein-type result establishes that a sufficient condition for two Boolean σ-algebras to be isomorphic is the existence of isomor-

2000 Mathematics Subject Classification: Primary 06C15, 06E05; Secondary 81P10.
Keywords: orthomodular lattice, σ-completeness, interval, center, Boolean σ-algebra, Cantor-Bernstein theorem.

The first author was supported by the European Union under project Miracle ICA 1-CT-2000-70002, PRIN "Analisi Reale e Teoria della Misura" of MIUR, and GNAMPA of INdAM.

The second author was supported by the Czech Ministry of Education under project MSM 212300013 "Decision Making and Control in Manufacturing" and by the grant 201/03/0455 of the Grant Agency of the Czech Republic.

phisms from each of them onto an interval in the other. Generalizations of this result have been obtained in other contexts as, for example, σ-OMLs ([3]), σ-complete MV-algebras ([2]) and more general structures ([5], [6]). In all generalizations it turned out to be necessary to introduce further requirements, as a rule certain restrictions on the bounds of the intervals in question. Unlike these results, we want to study the class of σ-OMLs which satisfy the Cantor-Bernstein theorem *without any additional assumption*. This class of σ-OMLs has already been introduced in [3] under the name of interval homogeneous OMLs. General examples of interval homogeneous OMLs are presented in [3].

The aim of this paper is to show that the class of interval homogeneous OMLs is much more rich than one would perhaps expect. We prove that any σ-OML may be embedded in an interval homogeneous one. Dually, each σ-OML can be obtained as a σ-epimorphic image of an interval homogeneous one. The constructions presented bring about a large class of interval homogeneous OMLs.

In Section 2 we first give basic notions, we then introduce the definition of interval homogeneous OML and recall some sufficient conditions for an OML to be interval homogeneous. Section 3 contains rather non-trivial examples of interval homogeneous OMLs. The last two sections are devoted to establish the size of the class of interval homogeneous OMLs.

2. Basic notions

We shall only deal with σ-OMLs, i.e. with those OMLs which are closed under the formation of countable suprema and infima (we refer to [1], [7] and [10] for the background on OMLs). We shall frequently use the elementary fact (see [10]) that an interval in a σ-OML constitutes, with the operations naturally inherited from the host OML, a σ-OML. Let us recall that an OML L is called *concrete* if it is isomorphic to a collection of subsets of a set, with set-theoretical complementation as orthocomplementation and orthogonal joins coinciding with disjoint unions (see [10] for more details).

If L is a σ-OML, we define its *center* $C(L)$ as the Boolean sub-σ-algebra consisting of all "absolutely compatible" elements, i.e., to be the set of all elements compatible to each element of L (see [10]). As known, L is a Boolean σ-algebra if and only if its elements are all central.

Let us recall that a sequence $(a_n)_{n\in\mathbb{N}}$ of pairwise orthogonal elements in the center of a σ-OML is called a *central partition of unity* if $\bigvee_{n\in\mathbb{N}} a_n = 1$.

Let us consider two σ-OMLs L and M. We recall that a *σ-homomorphism* between L and M is any mapping $f\colon L \to M$ which preserves the orthocomplementation and the countable lattice operations. If, moreover, it is surjective, we call it a *σ-epimorphism.* By an *isomorphism* between L and M we mean a bijective mapping $f\colon L \to M$ such that both f and f^{-1} are OML σ-homomorphisms (thus, in this case both f and f^{-1} preserve countable infima and suprema). We shall be interested in the class of those σ-OMLs L which, roughly speaking, satisfy the following homogeneity condition: If an interval in L is found isomorphic to L, then all its hyperintervals in L have to be isomorphic to L. Let us formally introduce this class in the following definition.

DEFINITION 2.1. ([3]) Let L be an OML. Then L is said to be *interval homogeneous* if it is σ-complete and enjoys the following property:

If, for some $a, b \in L$, $a \le b$, the interval $[a,b]_L$ is isomorphic to the entire L, then L is isomorphic to each interval $[c,d]_L$ with $c \le a$ and $d \ge b$ $(c,d \in L)$.

Let us denote the class of all interval homogeneous OMLs by Inthom. Our definition can be rephrased in a slightly simplified form.

PROPOSITION 2.2. ([3]) *A σ-OML L belongs to* Inthom *if and only if it enjoys the following property:*

If, for some $a \in L$, the interval $[0,a]_L$ is isomorphic to the entire L, then L is isomorphic to the interval $[0,b]_L$ for each $b \ge a$ $(b \in L)$.

The significance of Inthom to the Cantor-Bernstein theorem known for Boolean σ-algebras can be generalized to σ-OMLs. The following result states it.

PROPOSITION 2.3. ([3]) *A σ-OML is interval homogeneous if and only if it satisfies the following condition:*

If M is a σ-OML such that L is isomorphic to an interval $[0,b]_M$ in M and M is isomorphic to an interval $[0,a]_L$ in L, then L is isomorphic to M.

Let us start by recalling basic examples of σ-OMLs that belong to Inthom. As usual, let us call a maximal Boolean subalgebra of an OML a *block*. As we assume σ-completeness of the OMLs we deal with, each block of a σ-OML is σ-complete, too.

THEOREM 2.4. ([3]) *Let L be a σ-OML. Each of the following conditions is sufficient for L to be interval homogeneous:*

1. *Each block of L is finite,*
2. *L has finitely many blocks (in particular, L is Boolean),*
3. *L is not isomorphic to any its proper subinterval,*
4. *L is the lattice of projections of a separable Hilbert space.*

Remark 2.5. Notice that a necessary condition for L to be isomorphic to some of its proper subintervals is that there is a sequence of intervals $[0,b_i]_L$, $i \in \mathbb{N}$, which are nontrivial, mutually isomorphic and orthogonal. Thus, condition 1 of the latter theorem actually implies condition 3.

It is shown in [3] that there are OMLs which do not belong to Inthom. In what follows, we shall show that also the class of OMLs which are in Inthom is substantially large.

3. Some preliminaries

In this section we present some results which give an idea of the size of the class Inthom. We start by showing that none of the sufficient conditions of Theorem 2.4 is necessary. Recall that by MO_n ($n \in \mathbb{N}$) we mean the horizontal sum of n two-atomic Boolean algebras (MO_ω, ω any cardinal, is the obvious generalization, see [7] for details).

EXAMPLE 3.1. Let I be an uncountable set disjoint from $\mathbb{N}$. We define $L = MO_2^{\mathbb{N}} \times 2^I$. As usual, we will understand $f \in L$ as a mapping defined on $\mathbb{N} \cup I$, $f(r)$ being its rth coordinate. If f is such an element, it has $\operatorname{card} \mathbb{N}$ coordinates in MO_2 and $\operatorname{card} I$ coordinates in the two-element Boolean algebra.

Let $f \in L$ and let us suppose that $[0,f]_L \cong L$. Thus, $[0,f]_L \cong MO_2^{\mathbb{N}} \times 2^I$. The factors of $[0,f]_L$ isomorphic to MO_2 correspond to improper intervals in factors MO_2 of L, so

$$\operatorname{card}\bigl(f^{-1}(1) \cap \mathbb{N}\bigr) = \operatorname{card} \mathbb{N}. \tag{1}$$

The factors of $[0,f]_L$ isomorphic to 2 correspond either to improper intervals in factors 2 of L or to proper intervals in factors MO_2 of L, but there are only countably many of such proper intervals. As a result,

$$\operatorname{card}\bigl(f^{-1}(1) \cap I\bigr) = \operatorname{card} I. \tag{2}$$

We see that conditions (1), (2) are necessary and sufficient for $[0,f]_L \cong L$. Now, let $g \in L$, $g \geq f$. Then $f^{-1}(1) \subseteq g^{-1}(1)$, hence g also satisfies (1), (2) and $[0,g]_L \cong L$.

We have proved that $L \in$ Inthom. It is isomorphic to its proper subintervals. It has uncountably many uncountable blocks. Moreover, L is a complete concrete (= set-representable) modular ortholattice.

We shall need the following lemma. Recall that by the *height* of an OML we mean the supremum over the number of elements of its chains minus one. A horizontal sum of OMLs is called *nontrivial* if it has at least two arguments nonisomorphic to the trivial OML $\{0,1\}$. (The trivial horizontal sum of an OML L and $\{0,1\}$ is isomorphic to L.)

LEMMA 3.2. ([11], [8]) *To every graph we may assign a concrete OML of height* 3 *in such a way that nonisomorphic graphs correspond to nonisomorphic OMLs. Moreover, if the graph is connected, we obtain an OML which is not reducible to a nontrivial horizontal sum.*

COROLLARY 3.3. *There is a proper class of mutually nonisomorphic concrete OMLs of height* 3 *which are not reducible to nontrivial horizontal sums.*

Proof. It suffices to apply Lemma 3.2 to connected graphs. □

We are now ready to give an example with some interesting and useful properties.

LEMMA 3.4. *Let K be a σ-OML of height at least* 3 *and I be a set. Then there are σ-OMLs P_i, $i \in I$, such that*

1. $C(P_i) = \{0_{P_i}, 1_{P_i}\}$,
2. *K is a sub-σ-OML of P_i,*
3. *if P_i is isomorphic to a subinterval Q of some P_j, $j \in I$, then $j = i$ and $Q = P_i$.*

Proof. We apply Corollary 3.3 to find a collection of mutually nonisomorphic OMLs M_i, $i \in I$, such that each of them is

- of height 3,
- not reducible to a nontrivial horizontal sum,
- not isomorphic to a horizontal summand of a subinterval of K.

For $i \in I$, we take for P_i the horizontal sum of K and M_i.

Each proper subinterval of P_j is either a proper subinterval of K or a proper subinterval of M_j. The orthomodular lattice P_i can be neither a proper subinterval of M_j (because it is of height at least 3), nor of K (as P_i contains M_i as a horizontal summand of the improper subinterval). It remains to check the isomorphism of P_i with the improper subinterval, i.e., with P_j, but we have chosen these OMLs mutually nonisomorphic. □

Remark 3.5. In the latter lemma, instead of choosing M_i, $i \in I$, of height 3, we could have chosen MO_{ω_i} (which is of height 2) with mutually different cardinalities ω_i, $i \in I$. Then the verification of the properties is easier, but the whole construction becomes very extensive. Observe that each subinterval of MO_ω (ω arbitrary) is isomorphic either to MO_ω, to $\mathbf{2}$, or to the degenerate (one-element) Boolean algebra.

4. Epimorphic images of Inthom

In this section we study the relation of interval homogeneity to σ-epimorphic images. As a main result, we prove that each σ-OML is a σ-epimorphic image of an element of Inthom.

THEOREM 4.1. *Every σ-OML is a σ-epimorphic image of an interval homogeneous OML.*

Proof. Let K be a σ-OML. Suppose that $K \notin$ Inthom. According to Theorem 2.4, K is of infinite height.

Let I be an uncountable (indexing) set. We apply Lemma 3.4 and obtain σ-OMLs P_i, $i \in I$, which contain K as a sub-σ-OML. Let us form the product $P = \prod_{i \in I} P_i$. From P we select the sub-σ-OML L of all $f \in P$ for which there exists $e(f) \in K$ such that the set

$$\{i \in I : \ f(i) \neq e(f)\}$$

is (at most) countable. Notice that if such element $e(f)$ exists, it is unique. We claim that

1. L is a sub-σ-OML of P and $e\colon L \to K$ is a σ-epimorphism,
2. $L \in$ Inthom.

Obviously, L contains 0_P. Further, L is closed under the formation of orthocomplements. Indeed, if $f \in L$, then $e(f) \in K$ and we take $e(f^\perp) = e(f)^\perp$. Finally, let $(f_n)_{n \in \mathbb{N}} \in L^{\mathbb{N}}$. Denote by

$$f = \bigvee_{n \in \mathbb{N}} f_n$$

the (pointwise) supremum calculated in P. We have to prove that $f \in L$. As $e(f_n) \in K$, we may define $e(f)$ by

$$e(f) = \bigvee_{n \in \mathbb{N}} e(f_n)$$

(the supremum is taken in K). Then

$$\{i \in I : \ f(i) \neq e(f)\} \subseteq \bigcup_{n \in \mathbb{N}} \{i \in I : \ f_n(i) \neq e(f_n)\},$$

which is a countable set. Thus we proved that L is a sub-σ-OML of P and, as a by-product, that e is a σ-homomorphism. The surjectivity of e is immediate, thus it is a σ-epimorphism.

In order to prove that $L \in$ Inthom, we shall verify Condition 3 of Theorem 2.4. Assume that there exists an element $f \in L$ and an isomorphism $h\colon L \to [0, f]_L$. We shall prove that $f = 1_L$. The center of L is composed of all 0–1-valued functions of L, i.e., $C(L) = L \cap \{0,1\}^I$. More exactly, $C(L)$ contains all characteristic functions of countable and co-countable subsets of I.

Let $i \in I$. The element $u_i \in L$ defined by the requirement

$$u_i(j) = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

is an atom of $C(L)$. The interval $[0, u_i]_L$ is isomorphic to P_i. Its image under h, $[0, h(u_i)]_L$, must be isomorphic to P_i and its upper bound $h(u_i)$ has to be an atom of $C([0, f]_L)$. A central atom g in $[0, f]_L$ cannot have two nonzero coordinates $g(j)$ and $g(k)$: the element v defined by

$$v(m) = \begin{cases} g(j) & \text{if } m = j\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

would be a nonzero central element of $[0, f]_L$ strictly smaller than g, which contradicts the atomicity of g.

Applying the above argument to $h(u_i)$, we see that $h(u_i)$ has only one nonzero coordinate. Thus $[0, h(u_i)]_L$ is isomorphic to a subinterval of some P_j $(j \in I)$. But this may occur only if $j = i$ and $h(u_i) = 1_{P_i}$. Thus $f(i) = 1_{P_i}$, too. This means that all the coordinates $f(i)$ of the element f are equal to 1 (for all $i \in I$). This completes the proof. □

COROLLARY 4.2. *The class* Inthom *is not closed under the formation of σ-epimorphic images.*

We have already observed that not all "interesting" OMLs are in the class Inthom. Note also that the class of "non-interval homogeneous" OMLs is in fact quite large, too.

THEOREM 4.3. *Every σ-OML is a σ-epimorphic image of a σ-OML which is not interval homogeneous.*

The proof of the latter result is provided later since the most efficient construction yielding it is based on the following section.

5. The formation of subalgebras

The main result of this section reads as follows:

THEOREM 5.1. *Every σ-OML is a sub-σ-OML of an interval homogeneous OML.*

P r o o f . Let K be a σ-OML. Using Corollary 3.3, it is possible to find an orthomodular lattice M of height 3 such that M is not a horizontal summand of a subinterval in K. We form L, a horizontal sum of K and M. The only subinterval of L which contains M as a horizontal summand of a subinterval is the improper interval L. Thus, L satisfies condition 3 of Theorem 2.4 and $L \in$ Inthom. Note that L is concrete (resp. complete) if K is so. □

Remark 5.2. There is an alternative way to prove the latter result: It suffices to proceed like in Theorem 4.1 and notice that the constant elements of L in the proof also form a sub-σ-OML of L isomorphic to K. We wanted to present another proof since it preserves also completeness.

As a consequence, the latter proof presents some examples of σ-OMLs which do not satisfy the first two conditions of Theorem 2.4. Indeed, they have infinite blocks and infinitely many blocks.

Note that the interplay seen in Theorems 4.1 and 4.3 has an analogy here as well. In fact, the basic technical tool of the envisaged proof of Theorem 4.3 can be clearly seen in this way.

THEOREM 5.3. *Every σ-OML is a sub-σ-OML of a σ-OML which is not interval homogeneous.*

Proof. Take a σ-OML K. In the first step, using the construction from the proof of Theorem 4.1, we embed K into a σ-OML $P \in$ Inthom which is not isomorphic to any of its proper subintervals and which has a trivial center. Take $L = P^{\mathbb{N}}$, then L is isomorphic to the interval $\big[(0,0,0,\dots),(0,1,1,\dots)\big]_L$, a natural isomorphism being the "shift" mapping $h\colon P^{\mathbb{N}} \to P^{\mathbb{N}}$,

$$h(a_1,a_2,a_3,\dots,a_n,\dots) = (0,a_1,a_2,\dots,a_{n-1},\dots)\,.$$

On the other hand, L is easily seen to be nonisomorphic to the interval $J = \big[(0,0,0,\dots),(x,1,1,\dots)\big]_L$ for $x \notin \{0,1\}$: The element $(x,0,0,0,\dots)$ is an atom in the center of J, and therefore it should be the image of an atom in the center of L. This is impossible since the interval $[0,x]_P$ is not isomorphic to P. □

We are now ready to provide the proof of Theorem 4.3.

Proof of Theorem 4.3. Let K be an OML which is not interval homogeneous. Take an OML P which is not isomorphic to any of its proper subintervals (the description of the construction needed is contained in the proof of Theorem 5.3). Choose P in such a manner that it also contains a proper subinterval $[0,x]_P$ which has a trivial center and is not isomorphic to any interval of K. The product $L = K \times P^{\mathbb{N}}$ is isomorphic to its proper subinterval $\big[(0,0,0,\dots),(1,0,1,1,\dots)\big]_L$ by the shift applied on the factor $P^{\mathbb{N}}$. On the other hand, the interval $\big[(0,0,0,\dots),(1,x,1,1,\dots)\big]_L$ is not isomorphic to L. Indeed, it contains a factor $\big[(0,0,0,\dots),(0,x,0,0,\dots)\big]_L$ isomorphic to $[0,x]_P$ that is neither isomorphic to a subinterval of P or K, nor decomposable to a product of smaller OMLs. Obviously, K is a σ-epimorphic image of L under the canonical projection. □

Acknowledgement

The authors thank to Pavel Pták for his permanent inspiration.

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Received March 31, 2003

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