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Mathematica Slovaca, Vol. 39 (1989), No. 1, 31--41

Persistent URL: `http://dml.cz/dmlcz/128948`

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CANTOR EXTENSION OF AN ABELIAN CYCLICALLY ORDERED GROUP

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The cyclic order on a set P is a ternary relation $[x, y, z]$ on P with certain properties. Cyclically ordered sets where investigated by V. Novák and M. Novotný (e.g., [6], [7], [8]).

V. Novák [7] defined a completion of a cyclically ordered set that was constructed by means of regular cuts. The method is analogous to that of forming Dedekind cuts to obtain the MacNeille completion of an ordered set.

L. Rieger [11] introduced the notion of a cyclically ordered group (cf. also L. Fuchs [3]). Each linearly ordered group can be considered a cyclically ordered group. A representation theorem for cyclically ordered groups was proved by S. Swierczkowski [12].

Each cyclically ordered group G possesses a largest linearly ordered subgroup; this will be denoted by G_0 (see Pringerová [10]).

Let K be the additive group of all reals $a \in R$ such that $0 \leqq a < 1$, with the group operation defined as addition mod 1. For $a, b, c \in K$ we put $[a, b, c]$ if and only if $a < b < c$ or $b < c < a$ or $c < a < b$; then K is a cyclically ordered group.

Let G be an abelian cyclically ordered group. In the present paper we define the concept of a convergent (fundamental) sequence in G in such a way that it coincides with the concept of an o-convergent (o-fundamental) sequence provided G is a linearly ordered group. If every fundamental sequence in G converges, then G is called C-complete.

It will be proved that G is C-complete if and only if some of the following conditions is fulfilled:

(i) G is finite.

(ii) G is isomorphic to K.

(iii) $G_0 \neq \{0\}$ and G_0 is C-complete.

We next define the notion of the Cantor extension $Cant\, G$ of G. We prove that $Cant\, G$ does exist for each abelian cyclically ordered group G and that it is uniquely determined (up to isomorphisms leaving the elements of G fixed). Also a constructive description of $Cant\, G$ is given. The method is analogous to that which was used for lattice ordered groups by C. J. Everett [2] (cf. also L. Fuchs [3], F. Papangelou [9]) and in [1] by F. Dashiell, A. Hager and M. Henriksen.

Some questions concerning covergence in cyclically ordered groups were investigated by M. Harminc [4].

1. Preliminaries

Let us recall the definition of the lexicographic product of linearly ordered groups. Let A, B be linearly ordered groups. The cartesian product G of the groups A and B is made into a linearly ordered group as follows: if $(a_i, b_i) \in \in G$ $(i = 1, 2)$, then we put $(a_1, b_1) \leqq (a_2, b_2)$ if and only if $a_1 < a_2$ or $a_1 = a_2$, $b_1 \leqq b_2$. Then G is said to be the lexicographic product of linearly ordered groups A and B. We shall use the notation $G = A \circ B$.

Let G be a linearly ordered group, N the set of all positive integers, $g \in G$ and (g_n) a sequence in G (i.e. $g_n \in G$ for each $n \in N$). We say that (g_n) o-converges to g (or g is an o-limit of (g_n)) and we write $g_n \xrightarrow{o} g$ if for each $\varepsilon \in G$, $\varepsilon > 0$ there exists $n_0 \in N$ such that $g - \varepsilon < g_n < g + \varepsilon$ for each $n \in N$, $n \geqq n_0$. A sequence (g_n) is said to be o-fundamental if for each $\varepsilon \in G$, $\varepsilon > 0$ there exists $n_0 \in N$ such that $-\varepsilon < g_n - g_m < \varepsilon$ for each $m, n \in N$, $m, n \geqq n_0$ (see [5]). Every o-convergent sequence is o-fundamental. If every o-fundamental sequence is o-convergent, then G is called o-complete.

Now we describe the Cantor completion method of G (see [5]). A sequence (g_n) is said to be o-zero if $g_n \xrightarrow{o} 0$. Let $H^0(E^0)$ be the set of all o-fundamental (o-zero) sequences in G. For all (g_n), $(h_n) \in H^0$ we put $(g_n) + (h_n) = (g_n + h_n)$. Then H^0 is a group and E^0 is an invariant subgroup of H^0. The factor group H^0/E^0 can be made into a linearly ordered group by defining the order relation in the following way: $(g_n) + E^0 \leqq (h_n) + E^0$ if and only if there exists $n_0 \in N$ such that $g_n \leqq h_n$ for each $n \in N$, $n \geqq n_0$. This linearly ordered group will be denoted by $C(G)$ and called the Cantor extension of the linearly ordered group G. The coset of $C(G)$ containing a sequence $(g_n) \in H^0$ will be denoted by $(g_n)^0$. The following assertions hold true (see [5]):

(α) *$C(G)$ is o-complete.*

(β) *G is a subgroup (endowed with the induced order) of $C(G)$.*

(γ) *Every element of $C(G)$ is the o-limit of some o-fundamental sequence in G.*

(δ) *Let (g_n) be a sequence in G. If $g_n \xrightarrow{o} 0$ in G, then $g_n \xrightarrow{o} 0$ in $C(G)$.*

Let G be a group with the group operation $+$. A ternary relation $[x, y, z]$ which is defined on G is called a cyclic order on G if the following conditions are fulfilled:

I. If $x \neq y \neq z \neq x$, then either $[x, y, z]$ or $[z, y, x]$.

II. $[x, y, z]$ implies $[y, z, x]$.

III. $[x, y, z]$ and $[y, u, z]$ imply $[x, u, z]$.

IV. $[x, y, z]$ implies $[a+x+b, a+y+b, a+z+b]$ for each $a, b \in G$.

A group on which a cyclic order is defined will be called a cyclically ordered group.

Let L be a linearly ordered group. A cyclic order on L can be defined by

$$[x, y, z] \equiv x < y < z \quad \text{or} \quad y < z < x \quad \text{or} \quad z < x < y. \tag{1}$$

We say that the cyclic order on L defined by (1) is generated by the linear order on L. Therefore each linearly ordered group is at the same time a cyclically ordered group (with respect to the cyclic order generated by its linear order).

Let L and K be as above. We consider the cyclic order on L given by (1). There can be defined a cyclic order on the direct product of groups $L \times K$ as follows: let $u = (x, a)$, $v = (y, b)$, $w = (z, c)$ be elements of $L \times K$. We put $[u, v, w]$ if some of the following conditions is fulfilled:

(i) $[a, b, c]$;
(ii) $a = b \neq c$ *and* $x < y$;
(iii) $b = c \neq a$ *and* $y < z$;
(iv) $c = a \neq b$ *and* $z < x$;
(v) $a = b = c$ *and* $[x, y, z]$.

The group $L \times K$ with this cyclic order will be denoted by $L \otimes K$ (cf. [12]).

The isomorphism of cyclically ordered groups is defined in the natural way. Every subgroup of a cyclically ordered group is a cyclically ordered group (by the inherited cyclic order).

1.1. Theorem. ([12], *Theorem*) *If G is a cyclically ordered group, then there exists a linearly ordered group L such that G is isomorphic to a subgroup of $L \otimes K$.*

Let G, L, K be as in 1.1. In the whole paper f denotes an isomorphism of G into $L \otimes K$. Denote by G_0 the set of all $g \in G$ such that there exists $x \in L$ with the property $f(g) = (x, 0)$. Then G_0 is a subgroup of G. It can happen that $G_0 = \{0\}$. Let $G_0 \neq \{0\}$, $g \in G_0$, $g \neq 0$. Hence there exists $x \in L$ with $f(g) = (x, 0)$. If we put $g > 0$ if and only if $x > 0$, then G_0 is a linearly ordered group. The cyclic order on G_0 generated by this linear order coincides with the cyclic order on G_0 inherited from G. Next, G_0 is the largest linearly ordered subgroup with this property (see [10]).

Let G be a cyclically ordered group. The notion of c-convexity of subgroups of G is defined as follows. The subgroups $\{0\}$ and G are assumed to be c-convex in G. A proper subgroup G' of G is said to be c-convex in G if the following conditions are fulfilled (see [10]):

(i) $g' \in G'$, $g' \neq 0 \Rightarrow 2g' \neq 0$;

(ii) $g' \in G'$, $[-g', 0, g']$, $[-g', g, g'] \Rightarrow g \in G'$.

1.2. Lemma. ([10], *Chap.* III, 3.5) *Let G be a cyclically ordered group. Then G_0 is a c-convex subgroup in G.*

2. Convergent and fundamental sequences in an abelian cyclically ordered group

In what follows G will denote an abelian cyclically ordered group.

Let (g_n) be a sequence in G and $g \in G$. We say that (g_n) converges to g (g is a limit of (g_n)) in G and we write $g_n \to g$ if

(i) $\operatorname{card} G = 2$ and there exists $n_0 \in N$ such that $g_n = g$ for each $n \in N$, $n \geqq n_0$; or

(ii) $\operatorname{card} G \neq 2$ and for each $\varepsilon \in G$, $\varepsilon \neq 0$, with the property $[g - \varepsilon, g, g + \varepsilon]$ there exists $n_0 \in N$ such that $[g - \varepsilon, g_n, g + \varepsilon]$ for each $n \in N$, $n \geqq n_0$.

The sequence (g_n) is said to be fundamental in G if for each $\varepsilon \in G$ with $[-\varepsilon, 0, \varepsilon]$ there exists $n_0 \in N$ such that $[-\varepsilon, g_n - g_m, \varepsilon]$ for each $m, n \in N$, $m, n \geqq n_0$.

2.1. Lemma. *$g_n \to g$ if and only if $g_n - g \to 0$.*

Proof. Let $g_n \to g$, $\varepsilon \in G$ with $[-\varepsilon, 0, \varepsilon]$. From $[g - \varepsilon, g, g + \varepsilon]$ it follows that there exists $n_0 \in N$ such that $[g - \varepsilon, g_n, g + \varepsilon]$ for each $n \in N$, $n \geqq n_0$. Hence $[-\varepsilon, g_n - g, \varepsilon]$ and so $g_n - g \to 0$. The converse can be proved analogously.

By a zero sequence we understand a sequence which converges to 0. The set of all fundamental (zero) sequences in G will be denoted by $H(E)$.

It will be shown later that every convergent sequence in G is fundamental in G. The converse does not hold in general. If every fundamental sequence in G is convergent in G, then G is called C-complete.

Let G' be a subgroup of G and (g_n) a sequence in G'. Let us remark that it can happen that $g_n \to 0$ in G', but (g_n) does not converge to 0 in G.

Example. Let R be the additive group of all reals with the natural order, $G = R \circ R$, G' the set of all $g \in G$ of the form $g = (r, 0)$, $r \in R$. Then $g_n = \left(\frac{1}{n}, 0\right)$ is a sequence in G', $g_n \to 0$ in G', but (g_n) fails to converge to 0 in G (it suffices to put $\varepsilon = (0, 1)$).

A cyclically ordered group $\mathit{Cant}\, G$ is said to be a Cantor extension of G if the following conditions are satisfied:

(a) $\mathit{Cant}\, G$ is a C-complete abelian cyclically ordered group.

(b) G is a subgroup (ordered by the inherited cyclic order) of $\mathit{Cant}\, G$.

(c) Every element of $\mathit{Cant}\, G$ is the limit of some fundamental sequence in G.

(d) Let (g_n) be a sequence in G. If $g_n \to 0$ in G, then $g_n \to 0$ in $\mathit{Cant}\, G$.

It will be proved that for each abelian cyclically ordered group G there exists $\mathit{Cant}\, G$ and that it is uniquely determined (up to isomorphisms). We distinguish two cases: $G_0 \neq \{0\}$ and $G_0 = \{0\}$.

First we introduce some auxiliary results. In the following lemmas 2.2 and 2.3 we suppose that the cyclic order on a linearly ordered group A is generated by its linear order.

2.2. Lemma. *Let A be a linearly ordered group, (g_n) a sequence in A and $g \in A$. Then*

(i) $g_n \to g$ *if and only if* $g_n \xrightarrow{o} g$.

(ii) (g_n) *is fundamental if and only if* (g_n) *is o-fundamental.*

Proof. (i) Let $g_n \to g$, $\varepsilon \in A$, $\varepsilon > 0$. Hence $g - \varepsilon < g < g + \varepsilon$ and so $[g - \varepsilon, g, g + \varepsilon]$. The assumption implies that there exists $n_0 \in N$ such that $[g - \varepsilon, g_n, g + \varepsilon]$ for each $n \in N$, $n \geqq n_0$. Therefore $g - \varepsilon < g_n < g + \varepsilon$. Hence $g_n \xrightarrow{o} g$. The converse is analogous.

(ii) Let (g_n) be a fundamental sequence in A, $\varepsilon \in A$, $\varepsilon > 0$. Then $[-\varepsilon, 0, \varepsilon]$. By the assumption there exists $n_0 \in N$ such that $[-\varepsilon, g_n - g_m, \varepsilon]$ for each $m, n \in N$, $m, n \geqq n_0$. Hence $-\varepsilon < g_n - g_m < \varepsilon$ for each $m, n \in N$, $m, n \geqq n_0$ and thus (g_n) is an o-fundamental sequence in A. The converse is analogous.

From 2.2 we obtain immediately

2.3. Lemma. *Let A be a linearly ordered group. Then A is C-complete if and only if A is o-complete.*

3. The case $G_0 \neq \{0\}$

In the whole section we assume that $G_0 \neq \{0\}$. Then card $G \geqq \aleph_0$. Hence G is infinite.

Denote by H_0 (E_0) the set of all fundamental (zero) sequences in G_0. Let (g_n), (h_n), (t_n) be sequences in G, $f(g_n) = (x_n, a_n)$, $f(h_n) = (y_n, b_n)$, $f(t_n) = (z_n, c_n)$. Let n_0 be a fixed element of N. Denote $g_n^0 = g_{n_0 + n - 1}$ for each $n \in N$.

3.1. Lemma. *Let $(g_n) \in E$. Then there exists $n_0 \in N$ such that $g_n \in G_0$ for each $n \in N$, $n \geqq n_0$, and $(g_n^0) \in E_0$.*

Proof. Since $G_0 \neq \{0\}$, there exists $\varepsilon \in G_0$, $\varepsilon > 0$. From $-\varepsilon < 0 < \varepsilon$ it follows that $[-\varepsilon, 0, \varepsilon]$. The assumption implies that there exists $n_0 \in N$ such that $[-\varepsilon, g_n, \varepsilon]$ for each $n \in N$, $n \geqq n_0$. The c-convexity of G_0 in G implies that $g_n \in G_0$ for each $n \in N$, $n \geqq n_0$. Evidently, $(g_n^0) \in E_0$.

3.2. Lemma. *Let $g_n \to g$, $f(g) = (x, a)$. Then there exists $n_0 \in N$ such that $a_n = a$ for each $n \in N$, $n \geqq n_0$.*

Proof. By 2.1 and 3.1 there exists $n_0 \in N$ with $g_n - g \in G_0$ for each $n \in N$, $n \geqq n_0$. Hence $a_n = a$ for each $n \in N$, $n \geqq n_0$.

3.3. Lemma. *Every sequence in G has at most one limit in G.*

Proof. Let $g_n \to g$, $g_n \to h$, $f(g) = (x, a)$, $f(h) = (y, b)$. From 3.2 it follows that there exists $n_0 \in N$ such that $f(g_n) = (x_n, a)$, $f(g_n) = (x_n, b)$ for each $n \in N$, $n \geqq n_0$. Hence $a = b$ and so $g_n - h$, $g - h \in G_0$ for each $n \in N$, $n \geqq n_0$. From $g_n - h \to g - h$, $g_n - h \to 0$ in G it follows that $g_n^0 - h \to g - h$, $g_n^0 - h \to 0$ in G_0. According to 2.2 we have $g_n^0 - h \xrightarrow{o} g - h$, $g_n^0 - h \xrightarrow{o} 0$. Since o-limits are uniquely determined, we get $g = h$.

3.4. Lemma. *Let g, h, t be distinct elements of G, (g_n), (h_n), (t_n) sequences in G such that $g_n \to g$, $h_n \to h$, $t_n \to t$. Then $[g, h, t]$ if and only if there exists $n_0 \in N$ such that $[g_n, h_n, t_n]$ for each $n \in N$, $n \geqq n_0$.*

Proof. Let $[g, h, t]$, $f(g) = (x, a)$, $f(h) = (y, b)$, $f(t) = (z, c)$. Then in view of 3.2 there exist $n_0 \in N$ and $a, b, c \in K$ with $f(g_n) = (x_n, a)$, $f(h_n) = (y_n, b)$, $f(t_n) = (z_n, c)$ for each $n \in N$, $n \geqq n_0$.

First suppose that $[a, b, c]$. Hence $[g_n, h_n, t_n]$ for each $n \in N$, $n \geqq n_0$.

Now suppose that $a = b \neq c$, $x < y$. Assume that there exist $g' \in G$ and $x' \in L$ with $f(g') = (x', a)$, $x < x' < y$. We have $2x - x' < x < x'$, $x' < y < 2y - x'$. If we put $\varepsilon_1 = g' - g$, $\varepsilon_2 = h - g'$, then $[g - \varepsilon_1, g, g + \varepsilon_1]$, $[h - \varepsilon_2, h, h + \varepsilon_2]$. Hence there exists $n_1 \in N$ such that $[g - \varepsilon_1, g_n, g + \varepsilon_1]$, $[h - \varepsilon_2, h_n, h + \varepsilon_2]$ for each $n \in N$, $n \geqq n_1$. Therefore $2x - x' < x_n < x'$, $x' < y_n < 2y - x'$ and so $x_n < y_n$ for each $n \in N$, $n \geqq n_1$. Hence $[g_n, h_n, t_n]$ for each $n \in N$, $n \geqq n_1$. Assume that there does not exist elements $g' \in G$ and $x' \in L$ as above. In this case we have $2x - y < x < y$. If $\varepsilon = h - g$, then $[g - \varepsilon, g, g + \varepsilon]$. There exists $n_2 \in N$ such that $[g - \varepsilon, g_n, g + \varepsilon]$ for each $n \in N$, $n \geqq n_2$. Hence $2x - y < x_n \leqq x$. From this it follows that $x_n = x$ for each $n \in N$, $n \geqq n_2$. In fact, let there exist $n \in N$, $n \geqq n_2$ with $2x - y < x_n < x$. Then $x < y - x + x_n < y$. There exists an element $g'' \in G$, $f(g'') = (y - x + x_n, a)$, a contradiction. We get an analogous result for y_n. Therefore there exists $n_3 \in N$ such that $x_n < y_n$ for each $n \in N$, $n \geqq n_3$. We conclude that $[g_n, h_n, t_n]$ for each $n \in N$, $n \geqq n_3$.

Similar arguments can be used to prove the remaining cases.

Conversely, let there exist $n_0 \in N$ such that $[g_n, h_n, t_n]$ for each $n \in N$, $n \geqq n_0$. Assume, by way of contradiction, that $[t, h, g]$. Then there is $n_1 \in N$ with $[t_n, h_n, g_n]$ for each $n \in N$, $n \geqq n_1$, a contradiction.

Remark. If g, h, t are not distinct, then 3.4 need not hold in general. It suffices to put $G = R$, $g_n = \frac{1}{n}$, $h_n = \frac{2}{n}$, $t_n = \frac{3}{n}$.

From 2.2 it follows

3.5. Lemma. (i) $E_0 = E_0^0$. (ii) $H_0 = H_0^0$.

3.6. Lemma. (i) $E_0 \subseteq E$, (ii). $H_0 \subseteq H$.

Proof. (i) Let $(g_n) \in E_0$, $\varepsilon \in G$, $[-\varepsilon, 0, \varepsilon]$. If $\varepsilon \in G_0$, then there exists $n_0 \in N$ such that $[-\varepsilon, g_n, \varepsilon]$ for each $n \in N$, $n \geqq n_0$, If $\varepsilon \notin G_0$, we have $[-\varepsilon, g_n, \varepsilon]$ for each $n \in N$. Hence $(g_n) \in E$.

(ii) Let $(g_n) \in H_0$, $\varepsilon \in G$, $[-\varepsilon, 0, \varepsilon]$. If $\varepsilon \in G_0$, then $[-\varepsilon, g_n - g_m, \varepsilon]$ for each $m, n \in N$, $m, n \geqq n_0$. If $\varepsilon \notin G_0$, then $[-\varepsilon, g_n - g_m, \varepsilon]$ for each $m, n \in N$. We infer that $(g_n) \in H$.

3.7. Lemma. *If (g_n) is a convergent sequence in G, then $(g_n) \in H$.*

Proof. Let $g_n \to g$. By 2.1 we have $(g_n - g) \in E$. With respect to 3.1 there exists $n_0 \in N$ with $(g_n^0 - g) \in E_0$. Since $E_0^0 \subseteq H_0^0$, in view of 3.5 and 3.6 we obtain $(g_r^0 - g) \in H$. Therefore $(g_n^0) \in H$ and so $(g_n) \in H$.

3.8. Lemma. *Let $(g_n) \in H$. Then there exists $n_0 \in N$ such that $a_n = a_{n_0}$ for each $n \in N$, $n \geqq n_0$.*

Proof. Let $(g_n) \in H$, $\varepsilon \in G_0$, $\varepsilon > 0$. Hence $[-\varepsilon, 0, \varepsilon]$. There is $n_0 \in N$ such that $[-\varepsilon, g_n - g_m, \varepsilon]$ for each $m, n \in N$, $m, n \geqq n_0$. Since G_0 is c-convex in G, we conclude that $g_n - g_m \in G_0$ and so $g_n - g_{n_0} \in G_0$ for each $n \in N$, $n \geqq n_0$. Hence $a_n = a_{n_0}$ for each $n \in N$, $n \geqq n_0$.

3.9. Lemma. *G is C-complete if and only if G_0 is C-complete.*

Proof. Let G be C-complete, $(g_n) \in H_0$. According to 3.6 we have $(g_n) \in H$. There exists $g \in G$ such that $g_n \to g$ in G. It suffices to prove that $g \in G_0$. Let $\varepsilon \in G_0$, $[-\varepsilon, 0, \varepsilon]$. There exists $n_0 \in N$ such that $[-\varepsilon, g_n - g, \varepsilon]$ for each $n \in N$, $n \geqq n_0$. By the c-convexity of G_0 in G we have $g_n - g_0 \in G_0$ for each $n \in N$, $n \geqq n_0$. Therefore $g \in G_0$.

Conversely, let G_0 be C-complete, $(g_n) \in H$. Then according to 3.8 there are $n_0 \in N$, $a \in K$ with $f(g_n) = (x_n, a)$ for each $n \in N$, $n \geqq n_0$. There exist $x \in L$ and $v \in G$ with $f(v) = (x, a)$. Hence $(g_n^0 - v) \in H_0$. There is $g \in G_0$ with $g_n^0 - v \to g$ in G_0. With respect to 3.6 $g_n^0 - v \to g$ in G and $g_n^0 \to v + g$ in G. Thus $g_n \to v + g$ in G.

Define the operation $+$ in H by putting $(g_n) + (h_n) = (g_n + h_n)$ for each (g_n), $(h_n) \in H$.

3.10. Lemma. *H is a group.*

Proof. Let (g_n), $(h_n) \in H$. According to 3.8 there are $n_0 \in N$, $a, b \in K$ such that $f(g_n) = (x_n, a)$, $f(h_n) = (y_n, b)$ for each $n \in N$, $n \geqq n_0$. Let $x, y \in L$ and $v, w \in G$ such that $f(v) = (x, a)$, $f(w) = (y, b)$. Therefore $(g_n^0 - v)$, $(h_n^0 - w) \in H_0$. Since H_0^0 is a group, with respect to 3.5 H_0 is also a group and so $(g_n^0 - v) + (h_n^0 - w) \in H_0$. Therefore by 3.6 $(g_n^0) + (h_n^0) \in H$. Hence $(g_n) + (h_n) \in H$.

A similar argument may be applied to prove that if $(g_n) \in H$, then $-(g_n) \in H$.

3.11. Lemma. *E is a subgroup of H.*

Proof. Let (g_n), $(h_n) \in E$. By 3.1 there exists $n_0 \in N$ such that g_n, $h_n \in G_0$ for each $n \in N$, $n \geqq n_0$ and (g_n^0), $(h_n^0) \in E_0$. Because E_0^0 is a group, by 3.5 and 3.6 $(g_n^0) - (h_n^0) \in E$ holds. This implies $(g_n) - (h_n) \in E$.

We can form the factor group $\bar{G} = H/E$. The coset of $\bar{G}$ containing a sequence $(g_n) \in H$ will be denoted by $\overline{(g_n)}$.

Let $\overline{(g_n)}$, $\overline{(h_n)}$, $\overline{(t_n)}$ be distinct elements of $\bar{G}$. We put $[\overline{(g_n)}, \overline{(h_n)}, \overline{(t_n)}]$ in $\bar{G}$ if there exists $n_0 \in N$ such that $[g_n, h_n, t_n]$ in G for each $n \in N$, $n \geqq n_0$. We can easily verify that this definition is correct and that the conditions I—IV are satisfied. Hence $\bar{G}$ is an abelian cyclically ordered group.

If this definition is applied to $C(G_0)$, then the cyclic order of $C(G_0)$ is generated by the linear order of $C(G_0)$. The coset of $C(G_0)$ containing a sequence $(g_n) \in H_0^0 = H_0$ will be denoted by $(g_n)^*$.

Let φ: $G \to \bar{G}$ be a mapping defined by the rule $\varphi(g) = \overline{(g, g, \ldots)}$. Then φ is an isomorphism of the cyclically ordered group G into $\bar{G}$. We identify G and $\varphi(G)$. Then G is a subgroup of $\bar{G}$.

3.12. Lemma. *$\bar{G}$ is a Cantor extension of G.*

Proof. (a) It remains to show that $\bar{G}$ is C-complete. Let $\overline{(g_n^m)}$ be a fundamental sequence in $\bar{G}$ (a sequence is to be understood with respect to m), $f(g_n^m) = (x_n^m, a_n^m)$, $\varepsilon \in G_0$, $[-\varepsilon, 0, \varepsilon]$. Then $[-\overline{(\varepsilon, \varepsilon, \ldots)}, E, \overline{(\varepsilon, \varepsilon, \ldots)}]$. There exists $m_0 \in N$ such that $[-\overline{(\varepsilon, \varepsilon, \ldots)}, \overline{(g_n^m)} - \overline{(g_n^p)}, \overline{(\varepsilon, \varepsilon, \ldots)}]$ for each $m, p \in N$, $m, p \geqq m_0$. Hence there is $n_0 \in N$ with $[-\varepsilon, g_n^m - g_n^p, \varepsilon]$ for each $n \in N$, $n \geqq n_0$. Therefore $g_n^m - g_n^p \in G_0$ and so $a_n^m = a_n^p$ for each $n, m, p \in N$, $n \geqq n_0$, $m, p \geqq m_0$. Hence there are $n_0(m) \in N$, $a \in K$ with $a_n^m = a$ for each $m, n \in N$, $m \geqq m_0$, $n \geqq n_0(m)$. Let $x \in L$, $v \in G$ with $f(v) = (x, a)$ and let $g_n'^m = g_n^{m_0 + m - 1}$ for each $m \in N$, $m \geqq m_0$. Therefore $(g_n'^m - v) \in H_0$. According to (α) and 2.3 there exists $(g_n)^* \in C(G_0)$ such that $(g_n'^m - v)^* \to (g_n)^*$ in $C(G_0)$. We conclude that $\overline{(g_n'^m - v)} \to \overline{(g_n)}$ in $\bar{G}$ and so $\overline{(g_n^m)} \to \overline{(g_n + v)}$ in $\bar{G}$.

(c) Suppose that $\overline{(g_n)} \in \bar{G}$. There exists $n_0 \in N$ with $f(g_n) = (x_n, a)$ for each $n \in N$, $n \geqq n_0$. Let $v \in G$, $x \in L$ with $f(g) = (x, a)$. Hence $(g_n^0 - v) \in H_0$. Because of $(g_n^0 - v, g_n^0 - v, \ldots)^* \xrightarrow{o} (g_n^0 - v)^*$ in $C(G_0)$, in view of 2.2 we have $(g_n^0 - v, g_n^0 - v, \ldots)^* \to (g_n^0 - v)^*$. Therefore $\overline{(g_n - v, g_n - v, \ldots)} \to \overline{(g_n - v)}$ and $\overline{(g_n, g_n, \ldots)} \to \overline{(g_n)}$.

(d) Let (g_n) be a sequence in G, $g_n \to 0$ in G. By 3.1 there exists $n_0 \in N$ such that $g_n \in G_0$ for each $n \in N$, $n \geqq n_0$ and $(g_n^0) \in E_0$. With respect to (δ) and 2.2 we get $(g_n^0, g_n^0, \ldots)^* \to E_0$ in $C(G_0)$. Hence $\overline{(g_n^0, g_n^0, \ldots)} \to E$ and so $\overline{(g_n, g_n, \ldots)} \to E$, that is $g_n \to 0$ in $\bar{G}$.

Let $\hat{G}$ be a Cantor extension of G. From $G \subseteq \hat{G}$ we infer that $G_0 \subseteq (\hat{G})_0$ and so $(\hat{G})_0 \neq \{0\}$. Therefore all results obtained in this section may be used for $(\hat{G})_0$.

3.13. Proposition. *Let G_1 and G_2 be Cantor extensions of G. Then there exists an isomorphism α from the cyclically ordered group G_1 onto G_2 such that $\alpha(g) = g$ for each $g \in G$.*

Proof. With respect to (b) G is a subgroup of G_1 and G_2. Let $g^1 \in G_1$. By (c) there exists a fundamental sequence (g_n) in G such that $g_n \to g^1$ in G_1. With respect to (a) there exists $g^2 \in G_2$ with $g_n \to g^2$ in G_2. Define a mapping α from G_1 into G_2 by the rule $\alpha(g^1) = g^2$.

First we show that α is correctly defined. Let also (h_n) be a fundamental sequence in G with $h_n \to g^1$ in G_1. There exists $h^2 \in G_2$ such that $h_n \to h^2$ in G_2. According to 2.1 and 3.11 we get $g_n - h_n \to 0$ in G_1. Hence $g_n - h_n \to 0$ in G. In view of (d) we obtain $g_n - h_n \to 0$ in G_2. Again by 2.1 and 3.11 $g_n - h_n \to g^2 - h^2$ in G_2. By 3.3 we conclude that $g^2 = h^2$.

Let $g^2 \in G_2$. There exist a fundamental sequence (g_n) in G and $g^1 \in G_1$ with $g_n \to g^2$ in G_2 and $g_n \to g^1$ in G_1. Hence $\alpha(g^1) = g^2$. Therefore α is surjective.

Let $g^1, h^1 \in G_1$, $\alpha(g^1) = g^2$, $\alpha(h^1) = h^2$. There are fundamental sequences (g_n), (h_n) in G_1 such that $g_n \to g^1$, $h_n \to h^1$ in G_1 and $g_n \to g^2$, $h_n \to h^2$ in G_2. If $g^2 = h^2$, then $g_n - h_n \to 0$ in G_2 and thus $g_n - h_n \to 0$ in G. Hence $g_n - h_n \to 0$ in G_1. Since $g_n - h_n \to g^1 - h^1$ in G_1, we have $g^1 = h^1$. We conclude that α is a monomorphism.

Evidently, $\alpha(g^1 + h^1) = \alpha(g^1) + \alpha(h^1)$ for each $g^1, h^1 \in G_1$.

Let $g^1, h^1, t^1 \in G_1$, $\alpha(g^1) = g^2$, $\alpha(h^1) = h^2$, $\alpha(t^1) = t^2$. There are fundamental sequences (g_n), (h_n), (t_n) in G with $g_n \to g^1$, $h_n \to h^1$, $t_n \to t^1$ in G_1, $g_n \to g^2$, $h_n \to h^2$, $t_n \to t^2$ in G_2. Suppose that $[g^1, h^1, t^1]$ in G_1. By 3.4 there exists $n_0 \in N$ such that $[g_n, h_n, t_n]$ in G for each $n \in N$, $n \geqq n_0$. Hence again by 3.4 $[g^2, h^2, t^2]$ is valid in G_2. The converse is analogous.

Assume that $g \in G$. We have $(g, g, \ldots) \to g$ in G. By (d) $(g, g, \ldots) \to g$ in G_1 and in G_2 as well. Hence $\alpha(g) = g$.

4. The case $G_0 = \{0\}$

In this section it will be assumed that $G_0 = \{0\}$. Let f, x, a be as in section 3, $g \in G$, $f(g) = (x, a)$. Define the mapping f_1 from G into K as follows: $f_1(g) = a$. Let $h \in G$, $f(h) = (y, b)$. Then $f_1(h) = b$ is valid. If $a = b$, then $f(g - h) = (x - y, 0)$. Hence $g - h \in G_0$. As for $G_0 = \{0\}$, we get $g = h$. We conclude that f_1 is a one to one mapping from G into K. Therefore the following lemma is valid:

4.1. Lemma. *The mapping f_1 is an isomorphism from the cyclically ordered group G into K.*

Remark 1. From 4.1 it follows that G can be considered as a subgroup of a cyclically ordered group K.

Observe that if G is finite, then $G_0 = \{0\}$.

The following lemma obviously holds true:

4.2. Lemma. *Let G be a finite cyclically ordered group. Then G is C-complete.*

The natural linear order on R will be denoted by $<$.

4.3. Lemma. *Let G be an infinite cyclically ordered group. Then for each $a \in K$, $a \neq 0$ there exists $g \in G$, $0 < g < a$.*

Proof. Denote $x = \inf\{g_i - g_j \colon g_i, g_j \in G, g_j < g_i\}$ in R. Hence $x \geqq 0$. If $x > 0$, then $\operatorname{card} G \leqq \frac{1}{x}$. Therefore G is finite, a contradiction. From this it follows that $x = 0$. Let $a \in K$, $a \neq 0$. There are $g_1, g_2 \in G$, $g_1 < g_2$ with $0 < g_2 - g_1 < a$. We put $g = g_2 - g_1$.

4.4. Lemma. *Let G be an infinite cyclically ordered group, $a_1, a_2 \in K$, $a_1 < a_2$. Then there exists $g \in G$, $a_1 < g < a_2$.*

Proof. Let $a_1, a_2 \in K$, $a_1 < a_2$. With respect to 4.3 it suffices to consider the case $a_1 \neq 0$. We have $a_1 \leqq a_2 - a_1 < a_2$ or $a_2 - a_1 < a_1 < a_2$. According to 4.3 there exists $g \in G$ with $0 < g < a_2 - a_1$. Let $a_1 \leqq a_2 - a_1 < a_2$. Hence $a_1 < \frac{1}{2}$. If $g \leqq a_1$, then there is $n \in N$ such that $a_1 < ng < a_2$. If $g > a_1$, then the assertion is evident. Let $a_2 - a_1 < a_1 < a_2$. Then $a_2 - a_1 < \frac{1}{2}$. Therefore there exists $m \in N$ with $a_1 < mg < a_2$.

4.5. Lemma. *Let G be an infinite cyclically ordered group. Then K is a Cantor extension of G.*

Proof. (a) It is evident that K is an abelian C-complete cyclically ordered group.

By Remark 1, (b) is satisfied.

(c) The case $a = 0$ is obvious. If $a \in K$, $a \neq 0$, then there exists an increasing sequence (a_n) (i.e., $a_n < a_{n+1}$ for each $n \in N$) in K such that $a_n \to a$ in K. With respect to 4.4 for each $n \in N$ there exists $g_n \in G$ with $a_n < g_n < a_{n+1}$. Therefore $g_n \to a$ in K.

(d) Let (g_n) be a sequence in G, $g_n \to 0$ in G, $\varepsilon \in K$; $[-\varepsilon, 0, \varepsilon]$. Then $0 < < \varepsilon < -\varepsilon$. With respect to 4.3 there is $\varepsilon_1 \in G$, $0 < \varepsilon_1 < \varepsilon$. Hence $-\varepsilon < -\varepsilon_1$. We obtain $[-\varepsilon_1, 0, \varepsilon_1]$. There is $n_0 \in N$ such that $[-\varepsilon_1, g_n, \varepsilon_1]$ for each $n \in N$, $n \geqq n_0$. From this it follows that $g_n < \varepsilon_1 < -\varepsilon_1$ or $\varepsilon_1 < -\varepsilon_1 < g_n$ and so $g_n < \varepsilon < -\varepsilon$ or $\varepsilon < -\varepsilon < g_n$. Therefore $[-\varepsilon, g_n, \varepsilon]$. We infer that $g_n \to 0$ in K.

Remark 2. It is easy to prove that 3.13 is valid also in the case $G_0 = \{0\}$.

4.6. Lemma. *Let G be an infinite cyclically ordered group. Then G is C-complete if and only if G is isomorphic to K.*

Proof. Let G be C-complete. Hence G is a Cantor extension of G. By using 4.5 and Remark 2 we get that G is isomorphic to K. Conversely, let G be isomorphic to K. Since K is C-complete, the proof is finished.

Now let G be an arbitrary abelian cyclically ordered group.

By summarizing the above results, we infer from 4.2, 4.6 and 3.9 that the following theorem is valid:

4.7. Theorem. *Let G be an abelian cyclically ordered group. Then G is C-complete if and only if some of the following conditions is satisfied:*

(i) *G is finite.*

(ii) *G is isomorphic to K.*

(iii) *$G_0 \neq \{0\}$ and G_0 is C-complete.*

4.8. Corollary. *Let G be an abelian cyclically ordered group. Then G is C-complete if and only if G_0 is C-complete.*

From 3.12, 3.13, 4.5 and from Remark 2 we get

4.9. Theorem. *Let G be an abelian cyclically ordered group. Then*

(i) *there exists a Cantor extension of G,*

(ii) *if* G_1 *and* G_2 *are Cantor extensions of* G, *then there exists an isomorphism* α *from the cyclically ordered group* G_1 *onto* G_2 *such that* $\alpha(g) = g$ *for each* $g \in G$.

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Received May 18, 1987

Katedra matematiky VŠT
Švermova 9
040 01 Košice

КАНТОРОВСКОЕ РАСШИРЕНИЕ АБЕЛЕВОЙ ЦИКЛИЧЕСКИ УПОРЯДОЧЕННОЙ ГРУППЫ

Štefan Černák

Резюме

Пусть G-абелева циклически упорядоченная группа. В работе определено и построено канторовское расширение *Cant* G группы G методом фундаментальных последовательностей. Если *Cant* $G = G$, то G называется C-полной. Установлены необходимые и достаточные условия для того, чтобы G была C-полной.