Mathematica Slovaca

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Mathematica Slovaca, Vol. 55 (2005), No. 2, 141--154

Persistent URL: http://dml.cz/dmlcz/128944

Math. Slovaca, **55** (2005), No. 2, 141–154

Mathematica Slovaca
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Slovak Academy of Sciences

ON THE k-DOMINATING NUMBER OF CARTESIAN PRODUCTS OF TWO PATHS

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(*Communicated by Martin Škoviera*)

ABSTRACT. A subset $D \subset V(G)$ is called a *k-dominating set*, $k \geq 1$, if for every vertex y not in D, there exists at least one vertex $x \in D$ such that $d(x,y) \leq k$. For convenience we also say that D *k-dominates* G. The *k-domination number* $\gamma_k(G)$ is the cardinality of a smallest k-dominating set. The *1-domination number* is also called the *domination number*.

In this paper we determine the exact values of $\gamma_k(P_1 \square P_n), \dots, \gamma_k(P_3 \square P_n)$, 2-domination numbers $\gamma_2(P_4 \square P_n), \dots, \gamma_2(P_7 \square P_n)$, estimates for $\gamma_k(P_m \square P_n)$ when $k \geq m-1$ and $\lim\limits_{m,n\to\infty} \frac{\gamma_k(P_m \square P_n)}{mn}$ where P_n denote the path of length n.

1. Introduction and terminology

For any graph G we denote the vertex-set and the edge-set of G by $V(G)$ and $E(G)$, respectively.

The Cartesian product of two graphs G, H is a graph with vertex set $V(G) \times V(H)$ and $\big((g_1,h_1),(g_2,h_2)\big) \in E(G \square H)$ if and only if either $g_1 = g_2$ and $(h_1,h_2) \in E(H)$, or $(g_1,g_2) \in E(G)$ and $h_1 = h_2$.

The study of domination numbers of products of graphs was initiated by Vizing [16]. He conjectured that the domination number of the Cartesian product of two graphs is always greater than or equal to the product of the domination numbers of two factors; a conjecture which is still unproven.

Domination numbers of Cartesian products were intensively investigated in the past (see e.g. [1], [2], [5], [6], [10]).

In this paper we extend these investigations to k-domination for $k \geq 2$.

2000 Mathematics Subject Classification: Primary 05C69.
Keywords: k-dominating number, Cartesian product, path.

k-Domination numbers of $P_2 \square P_n$ and $P_3 \square P_n$

To fix terminology for the proofs of our results we need some more definitions.

DEFINITION 1. Let $1, \dots, k$ and $1, \dots, n$ be vertices of paths P_k and P_n, respectively.

For a fixed m, $1 \le m \le n$, the set $(P_k)_m = \{(1,m), (2,m), (3,m), \dots, (k,m)\}$ is called a *column* of $P_k \square P_n$. The set $(P_n)^l = \{(l,1), (l,2), (l,3), \dots, (l,n)\}$, $1 \le l \le k$, is called a *row* of $P_k \square P_n$.

Any set $B = \{(P_k)_m, (P_k)_{m+1}, \dots, (P_k)_{m+l} : \ l \ge 0, \ m \ge 1, \ m+l \le n\}$ of consecutive columns is called a *block of size* $k \times (l+1)$ of $P_k \times P_n$. If another block B' contains column $(P_k)_{m-1}$ (but it does not contain the column $(P_k)_m$), or the column $(P_k)_{m+l+1}$ (but not column $(P_k)_{m+l}$), then B' is said to be *adjacent* to B. Block B is called *internal* if it is adjacent to two other blocks. It is called *external*, if it is adjacent only to one block.

The following observations will be frequently used in the sequel.

OBSERVATION 1. *Let C_n and P_n denote the cycle and path with n vertices, respectively. Then*

$$\gamma_k(C_n) = \gamma_k(P_n) = \left\lceil \frac{n}{2k+1} \right\rceil .$$

The following theorem was shown by Jacobson and Kinch [5].

THEOREM 1. *We have*

$$\gamma_1(P_2 \square P_n) = \left\lceil \frac{n+1}{2} \right\rceil ,$$
$$\gamma_1(P_3 \square P_n) = n - \left\lfloor \frac{n-1}{4} \right\rfloor .$$

These two results can be easily extended to all $k \ge 1$.

THEOREM 2. *Let $k \ge 1$. Then*

$$\gamma_k(P_2 \square P_n) = \begin{cases} \frac{n}{2k} + 1, & n \equiv 0 \pmod{2k}, \\ \left\lceil \frac{n}{2k} \right\rceil, & \textit{otherwise.} \end{cases}$$

Proof. It is obvious that for $k = 1$ our result reduces to the result given in Theorem 1. We consider the set

$$S = \left\{(1, k+4kl) : \ l = 0, 1, \dots, \left\lfloor \tfrac{n-k}{4k} \right\rfloor \right\} \cup \left\{(2, 3k+4kl) : \ l = 0, 1, \dots, \left\lfloor \tfrac{n-3k}{4k} \right\rfloor \right\} .$$

It is easy to check that S is a k-dominating set for $n \equiv k, \dots, (2k-1) \pmod{2k}$. Also in these cases each vertex of $P_2 \square P_n$ is k-dominated by exactly one vertex of S. For all these cases $|S| = \left\lceil \frac{n}{2k} \right\rceil$.

Proof of minimality follows from the fact that on $P_2 \square P_{2k-1}$ only one k-dominating vertex can k-dominate all vertices, but on $P_2 \square P_{2k}$ we need at least two vertices. (For each vertex on $P_2 \square P_{2k}$ there exists at least one other vertex such that distance between them is $\geq k+1$.)

For $n \equiv 0, 1, \dots, (k-1) \pmod{2k}$ we take $S_1 = S \cup \{(2,n)\}$. It can be easily seen that for these n, S_1 is a k-dominating set, and

$$\gamma_k(P_2 \square P_n) = \begin{cases} \frac{n}{2k} + 1, & n \equiv 0 \pmod{2k}, \\ \left\lceil \frac{n}{2k} \right\rceil, & \textit{otherwise.} \end{cases}$$

Proof of minimality follows from previous and from the fact that for S on $P_2 \square P_n$ at least one vertex is not k-dominated. $\square$

In the sequel we investigate k-dominating sets on $P_3 \square P_n$.

Lemma 1. *Let $k \geq 2$, $n \geq 2$. Then there exists a minimum k-dominating set D of $P_3 \square P_n$ such that for every $i \in \{1, \dots, n\}$, $|(P_3)_i \cap D| \leq 1$.*

Proof. Let D be a minimum k-dominating set of $P_3 \square P_n$. We first assume that $|(P_3)_1 \cap D| \geq 2$ holds. Without loss of generality, let $(1,1) \in D$ and let M denote the set of vertices which are k-dominated by the vertices of $(P_3)_1 \cap D$. Then $D' = \big(D \cup \{(1,2)\}\big) \setminus \{(1,1)\}$ also k-dominates at least vertices of M. (If $(1,2) \in D$, then D is not minimal.) Hence, we can conclude that $(P_3)_1$ contains at most one vertex of D and the same holds for $(P_3)_n$.

Assume $|(P_3)_i \cap D| = 3$ holds for some $(P_3)_i$, $2 \leq i \leq n-1$. If M denotes the set of vertices k-dominated by the vertices of $(P_3)_i$, then $D' = \big(D \setminus \{(2,i),(3,i)\}\big) \cup \{(3,i{+}1),(3,i{-}1)\}$ also k-dominates all vertices of M and therefore k-dominates $P_3 \square P_n$.

If now either $(P_3)_{i-1}$ or $(P_3)_{i+1}$ contains three vertices of D', then we repeat this process and finally obtain either a contradiction to the minimality of D or a k-dominating set with at most two vertices of one column of $P_3 \square P_n$.

We now assume that $|(P_3)_i \cap D| = 2$ holds for some i, $2 \leq i \leq n-1$, and no $(P_3)_j$, $1 \leq j \leq n$, contain more than 2 vertices of D.

a) We first consider the case that $\{(1,i),(2,i)\} \subset D$. Then $\{(1,i),(2,i)\}$ k-dominate vertices of $M = \{(j,i{-}1),(j,i{-}2),\dots,(j,i{-}k)\} \cup \{(j,i{+}1),(j,i{+}2),\dots,(j,i{+}k)\} \cup \{(3,i{-}1),(3,i{-}2),\dots,(3,i{-}k{+}1)\} \cup \{(3,i),(3,i{+}1),(3,i{+}2),\dots,(3,i{+}k{-}1)\}$, where $j \in \{1,2\}$.

But vertices $\{(2,i{-}1),(2,i{+}1)\}$ also k-dominate M. Hence, $D' = \big(D \setminus \{(1,i),(2,i)\}\big) \cup \{(2,i{-}1),(2,i{+}1)\}$ also k-dominate $P_3 \square P_n$. If now $(P_3)_{i-1}$ or $(P_3)_{i+1}$ contains three vertices from D', then D' and therefore D is not minimum. D is also not minimum if $(2,i{-}1)$ or $(2,i{+}1)$ is already contained in D.

Now assume that $|(P_3)_{i+1} \cap D| = 1$ ($|(P_3)_{i+1} \cap D'| = 2$) holds. From previous notations it follows that $(1, i+1)$ or $(3, i+1) \in D$, say $(1, i+1)$.

Then for $i \leq n-3$ we set $D'' = \big(D' \setminus \{(1, i+1)\}\big) \cup \{(2, i+3)\}$ and repeat the above arguments. If $n-2 \leq n \leq n-1$, then $D'' = \big(D' \setminus \{(1, i+1)\}\big)$.

b) For $\{(2,i),(3,i)\} \subset D$ and $\{(1,i),(3,i)\} \subset D$ analogously as in a). □

THEOREM 3. *For every path P_n, $n \geq 2$, and $k \geq 2$*

$$\gamma_k(P_3 \,\square\, P_n) = \left\lceil \frac{n}{2k-1} \right\rceil .$$

Proof. We consider the set

$$S = \Big\{\big(2, k+(2k-1)l\big) : \ l = 0, 1, \dots, \lfloor \tfrac{n}{2k-1} \rfloor - 1\Big\} .$$

For $n \equiv 0 \pmod{(2k-1)}$, S is a k-dominating set and

$$|S| = \frac{n}{2k-1} = \left\lceil \frac{n}{2k-1} \right\rceil .$$

For $n \equiv k, \dots, (2k-2) \pmod{(2k-1)}$, $S_1 = S \cup \Big\{\big(2, k+(2k-1)\cdot\lfloor \tfrac{n}{2k-1} \rfloor\big)\Big\}$ is a k-dominating set and

$$|S_1| = \left\lceil \frac{n}{2k-1} \right\rceil .$$

For $n \equiv 1, \dots, (k-1) \pmod{(2k-1)}$, $S_1 = S \cup \{(2, n)\}$ is a k-dominating set and

$$|S_1| = \left\lceil \frac{n}{2k-1} \right\rceil .$$

It follows that $\gamma_k(P_3 \,\square\, P_n) \leq \lceil \frac{n}{2k-1} \rceil$.

We now prove that these sets are also minimum k-dominating sets.

Let D be a minimum k-dominating set which satisfies Lemma 1. Let s be the number of columns which contain no vertex of D. Then, since at most $(2k-2)$ empty columns can be adjacent,

$$s \leq \left\lfloor \frac{n}{2k-1} \right\rfloor (2k-2) .$$

Then for every k-dominating set D there holds

$$|D| \geq n - s \geq n - \left\lfloor \frac{n}{2k-1} \right\rfloor \cdot (2k-2) .$$

If $n \equiv 0 \pmod{(2k-1)}$,

$$|D| \geq \left\lfloor \frac{n}{2k-1} \right\rfloor \cdot (2k-1) - \left\lfloor \frac{n}{2k-1} \right\rfloor \cdot (2k-2) = \left\lfloor \frac{n}{2k-1} \right\rfloor = \left\lceil \frac{n}{2k-1} \right\rceil .$$

If $n \equiv t \pmod{(2k-1)}$, $1 \leq t \leq 2k-2$,

$$|D| \geq \left\lfloor \frac{n}{2k-1} \right\rfloor \cdot (2k-1) + t - \left\lfloor \frac{n}{2k-1} \right\rfloor \cdot (2k-2) = \left\lfloor \frac{n}{2k-1} \right\rfloor + t \geq \left\lceil \frac{n}{2k-1} \right\rceil .$$

□

3. 2-Domination numbers

Of course, the situation is much more complex if we consider $P_j \,\square\, P_n$ for some $j \geq 4$. Even the determination of the k-domination number for $k = 1$ is difficult if $j \geq 5$ holds. Hence, we only consider 2-domination for $P_j \,\square\, P_n$, $4 \leq j \leq 7$.

THEOREM 4. *For $n \geq 2$,*

$$\gamma_2(P_4 \,\square\, P_n) = \begin{cases} 3\lfloor \frac{n}{7} \rfloor + 2\,, & n \equiv 1,2,3 \pmod 7\,, \\ 3\lceil \frac{n}{7} \rceil\,, & n \equiv 4,5 \pmod 7\,, \\ 3\lceil \frac{n}{7} \rceil + 1\,, & n \equiv 6,0 \pmod 7\,, \\ 4\,, & n = 8\,. \end{cases}$$

Proof. We consider the set

$$\begin{aligned} S = \ & \big\{(1,7{+}14k) : \ k{=}0,1,\dots,\lfloor \tfrac{n-7}{14} \rfloor\big\} \cup \big\{(1,13{+}14k) : \ k{=}0,1,\dots,\lfloor \tfrac{n-13}{14} \rfloor\big\} \\ & \cup \big\{(2,3{+}14k) : \ k{=}0,1,\dots,\lfloor \tfrac{n-3}{14} \rfloor\big\} \cup \big\{(3,10{+}14k) : \ k{=}0,1,\dots,\lfloor \tfrac{n-10}{14} \rfloor\big\} \\ & \cup \big\{(4,6{+}14k) : \ k{=}0,1,\dots,\lfloor \tfrac{n-6}{14} \rfloor\big\} \cup \big\{(4,14{+}14k) : \ k{=}0,1,\dots,\lfloor \tfrac{n}{14} \rfloor{-}1\big\} \\ & \cup \big\{(3,1)\big\}. \end{aligned}$$

See Figure 1.

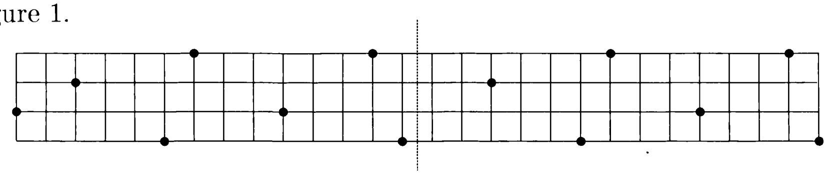

FIGURE 1.

Then S is a 2-dominating set for $n \equiv 0,3 \pmod 7$.

Let $n \geq 2$ and $S_1 = S \cup \big\{(2,n)\big\}$. S_1 is a 2-dominating set for $n \equiv 1,2,4 \pmod 7$.

If the number of $4 \,\square\, 7$ blocks is odd, let $S_2 = S \cup \big\{(2,n)\big\}$; otherwise, let $S_2 = S \cup \big\{(3,n)\big\}$. Then S_2 is a 2-dominating set for $n \equiv 5 \pmod 7$.

For $n \equiv 6 \pmod 7$ a 2-dominating set S_3 is given by $S_3 = S \cup \big\{(2,n)\big\}$ if the number of $4 \,\square\, 7$ blocks is even, and by $S_3 = S \cup \big\{(3,n)\big\}$ if the number of $4 \,\square\, 7$ blocks is odd.

It follows that

$$\gamma_2(P_4 \,\square\, P_n) \leq |S| = \begin{cases} 3\lfloor \frac{n}{7} \rfloor + 2\,, & n \equiv 1,2,3 \pmod 7\,, \\ 3\lceil \frac{n}{7} \rceil\,, & n \equiv 4,5 \pmod 7\,, \\ 3\lceil \frac{n}{7} \rceil + 1\,, & n \equiv 6,0 \pmod 7\,. \end{cases}$$

We now prove that $\gamma_2(P_4 \,\square\, P_n) \geq |S|$.

Definition 2. We partition $P_4 \square P_n$ into $\lfloor \frac{n}{7} \rfloor$ blocks of size $4 \square 7$, where the first $4 \square 7$ block contains $(P_4)_1$. If $n \equiv k \pmod 7$, where $k \neq 0$, then we also have a block B', which is a $4 \square k$ block. Without loss of generality, we always assume that $B' = \{(P_4)_n, \dots, (P_4)_{n-k+1}\}$.

Lemma 2. *If B is an external $4 \square 7$ block, for every 2-dominating set D, there holds $|D \cap B| \geq 3$.*

Proof. Without loss of generality, assume that B is the first block in the graph $P_4 \square P_n$. (Only if $n \equiv 0 \pmod 7$ there are 2 external $4 \square 7$ blocks.) If columns $(P_4)_6$ and $(P_4)_7$ are 2-dominated by vertices from the adjacent block, there is still one undominated block of size $4 \square 5$. To 2-dominate these vertices, we need at least three vertices from block B. □

Lemma 3. *$|D \cap B| \geq 2$ holds for every internal block B.*

Proof. Let $B = \{(P_4)_j, (P_4)_{j+1}, \dots, (P_4)_{j+6}\}$, $j \geq 8$, be some internal block. Only vertices of columns $(P_4)_j$, $(P_4)_{j+1}$ and $(P_4)_{j+5}$, $(P_4)_{j+6}$ can be 2-dominated by vertices of adjacent blocks. To 2-dominate vertices of columns $(P_4)_{j+2}$, $(P_4)_{j+3}$, $(P_4)_{j+4}$, we always need at least two vertices which are contained in B. □

Lemma 4. *Let $n \geq 21$. If $|D \cap B_k| = 2$ for some internal $4 \square 7$ block B_k, then $|D \cap B_{k-1}| \geq 4$, and $|D \cap B_{k+1}| \geq 4$. If B_{k-1} (B_{k+1}) is external, then $|D \cap B_{k-1}| \geq 5$ ($|D \cap B_{k+1}| \geq 5$).*

Proof. Let $B_k = \{(P_4)_j, (P_4)_{j+1}, \dots, (P_4)_{j+6}\}$, $j = 7(k-1)+1$, $k \in \{2, \dots, \lfloor \frac{n}{7} \rfloor - 1\}$. Let $B_k \cap D = M$ and let $|M| = 2$. Then both vertices of M must be contained in the set $L = (P_4)_{j+2} \cup (P_4)_{j+3} \cup (P_4)_{j+4}$ since no vertex outside B_k can 2-dominate a vertex of these three columns. This also implies that vertices of M are contained in different columns. Hence, vertices of M 2-dominate at most one vertex of $(P_4)_j$ and at most one vertex of $(P_4)_{j+6}$.

Case a):

Vertices of M 2-dominate exactly one vertex of $(P_4)_{j+6}$ and exactly one vertex of $(P_4)_j$.

In this case two vertices of M must be contained in $(P_4)_{j+2}$ and $(P_4)_{j+4}$, respectively. Then three vertices of $(P_4)_j$, $(P_4)_{j+6}$, at least one vertex of $(P_4)_{j+1}$ and at least one vertex of $(P_4)_{j+5}$ remain undominated by vertices of M. We first consider B_{k+1}. Since three vertices of $(P_4)_{j+6}$ and at least one vertex of $(P_4)_{j+5}$ are not 2-dominated by vertices of M, the first column of B_{k+1} (i.e. the column $(P_4)_{j+7}$) contains at least two vertices of D.

But these two vertices cannot 2-dominate any vertex of the columns $(P_4)_{j+10}$ and $(P_4)_{j+11}$. Vertices of $(P_4)_{j+10}$ and $(P_4)_{j+11}$ cannot be 2-dominated by vertices contained in B_{k+2}. To 2-dominate these two columns, we need at least two vertices of D which are contained in B_{k+1}.

If B_{k+1} is the last block, we need at least one more vertex to 2-dominate the remaining vertices on B_{k+1}.

Of course the same holds for B_{k-1}.

Case b):
Vertices of M 2-dominate exactly one vertex of $(P_4)_{j+6}$ and no vertex of $(P_4)_j$. In this case it is obvious that $|D \cap B_{k-1}| \geq 4$ holds.

The situation on B_{k+1} is the same as in the Case 1.

Case c):
Vertices of M 2-dominate no vertex of $(P_4)_j$ and no vertex of $(P_4)_{j+6}$. In this case our result obviously holds. □

Applying Lemma 4, it is now possible to prove Theorem 4 in the case each n is a multiple of 7.

Case 1: $n = 7m$.
We first assume that $n \geq 21$.

Let D be any 2-dominating set. $|D \cap B_k| \geq 2$ holds for each block B_k, $1 \leq k \leq \frac{n}{7}$, by Lemma 3. Assume that there are s $(4 \square 7)$-blocks which contain only two vertices of D. By Lemma 2, these blocks are internal. Then, by Lemma 4, there are at least $s+1$ $(4 \square 7)$-blocks which contain at least four vertices of D. Let B_{i_j}, $1 \leq j \leq 2s+1$, denote these blocks which contain either two or four vertices. Then $\mathcal{B} = \bigcup_{j=1}^{2s+1} B_{i_j}$ contains at least $6s+4$ vertices of D. Together we have $m = \frac{n}{7}$ $(4 \square 7)$-blocks. $2s+1$ blocks of $\mathcal{B}$ contain $3(2s+1)+1$ vertices of D, the remaining $r = m - 2s - 1$ $(4 \square 7)$-blocks at least $3r$ vertices of D. Therefore $|D| \geq 3m + 1 = |S|$, which completes the proof in this case.

Let $n = 14$. $|D \cap B_k| \geq 3$ holds for each block B_k, $k = 1, 2$, by Lemma 2. If $|D \cap B_1| = 3$, at least one vertex of B_1 is 2-dominated by vertices of B_2. Then it is obvious that $|D \cap B_2| \geq 4$ and therefore $|D| \geq |S|$ holds.

Case 2: $n = 7m + 1$.

Lemma 5. $|D \cap (B_m \cup B')| \geq 4$ *for any 2-dominating set* D.

Proof. $B_m \cup B'$ is a $4 \square 8$ block. If the first two columns of B_m are 2-dominated by vertices of B_{m-1}, there is still an undominated block of size $4 \square 6$. To 2-dominate vertices of this block we need at least four vertices which are contained in $B_m \cup B'$. □

Lemma 6. *Let* $n \geq 15$. *If* $|D \cap (B_m \cup B')| = 4$, *then* $|D \cap B_{m-1}| \geq 3$, *and if* B_{m-1} *is external, then* $|D \cap B_{m-1}| \geq 4$.

P r o o f . $B_m \cup B'$ is a $4\square 8$ block, and, by Lemma 5, it contains at least 4 vertices of D. If $|B_m \cup B'| = 4$ holds, then $(P_4)_{n-7} \cap D = \emptyset$ and $|(P_4)_{n-6} \cap D| \leq 1$ must hold.

If B_{m-1} is an internal block, then at most the first two columns of B_{m-1} can be 2-dominated by vertices of B_{m-2}.

By the same arguments as in the proof of Lemma 4, we obtain that $|D \cap B_{m-1}| \geq 3$ holds. If B_{m-1} is already external (i.e. $n = 15$), then we need at least four vertices to 2-dominate all vertices of B_{m-1}, which means that $|D \cap B_{m-1}| \geq 4$. □

Let D be any 2-dominating set. Again we assume that there are s blocks containing only two vertices of D. From Lemma 6, if the block $B_m \cup B'$ contains only four vertices of D, then B_{m-1} cannot be a block containing only two vertices of D. Therefore, we can again apply Lemma 4 to show that $|D| \geq 3\lfloor \frac{n}{7} \rfloor + 2 = |S|$.

The case $n = 8$ can be checked directly.

If $n = 7m + 2$ or $n = 7m + 3$, we cannot have a minimal 2-dominating set with less vertices than in the case $n = 7m + 1$. Therefore, our result also holds in these cases.

Case 3: $n = 7m + 4$.

Lemma 7. *$|D \cap B'| \geq 2$ for any 2-dominating set D.*

P r o o f . B' is a $4\square 4$ block. At most the first two columns can be 2-dominated by vertices from B_{m-1}. To 2-dominate the remaining vertices, we need at least two vertices which are contained in B'. □

Lemma 8. *Let $|D \cap B'| = 2$. Then $|D \cap B_m| \geq 3$ if B_m is internal, and $|D \cap B_m| \geq 4$ if B_m is external.*

P r o o f . The same kind of argument as in the proofs of the above lemmas immediately lead to this result. □

We now assume that there exist s blocks B_{j_i}, $1 \leq s$, $j_i < m - 1$, with $|B_{j_i} \cap D| = 2$. Let $|B_m \cap D| = 3$. Then, by Lemma 4, there are also $s+1$ blocks B_{k_i}, with $|B_{k_i} \cap D| \geq 4$. This, together with Lemma 8, is sufficient to show that $|D| \geq |S|$ holds for every 2-dominating set D.

If $|D \cap B_m| \geq 4$ holds, we again assume that there are s blocks B_{j_i}, $j_i \leq m - 1$, which contain only two vertices of D. As above, Lemma 4 now immediately implies that there are also s blocks B_{k_i} with $|B_{k_i} \cap D| \geq 4$, and then again $|D| \geq |S|$.

For $n = 7m + 5$ minimality follows directly from the fact that we need at least as many vertices to 2-dominate $P_4 \square P_n$ as in the case of $n = 7m + 4$.

Case 4: $n = 7m + 6$.

LEMMA 9. *$|D \cap B'| \geq 3$ for any 2-dominating set D.*

Proof. The same as in Lemma 7. □

LEMMA 10. *If $|D \cap B'| = 3$, then $|D \cap B_m| \geq 3$.*

Proof. The same as in the Lemma 6. □

If B' contains at least four vertices, then we can again apply Lemma 4 as above to obtain that $|D| \geq |S|$ holds. If $|B' \cap D| = 3$ holds, then B_m contains more than two vertices of D, and Lemma 4 again completes the proof. □

The results about $\gamma_2(P_5 \square P_n)$, $\gamma_2(P_6 \square P_n)$ and $\gamma_2(P_7 \square P_n)$ are given without proof of minimality, because these proofs are long and tedious. They go along similar lines as for $\gamma_2(P_4 \square P_n)$. We partition graph into blocks. Then we consider how many vertices we must at least have on some block. On $P_5 \square P_n$ we have $5 \square 6$ blocks, on $P_6 \square P_n$ $6 \square 5$ blocks and on $P_7 \square P_n$ $7 \square 6$ blocks.

THEOREM 5. *For $n \geq 2$*

$$\gamma_2(P_5 \square P_n) = \begin{cases} 3\lfloor \frac{n}{6} \rfloor + 1, & n \equiv 1 \pmod 6, \\ 3\lfloor \frac{n}{6} \rfloor + 2, & n \equiv 2 \pmod 6, \\ 3\lfloor \frac{n}{6} \rfloor + 3, & n \equiv 3,4 \pmod 6, \\ 3\lceil \frac{n}{6} \rceil + 1, & n \equiv 5,0 \pmod 6. \end{cases}$$

Proof. We consider the set

$$S = \left\{(1, 4{+}6k) : \ k = 0, 1, \dots, \lfloor \tfrac{n-4}{6} \rfloor\right\} \cup \left\{(3, 1{+}6k) : \ k = 0, 1, \dots, \lfloor \tfrac{n-1}{6} \rfloor\right\}$$
$$\cup \left\{(5, 4{+}6k) : \ k = 0, 1, \dots, \lfloor \tfrac{n-4}{6} \rfloor\right\}.$$

S is a 2-dominating set of $P_5 \square P_n$ for $n \equiv 1,4 \pmod 6$.

$$S_1 = \begin{cases} S \cup \{(3,n)\}, & n \equiv 2,5,0 \pmod 6, \\ S \cup \{(1,n),(5,n)\}, & n \equiv 3 \pmod 6. \end{cases}$$

S_1 is a 2-dominating set for $n \equiv 2,3,5,0 \pmod 6$.

Hence, we have 2-dominating sets with the following cardinalities:

$$\begin{cases} 3\lfloor \frac{n}{6} \rfloor + 1 & \text{if } n \equiv 1 \pmod 6, \\ 3\lfloor \frac{n}{6} \rfloor + 2 & \text{if } n \equiv 2 \pmod 6, \\ 3\lfloor \frac{n}{6} \rfloor + 3 & \text{if } n \equiv 3,4 \pmod 6, \\ 3\lceil \frac{n}{6} \rceil + 1 & \text{if } n \equiv 5,0 \pmod 6. \end{cases}$$

□

THEOREM 6. *For* $n \geq 2$,

$$\gamma_2(P_6 \,\Box\, P_n) = \begin{cases} 3\lfloor \frac{n}{5} \rfloor + 1\,, & n \equiv 0 \pmod 5\,, \\ 3\lfloor \frac{n}{5} \rfloor + 2\,, & n \equiv 1,2 \pmod 5\,, \quad n \neq 6,7\,, \\ 3\lfloor \frac{n}{5} \rfloor + 3\,, & n \equiv 3,4 \pmod 5\,, \quad n \neq 3,4\,, \\ 2\,, & n = 3\,, \\ 4\,, & n = 4,6\,, \\ 6\,, & n = 7\,. \end{cases}$$

Proof. We consider the set

$$\begin{aligned} S = \ & \{(1,1+10k):\ k=0,1,\ldots,\lfloor \tfrac{n-1}{10} \rfloor\} \cup \{(1,7+10k):\ k=0,1,\ldots,\lfloor \tfrac{n-7}{10} \rfloor\} \\ & \cup \{(3,4+10k):\ k=0,1,\ldots,\lfloor \tfrac{n-4}{10} \rfloor\} \cup \{(4,9+10k):\ k=0,1,\ldots,\lfloor \tfrac{n-9}{10} \rfloor\} \\ & \cup \{(6,6+10k):\ k=0,1,\ldots,\lfloor \tfrac{n-6}{10} \rfloor\} \cup \{(6,12+10k):\ k=0,1,\ldots,\lfloor \tfrac{n-2}{10} \rfloor - 1\} \\ & \cup \{(5,2)\}\,. \end{aligned}$$

S is 2-dominating set for $n \equiv 4 \pmod 5$, $n \neq 4$.

The 2-dominating set for other n (when there are even number of $6 \,\Box\, 5$ blocks) is

$$\begin{aligned} S_1 &= S \cup \{(6,n)\} && \text{for} \quad n \equiv 1 \pmod 5\,, \\ S_1 &= \big(S \setminus \{(1,n-1)\}\big) \cup \{(2,n-1)\} && \text{for} \quad n \equiv 2 \pmod 5\,, \\ S_1 &= S \cup \{(3,n)\} && \text{for} \quad n \equiv 3 \pmod 5\,, \\ S_1 &= \big(S \setminus \{(6,n-4),(4,n-1)\}\big) \cup \{(5,n-4),(3,n),(6,n-2)\} \\ & && \text{for} \quad n \equiv 0 \pmod 5\,. \end{aligned}$$

When there are odd number of $6 \Box 5$ blocks, we have the symmetrical case. Then in S_1 instead $(6,n)$ we must take $(1,n)$, and so on.

It follows that

$$\gamma_2(P_6 \,\Box\, P_n) \leq |S| = \begin{cases} 3\lfloor \frac{n}{5} \rfloor + 1\,, & n \equiv 0 \pmod 5\,, \\ 3\lfloor \frac{n}{5} \rfloor + 2\,, & n \equiv 1,2 \pmod 5\,, \quad n \neq 6,7\,, \\ 3\lfloor \frac{n}{5} \rfloor + 3\,, & n \equiv 3,4 \pmod 5\,, \quad n \neq 3,4\,, \\ 2\,, & n = 3\,, \\ 4\,, & n = 4,6\,, \\ 6\,, & n = 7\,, \end{cases}$$

□

Theorem 7. *We have*

$$\gamma_2(P_7 \,\Box\, P_n) = \begin{cases} 2\,, & n = 2\,, \\ 4\lfloor \frac{n}{6} \rfloor + 2\,, & n \equiv 0,1 \pmod 6\,, \\ 4\lfloor \frac{n}{6} \rfloor + 3\,, & n \equiv 2 \pmod 6\,,\ \ n \neq 2\,, \\ 4\lfloor \frac{n}{6} \rfloor + 4\,, & n \equiv 3,4 \pmod 6\,, \\ 4\lfloor \frac{n}{6} \rfloor + 5\,, & n \equiv 5 \pmod 6\,. \end{cases}$$

Proof. We consider the set

$$\begin{aligned} S = \ & \big\{(1,4{+}6k) :\ k = 0,1,\dots,\lfloor \tfrac{n-4}{6} \rfloor\big\} \cup \big\{(3,1{+}6k) :\ k = 0,1,\dots,\lfloor \tfrac{n-1}{6} \rfloor\big\} \\ & \cup \big\{(5,4{+}6k) :\ k = 0,1,\dots,\lfloor \tfrac{n-4}{6} \rfloor\big\} \cup \big\{(7,7{+}6k) :\ k = 0,1,\dots,\lfloor \tfrac{n-1}{6} \rfloor - 1\big\} \\ & \cup \big\{(6,2)\big\} \end{aligned}$$

S is a 2-dominating set of $P_7 \,\Box\, P_n$ for $n \equiv 1,4 \pmod 6$, $n \neq 1$.

$$S_1 = \begin{cases} S \cup \big\{(3,n)\big\}\,, & n \equiv 2 \pmod 6\,, \\ S \cup \big\{(1,n),(5,n)\big\}\,, & n \equiv 3 \pmod 6\,, \\ S \cup \big\{(5,n)\big\}\,, & n \equiv 5 \pmod 6\,, \\ S \cup \big\{(3,n),(7,n)\big\}\,, & n \equiv 0 \pmod 6\,. \end{cases}$$

S_1 is 2-dominating set for $n \equiv 2,3,5,0 \pmod 6$.

Hence we have 2-dominating sets with the following cardinalities:

$$\begin{cases} 2\,, & n = 2\,, \\ 4\lfloor \frac{n}{6} \rfloor + 2\,, & n \equiv 0,1 \pmod 6\,, \\ 4\lfloor \frac{n}{6} \rfloor + 3\,, & n \equiv 2 \pmod 6\,,\ \ n \neq 2\,, \\ 4\lfloor \frac{n}{6} \rfloor + 4\,, & n \equiv 3,4 \pmod 6\,, \\ 4\lfloor \frac{n}{6} \rfloor + 5\,, & n \equiv 5 \pmod 6\,. \end{cases}$$

□

4. Some general results

Theorem 8. *For m odd and $k \geq m - 1$,*

$$\gamma_k(P_m \,\Box\, P_n) \leq \left\lceil \frac{n}{2k - m + 2} \right\rceil .$$

Proof. We consider the set

$$S = \Big\{\big(\lfloor \tfrac{m}{2} \rfloor{+}1,\ k{-}\lfloor \tfrac{m}{2} \rfloor{+}1{+}(2k{-}m{+}2)l\big) :\ l = 0,1,\dots,\lfloor \tfrac{n}{2k-m+2} \rfloor{-}1\Big\}.$$

For $n \equiv 0 \pmod{(2k-m+2)}$, S is a k-dominating set and $|S| = \frac{n}{2k-m+2}$.

For $n \equiv k-\lfloor\frac{m}{2}\rfloor+1, \dots, (2k-m+1) \pmod{(2k-m+2)}$,

$$S_1 = S \cup \Big\{ \big(\lfloor\tfrac{m}{2}\rfloor+1,\ k-\lfloor\tfrac{m}{2}\rfloor+1+(2k-m+2)\cdot\lfloor\tfrac{n}{2k-m+2}\rfloor \big) \Big\}$$

is a k-dominating set and

$$|S| = \left\lceil \frac{n}{2k-m+2} \right\rceil .$$

For $n \equiv 1, \dots, \big(k-\lfloor\frac{m}{2}\rfloor\big) \pmod{(2k-m+2)}$, $S_1 = S \cup \Big\{ \big(\lfloor\frac{m}{2}\rfloor+1, n \big) \Big\}$ is a k-dominating set and $|S_1| = \lceil \frac{n}{2k-m+2} \rceil$. □

THEOREM 9. *For m even and $k \geq m-1$*

$$\gamma_k(P_m \,\Box\, P_n) \leq \begin{cases} \frac{n}{2k-m+2} + 1, & n \equiv 0 \pmod{(2k-m+2)}, \\ \lceil \frac{n}{2k-m+2} \rceil, & \textit{otherwise.} \end{cases}$$

Proof. We consider the set

$$\begin{aligned} S = \ & \Big\{ \big(\tfrac{m}{2}, k-\tfrac{m}{2}+1+(2k-m+2)2l\big) :\ l = 0,1,\dots, \Big\lfloor \tfrac{1}{2}\Big(\tfrac{n-k+\frac{m}{2}-1}{2k-m+2}\Big) \Big\rfloor \Big\} \\ & \cup \Big\{ \big(\tfrac{m}{2}+1, 3k-\tfrac{3m}{2}+3+(2k-m+2)2l\big) :\ l = 0,1,\dots, \Big\lfloor \tfrac{1}{2}\Big(\tfrac{n-k+\frac{m}{2}-1}{2k-m+2}\Big) \Big\rfloor - 1 \Big\}. \end{aligned}$$

For $n \equiv k-\frac{m}{2}+1, \dots, 2k-m+1 \pmod{(2k-m+2)}$, S is a k-dominating set and

$$|S| = \left\lceil \frac{n}{2k-m+2} \right\rceil .$$

For $n \equiv 0 \pmod{(2k-m+2)}$, $S_1 = S \cup \big\{ \big(\frac{m}{2}, n\big) \big\}$ is a k-dominating set and $|S_1| = \frac{n}{2k-m+2} + 1$.

For $n \equiv 1, \dots, k-\frac{m}{2} \pmod{(2k-m+2)}$, also $S_1 = S \cup \big\{ \big(\frac{m}{2}, n\big) \big\}$ is a k-dominating set and $|S_1| = \lceil \frac{n}{2k-m+2} \rceil$. □

PROPOSITION 1. *For any two paths P_m, P_n, $m,n \geq 2$,*

$$\lim_{m,n\to\infty} \frac{\gamma_k(P_m \,\Box\, P_n)}{mn} = \frac{1}{2k^2+2k+1}.$$

Proof. We follow the ideas used in [15] for the cardinal product.

We consider the set $H = \big\{ (i,j) :\ j \equiv (2k+1)i \pmod{(2k^2+2k+1)} \big\}$. H contains $\left\lceil \frac{nm}{2k^2+2k+1} \right\rceil$ vertices.

Vertex (i,j) k-dominates all vertices on a $(k+1)\Box(k+1)$ block on $P_m \Box P_n$. There are $2k^2+2k+1$ such vertices.

We take $(i,j) \in H$ $(j \equiv (2k+1)i \pmod{(2k^2+2k+1)})$. This vertex can k-dominate all vertices at distance $\leq k$.

The vertices at distance $k+1$ from (i,j) are

$$\begin{aligned}\{&(i,j-k-1),(i+1,j-k),\dots,(i+k,j-1),\\ &(i+k+1,j),(i+k,j+1),\dots,(i+1,j+k),\\ &(i,j+k+1),(i-1,j+k),\dots,(i-k,j+1),\\ &(i-k-1,j),(i-k,j-1),\dots,(i-1,j-k)\}.\end{aligned}$$

It is easy to see (by the same methods as in [13]) that all these vertices are k-dominated by vertices of H or vertices of the kind $(1,r),(m,r),(s,1),(s,n)$, $1 \leq r \leq n$, $1 \leq s \leq m$.

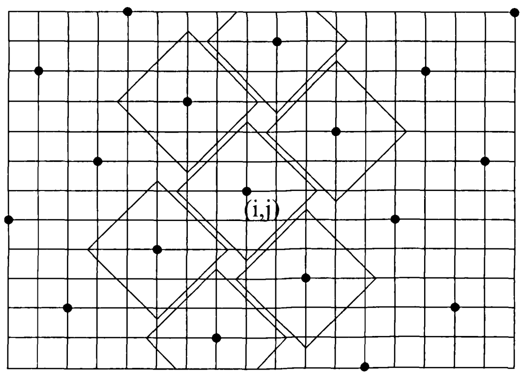

FIGURE 2. $(k=2)$.

Then $D = H \cup \{(1,s),(m,s),(r,1),(r,n) : \ 1 \leq s \leq n, \ 1 \leq r \leq m\}$ is a k-dominating set and

$$|D| = \left\lceil \frac{nm}{2k^2+2k+1} \right\rceil + 2m + 2n .$$

From the fact that one vertex can k-dominate at most $2k^2+2k+1$ vertices it follows that D must contain at least $\frac{mn}{2k^2+2k+1}$ vertices. Then

$$\frac{mn}{2k^2+2k+1} \leq \gamma_k(P_m \,\Box\, P_n) \leq \left(\frac{mn}{2k^2+2k+1} + 2m + 2n\right),$$

$$\frac{1}{2k^2+2k+1} \leq \frac{\gamma_k(P_m \,\Box\, P_n)}{mn} \leq \frac{1}{mn}\cdot\left(\frac{mn}{2k^2+2k+1} + 2m + 2n\right).$$

For $m,n \to \infty$ the right hand side of this inequality tends to $\frac{1}{2k^2+2k+1}$. Therefore

$$\lim_{m,n\to\infty} \frac{\gamma_k(P_m \,\Box\, P_n)}{mn} = \frac{1}{2k^2+2k+1}.$$

□

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Received October 18, 2001
Revised February 9, 2004

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