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NEARLY REGULAR CELL-DECOMPOSITIONS OF ORIENTABLE 2-MANIFOLDS WITH AT MOST TWO EXCEPTIONAL CELLS

MIRKO HORŇÁK—ERNEST JUCOVIČ

1. Introduction

Convex 3-polytopes whose valencies of all vertices are multiples of m and numbers of edges of all faces are multiples of k, m, $k<6$, appear as especially interesting in the study of combinatorial properties of polytopes (cf. Grünbaum [4], Gallai [3]). They are sometimes called *nearly regular* (cf. Crowe [2]).

Much effort has been devoted to the study of the nearly regular decompositions of the sphere with one or two vertices or faces exceptional in the sense that the numbers of edges they are incident with are not multiples of m or k, respectively. Some of the results dealing with boundary complexes of 3-polytopes were obtained by investigating much more general decompositions of the sphere than those which are complexes. (For definitions concerning complexes cf. e.g. Grünbaum [5].) And, inspired by Grünbaum [4], Malkevitch [11] answers, for all pairs (m, k) the question of existence of a decomposition of the sphere (not necessarily a complex) with at most two exceptional elements (vertices or faces). He shows that for most pairs (m, k) and required numbers of the exceptional elements such decompositions do not exist. The aim of the present paper is to show how things are on 2-manifolds of higher genus and to give general assertions. It appears, analogously as in some other problems concerning cell-decompositions of 2-manifolds (cf. Jucovič—Trenkler [10], Jucovič [8]), that manifolds of low genus are exceptional in the sense that decompositions of some type exist only on manifolds of sufficiently great genus. But let us first introduce the necessary notions and notations.

We investigate 2-dimensional orientable 2-manifolds with no boundary only and decompositions of them which are cell-complexes. For a 2-dimensional cell-complex N let $p_i(N)$ or $v_i(N)$ denote the number of 2-cells *(faces)* or 0-cells *(vertices)*, respectively, which are incident with precisely i 1-cells *(edges)*.

Let m, k be integers greater than 1, ε_v, ε_f, g non-negative integers. By $M(m, k; \varepsilon_v, \varepsilon_f; g)$ we denote the class of decompositions of the manifold of genus g with the following properties:

a) the valencies of all vertices with the exception of precisely ε_v vertices *(exceptional vertices)* are multiples of m;

b) the numbers of edges of all faces, with the exception of precisely ε_f faces *(exceptional faces)* are multiples of k.

If $\varepsilon_v + \varepsilon_f = 2$, the class $M(m, k; \varepsilon_v, \varepsilon_f; g)$ can be divided into subclasses by prescribing the distance between the two exceptional cells c_1, c_2 of the decomposition; by the distance of the cells c_1, c_2 it is meant the (graphical) length of the shortest path joining a vertex of c_1 with a vertex of c_2. A subclass of $M(m, k; \varepsilon_v, \varepsilon_f; g)$ with the distance d of the exceptional cells will be denoted by $M(m, k; \varepsilon_v, \varepsilon_f; g, d)$.

For $d=0$ it is worth while to distinguish the classes $M(m, k; 0, 2; g, 0)$ and $M(m, k; 0, 2; g, \bar{0})$. In a complex of the first class the exceptional faces have only one vertex in common, in a complex of the second class the exceptional faces have one edge in common.

In the following sections there are given solutions of existence problems concerning some types of the nearly regular cell-decompositions of orientable 2-manifolds with at most two exceptional cells. To show that some class of complexes is non-empty a member of it is constructed. To simplity the description of complexes, many pictures are used; in them the exceptional vertices or faces are marked u_i or U_i, respectively ($i \in \{1, 2\}$). In the assertions infinite numbers of non-negative integers g, d appear; therefore inductive constructive steps are employed increasing the genus g of the manifold or the distance d of the exceptional cells.

2. Decompositions with at most one exceptional cell

Lemma 1. *Let $(m, k) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$, ε_v, ε_f and g_0 be non-negative integers. If the class $M(m, k; \varepsilon_v, \varepsilon_f; g_0)$ is non-empty, so is the class $M(m, k; \varepsilon_v, \varepsilon_f; g)$ for every integer $g > g_0$.*

Proof. Let the complex $D \in M(m, k; \varepsilon_v, \varepsilon_f; g_0)$ be realized on the surface S. Choose its arbitrary face W and change it according Fig. 1a where the subcomplex D_k^m means the boundary complex of the regular 3-polytope with m-valent vertices and k-gonal faces in which one face is omitted. As an example, in Fig. 1b for $(m, k) = (3, 4)$ a quadrangle is subdivided using graphs of four cubes. Clearly the new decomposition D' of surface S is again a complex. The n-gon W is subdivided into the k-gons $X_1, \dots, X_n$, the k-gons belonging to the subcomplexes D_k^m and the n-gon W'. The valencies of the vertices w_i have increased by m, the vertices w_i' are $2m$-valent and the remaining new vertices are m-valent. So the complex D' has ε_v exceptional vertices and ε_f exceptional faces; this means it belongs to the class $M(m, k; \varepsilon_v, \varepsilon_f; g_0)$ as well as the complex D.

Now it is clear that from among the k-gons belonging to the subcomplexes D_k^m two disjoint k-gons X, Y with vertices $x_1, \ldots, x_k, y_1, \ldots, y_k$, respectively, can be chosen. These k-gons are used as openings for setting a handle to the surface S.

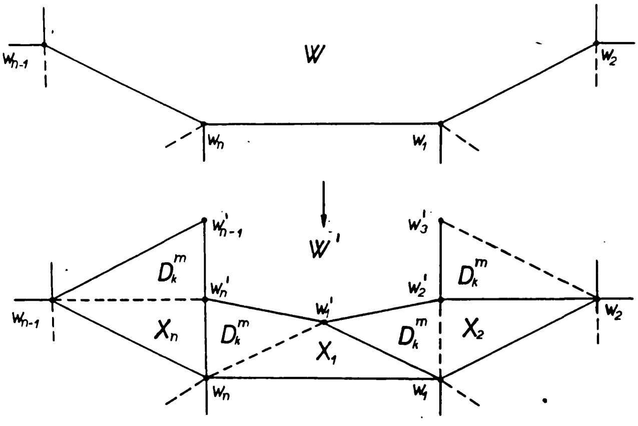

Fig. 1a

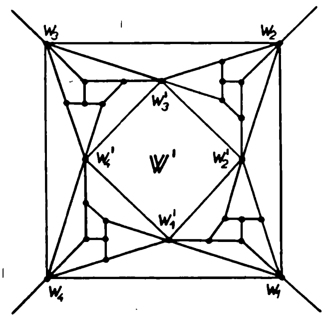

Fig. 1b

This is done in the obvious way and the handle is decomposed as marked in Fig. 2, 3, 4, 5 or 6 depending on whether $(m, k) = (3, 3)$, $(3, 4)$, $(4, 3)$, $(3, 5)$ or $(5, 3)$, respectively. The new decomposition D_1 of the manifold of genus $g_0 + 1$ is again a

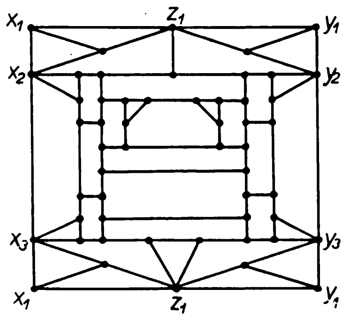

Fig. 2

complex (there is no face on the handle meeting simultaneously a vertex of X and a vertex of Y). At the passage from D' to D_1 the numbers of the exceptional vertices and faces are preserved; so the complex D_1 is a member of $M(m, k; \varepsilon_v, \varepsilon_f; g_0+1)$. This procedure can continue until a member of $M(m, k; \varepsilon_v, \varepsilon_f; g)$ is obtained for any integer $g > g_0$.

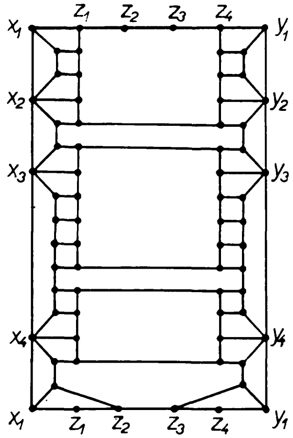

Fig. 3

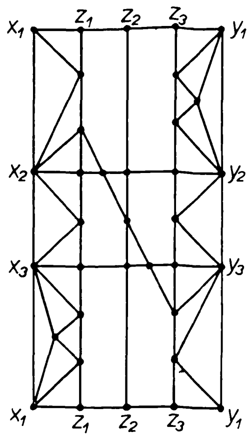

Fig. 4

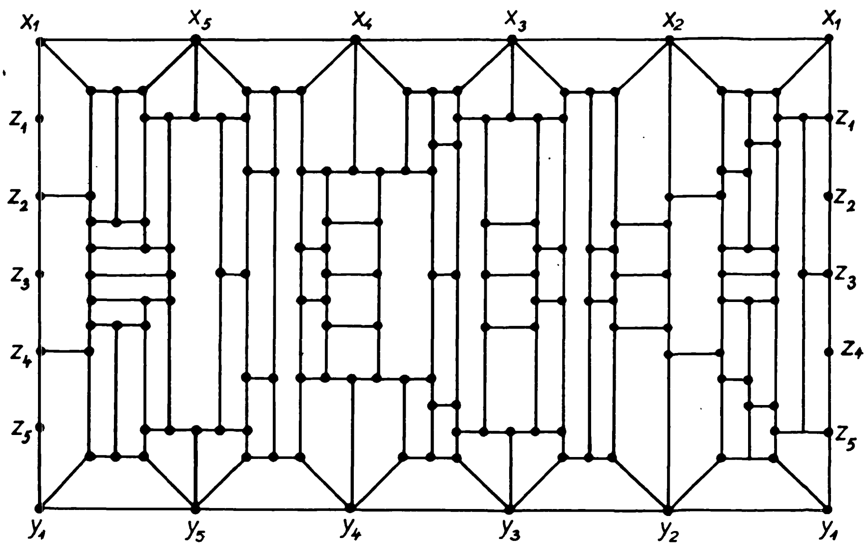

Fig. 5

Theorem 1. *Let g be a non-negative integer.*

a) If $(m, k) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$, the class $M(m, k; 0, 0; g)$ is non-empty.

b) The class $M(4,4;\ 0,0;\ g)$ is empty iff $g=0$.

Proof. a) As the boundary complex of the regular 3-polytope with m-valent vertices and k-gonal faces is a member of the class $M(m, k; 0, 0; 0)$, it is sufficient to apply Lemma 1.

b) The emptiness of the class $M(4, 4;\ 0, 0;\ 0)$ is implied by the following corollary of Euler's theorem concerning a cell-decomposition N of the 2-manifold of genus g:

$$(*) \qquad \sum_{i\geq 3}(4-i)\,(p_i(N)+v_i(N))=8(1-g)\,.$$

The well-known cell-complex decomposing the torus into quadrangles with four-valent vertices belongs to the class $M(4, 4;\ 0, 0;\ 1)$.

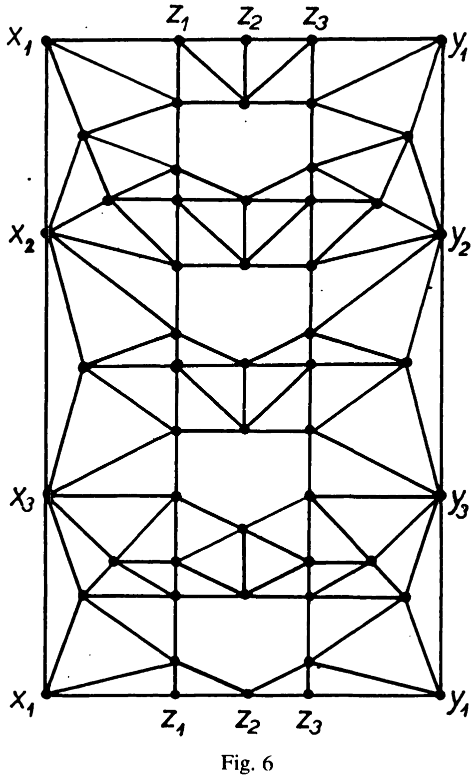

Fig. 6

The required complex for $g>1$ is constructed as follows: First a decomposition of the torus as in Fig. 7 is performed (by the identification of equally marked points). There we have one $(8g-4)$-gon W, $2(g-1)$ hexagons $H_1, H_2, \ldots, H_{2g-2}$ with trivalent vertices only, and quadrangles. The hexagons H_i and H_j, $i+j=$

$2g-1$, are joined then by $g-1$ handles decomposed into quadrangles (cf. Jucovič—Trenkler [10]). A manifold of genus g decomposed by a complex belonging to $M(4,4;\ 0,0;\ g)$ is reached.

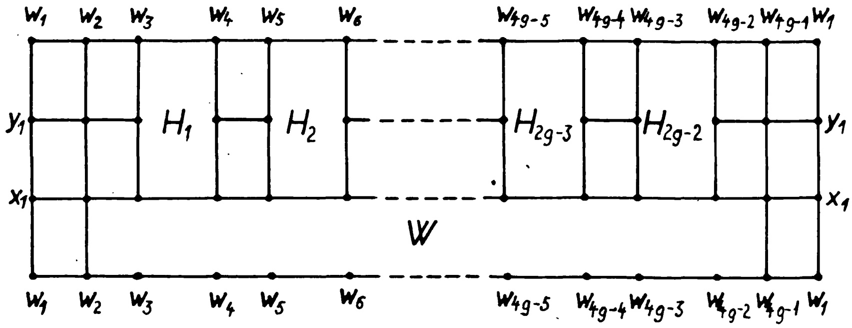

Fig. 7

Theorem 2. *Let g be a non-negative integer.*

a) The class $M(m, k;\ 0,1;\ g)$ as well as the class $M(m, k;\ 1,0;\ g)$ is empty for $m=k$.

b) Let $(m, k) \in \{(3,4), (4,3), (3,5), (5,3)\}$. The classes $M(m, k;\ 1,0;\ g)$, $M(m, k;\ 0,1;\ g)$ are empty iff $g=0$.

Proof of Theorem 2.

a) For a cell-complex N we obviously have:

$$(**) \qquad \sum_{i\geq 3} ip_i(N) = \sum_{i\geq 3} iv_i(N) = 2e(N)$$

where $e(N)$ is the number of edges of N. From the existence of precisely one exceptional cell in N, e.g. face, a contradiction would follow:

$$\sum_{i\geq 3} ip_i(N) \not\equiv 0 \pmod{m} \qquad \text{and} \qquad \sum_{i\geq 3} iv_i(N) \equiv 0 \pmod{m}\,.$$

b) The emptiness of the classes treated for $g=0$ is proved in Malkevitch [11]. Figures 8, 9, 10 and 11 present complexes belonging to the classes $M(4,3;\ 1,0;\ 1)$, $M(5,3;\ 1,0;\ 1)$, $M(4,3;\ 0,1;\ 1)$ and $M(5,3;\ 0,1;\ 1)$, respectively. Their duals belong to the classes $M(3,4;\ 0,1;\ 1)$, $M(3,5;\ 0,1;\ 1)$, $M(3,4;\ 1,0;\ 1)$, $M(3,5;\ 1,0;\ 1)$, respectively. Now using Lemma 1 we get complexes from all classes whose non-emptiness is asserted in Theorem 2.

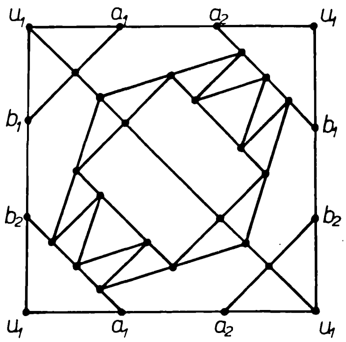

Fig. 8

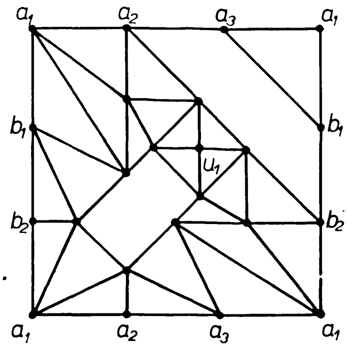

Fig. 9

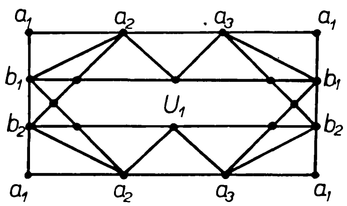

Fig. 10

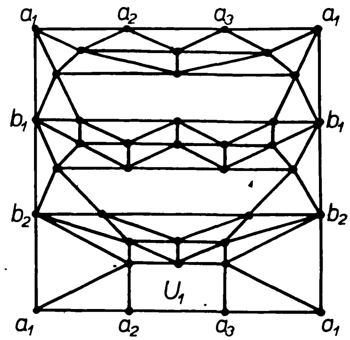

Fig. 11

3. Decompositions with two exceptional cells

Lemma 2. *Let* $(m, k) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$, $(\varepsilon_v, \varepsilon_f) \in \{(1, 1), (0, 2), (2, 0)\}$, *$g$ be a non-negative integer and* $d \in \{\bar{0}, 0, 1, 2, \dots\}$. *If the class* $M(m, k; \varepsilon_v, \varepsilon_f; g_0, d)$ *is non-empty, so is the class* $M(m, k; \varepsilon_v, \varepsilon_f; g; d)$ *for every integer* $g > g_0$.

The proof proceeds as that of Lemma 1, but in this case it is moreover necessary to preserve the distance between the exeptional cells c_1, c_2 at the passage a) from D to D' and b) from D' to D_1.

a) For this purpose the face W chosen from D must be multi-k-gonal. Then the distance of c_1, c_2 will not decrease because the distance between no two vertices w_i, w_j of the face W will decrease. As all edges of D remain edges of D', the distance of c_1, c_2 will not increase, so it is d in D', too. Therefore D' belongs to $M(m, k; \varepsilon_v, \varepsilon_f; g_0, d)$.

b) Clearly the k-gons X, Y can be chosen so that no path joining c_1, c_2 and containing inner vertices of the handle is shorter than the shortest path joining c_1 and c_2 in D' (e.g. it is sufficient to take the k-gons containing the edges w_1w_1' and $w_2'w_3'$, respectively). So the complex D_1 belongs to the class $M(m, k; \varepsilon_v, \varepsilon_f; g_0+1, d)$ and it is possible to apply induction.

Lemma 3. *Let g, d_0 be non-negative integers, $(m, k) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$, $(\varepsilon_v, \varepsilon_f) \in \{(0, 2), (1, 1)\}$. If the class $M(m, k; \varepsilon_v, \varepsilon_f; g, d_0)$ is non-empty, so is the class $M(m, k; \varepsilon_v, \varepsilon_f; g, d)$ for every integer $d > d_0$.*

Proof. Let the complex $D \in M(m, k; \varepsilon_v, \varepsilon_f; g, d_0)$, let W be its exceptional face and C the second exceptional cell. Arrange the face W analogously as in the proof of Lemma 1 in accordance with Fig. 1a; we get a complex D'. If P is a path of length d_0 joining a vertex w_i of W with a vertex of C in the complex D, then the path $P' = \{P, w_iw_i', w_i'\}$ is clearly the shortest path joining a vertex of W' with a vertex of C in the complex D'. As its length equals d_0+1, and the cells W', C are the only exceptional cells in D', the complex $D' \in M(m, k; \varepsilon_v, \varepsilon_f; g, d_0+1)$ and the induction works.

Theorem 3. *Let g and d be non-negative integers.*

a) The class $M(3, 3; 1, 1; g, d)$ is empty iff $(g, d) = (0, 0)$.

b) For $(m, k) \in \{(3, 4), (4, 3), (3, 5), (5, 3)\}$ the class $M(m, k; 1, 1; g, d)$ is empty iff $g = 0$.

Proof.

a) The emptiness of the class $M(3, 3; 1, 1; 0, 0)$ has been proved by Malkevitch [11].

In Fig. 12 a complex from the class $M(3, 3; 1, 1; 1, 0)$ is drawn. Using Lemma 2 we get a member of the class $M(3, 3; 1, 1; g, 0)$ for every integer $g > 1$.

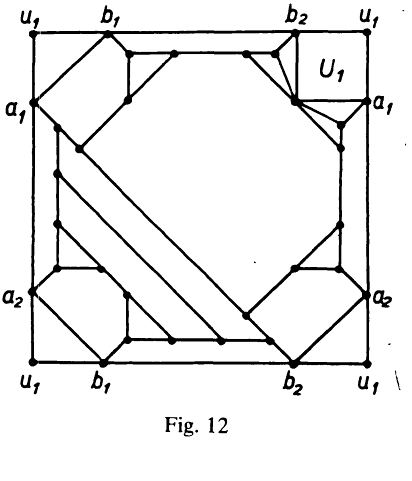

Fig. 12

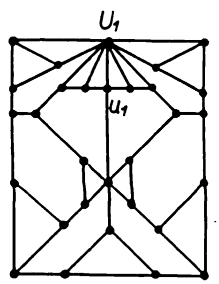

Fig. 13

Fig. 13 shows a complex belonging to the class $M(3, 3;\ 1, 1;\ 0, 1)$; so by Lemma 3 it is guaranteed the existence of complexes from classes $M(3, 3;\ 1, 1;\ 0, d)$ for all positive integers d. Now using Lemma 2 we get the remaining required complexes.

b) Again the emptiness of the classes $M(m, k;\ 1, 1;\ 0, d)$ has been proved by Malkevitch [11].

In Fig. 14 and 15 there are complexes belonging to the classes $M(4, 3;\ 1, 1;\ 1, 0)$ and $M(3, 5;\ 1, 1;\ 1, 0)$, respectively. The duals of these complexes belong to the classes $M(3, 4;\ 1, 1;\ 1, 0)$ and $M(5, 3;\ 1, 1;\ 1, 0)$, respectively. Combining procedures from Lemmas 2 and 3 our Theorem is proved.

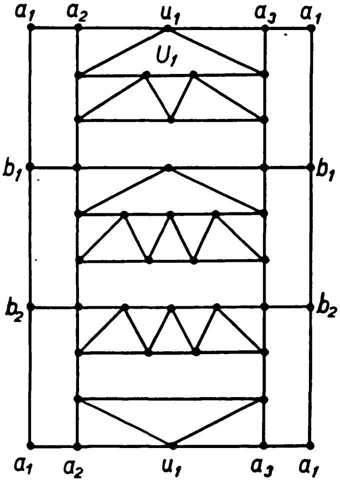

Fig. 14

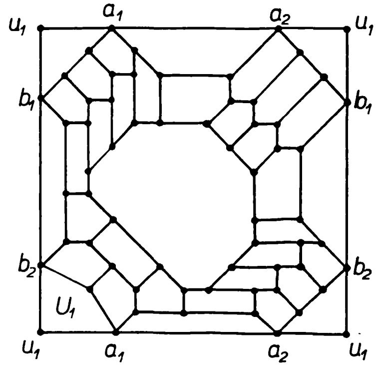

Fig. 15

Theorem 4. *Let g be a non-negative integer.*

a) For $(m, k) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$ the class $M(m, k;\ 0, 2;\ g, \bar{0})$ is empty iff $g = 0$.

b) If $(m, k) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$ and d is a non-negative integer, the class $M(m, k;\ 0, 2;\ g, d)$ is non-empty.

c) For $d \in \{\bar{0}, 1, 2, \ldots\}$ the class $M(4, 4;\ 0, 2;\ g, d)$ is empty iff $g < 2$.

Proof.

a) The emptiness of the classes mentioned in the statement is proved in Malkevitch [11]. Examples of complexes belonging to $M(m, k;\ 0, 2;\ 1, \bar{0})$ are drawn in Fig. 16, 17, 18, 19 and 20 for $(m, k) = (3, 3), (3, 4), (4, 3), (3, 5)$ and $(5, 3)$, respectively. Now use Lemma 2.

b) Fig. 21, 22, 23, 24 and 25 show complexes from $M(m, k; 0, 2; 0, 0)$ for $(m, k)=(3, 3)$, $(3, 4)$, $(4, 3)$, $(3, 5)$ and $(5, 3)$, respectively. Now use Lemmas 2 and 3.

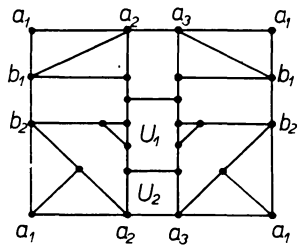

Fig. 16

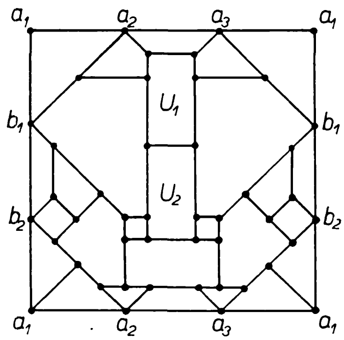

Fig. 17

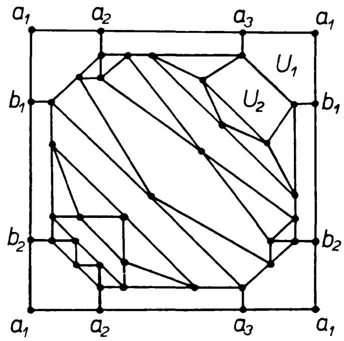

Fig. 18

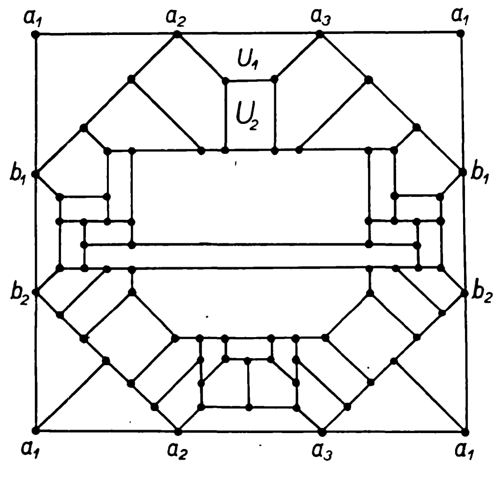

Fig. 19

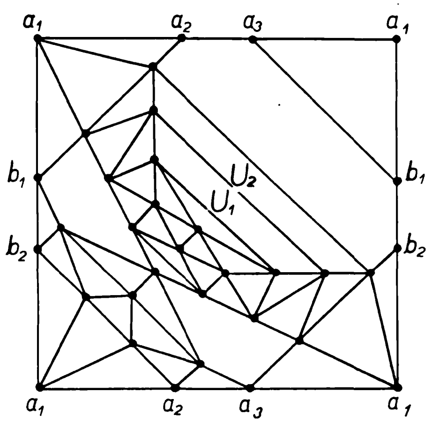

Fig. 20

c) The emptiness of the class $M(4, 4; 0, 2; 0, d)$ follows, for every d, from the relation (*). For $g = 1$ from the same relation and from the obvious equality (**) it follows that in a complex from the class $M(4, 4; 0, 2; 1, d)$ the exceptional faces could be only a triangle and a pentagon with remaining faces being quadrangles and all vertices being 4-valent. In Barnette—Jucovič—Trenkler [1] it is proved that such decompositions of the torus do not exist. So it remains to show that for all pairs (g, d), $g \in \{2, 3, \dots\}$, $d \in \{(\bar{0}, 1, 2, \dots)$, the class $M(4, 4; 0, 2; g, d)$ is non-empty.

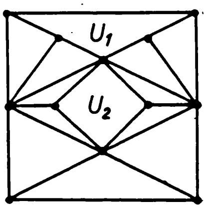

Fig. 21

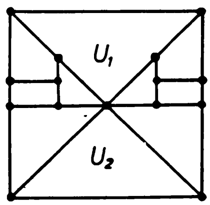

Fig. 22

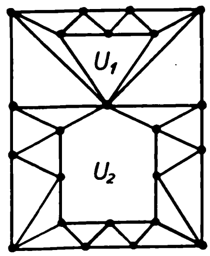

Fig. 23

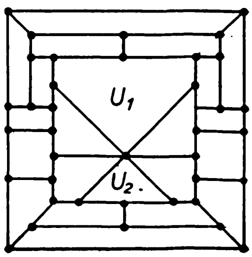

Fig. 24

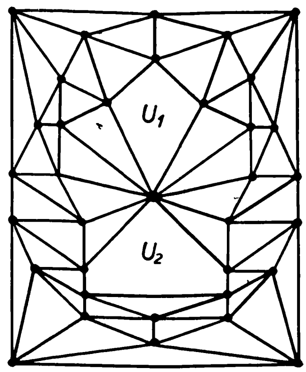

Fig. 25

First a similar decomposition of the torus as in the proof of Theorem 1 is constructed, see Fig. 26 (full lines). In it we have two octagons H_1, H_{2g-2} and hexagons $H_2, H_3, \dots, H_{2g-3}$ as openings for handles forming, the adjacent exceptional faces U_1, U_2 (hexagons), one $8(g-1)$-gon W and quadrangles. The complex constructed so far belongs to $M(4, 4; 0, 2; g, \bar{0})$.

A member of $M(4, 4; 0, 2; g, d)$ can be obtained by adding d strips containing three quadrangles each at the "end" of the map in Fig. 26 (dotted lines). To avoid the shortest path which joins the exceptional faces going, for a sufficiently great d,

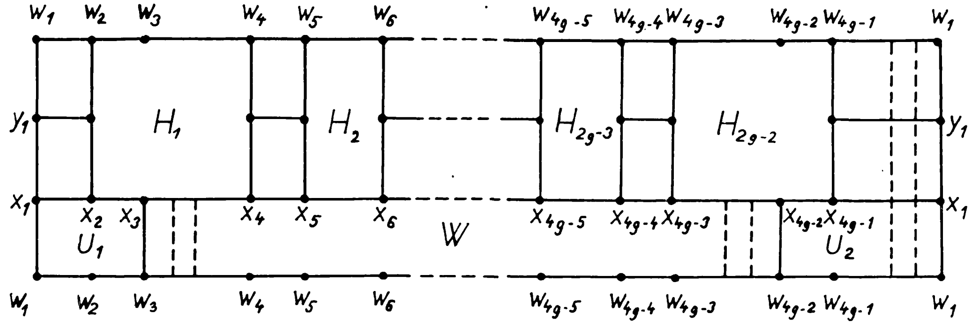

Fig. 26

through edges of the face W, a "great" number ($\geq d/2$) of edges parallel to x_3w_3 and to $x_{4g-2}w_{4g-2}$ meeting the openings H_1, H_{2g-2} have been added. The decomposition of the handles can be done so that the shortest path joining the exceptional faces does not meet vertices on the handles.

Lemma 4. *Let g be a non-negative, d_0 a positive integer and $(m, k) \in \{(3, 3), (4, 3), (5, 3)\}$. If the class $M(m, k;\ 2, 0;\ g, d_0)$ is non-empty, so is the class $M(m, k;\ 2, 0;\ g, d)$ for every integer $d > d_0$.*

Proof.

Let the complex $D \in M(m, k;\ 2, 0;\ g, d_0)$. The edges $e_1 = u_1w_1$, $e_2 = u_1w_2$, ..., $e_n = u_1w_n$ incident with the exceptional vertex u_1 can be marked so that e_n belongs to the shortest path P joining u_1 with the second exceptional vertex u_2. Inside the edge e_1 two points x_1, y_1 are chosen, both are joined with w_2, w_n forming four triangles. Two of these triangles are subdivided to form two subcomplexes D_k^m employed already in the proof of Lemma 1 (see Fig. 27a). In Fig. 27b as an example for $(m, k) = (4, 3)$ the result of this procedure is drawn. In the new complex D_1 the valencies of the vertices w_2, w_n have increased by m, the valencies of the vertices x_1, y_1 are $2m$, the faces W_1', W_2' are multi-3-gonal (W_i' has the same number of edges as W_i in the complex D), the remaining added faces are triangles, all added vertices are m-valent. The only exceptional cells remain the vertices u_1 and u_2. If P' is a part of the path P joining the vertices u_2 and w_n, then the path $P_1 = \{P',\ w_nx_1,\ x_1,\ x_1u_1,\ u_1\}$ has length $d_0 + 1$.

The next arrangement follows the goal to get a complex with the only exceptional cells u_1, u_2 joined by P_1 as the shortest path. For this purpose inside e_2 two new vertices x_2, y_2 are chosen and the whole procedure above with the four triangles and two subcomplexes D_k^m is performed; we get a complex D_2. In D_2 the same is done with the edge e_3 and so is proceeded with all edges e_i. A sequence of complexes D, D_1, D_2, ..., D_n is reached whose last member belongs to $M(m, k;\ 2, 0;\ g, d_0 + 1)$; in it P_1 is the shortest path joining u_1 and u_2. The induction can be applied.

Lemma 5. *Let g be a non-negative, d_0 a positive integer and $(m, k) \in \{(3, 4), (3, 5)\}$. If the class $M(m, k; 2, 0; g, d_0)$ is non-empty, so is the class $M(m, k; 2, 0; g, d_0 + 2z)$ for every positive integer z.*

The proof proceeds equally as that of Lemma 4. All edges e_i are subdivided into three edges and the subcomplexes D_k^m of the cube and the regular dodecahedron are used.

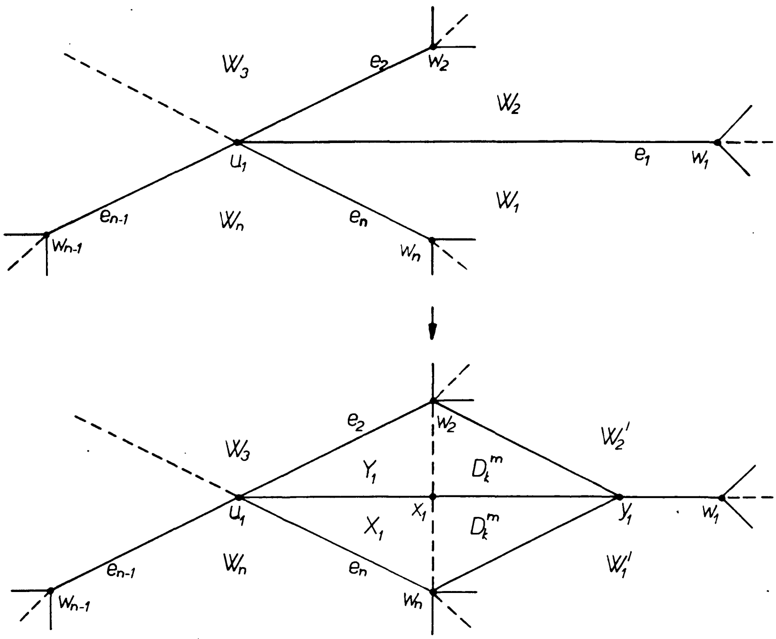

Fig. 27a

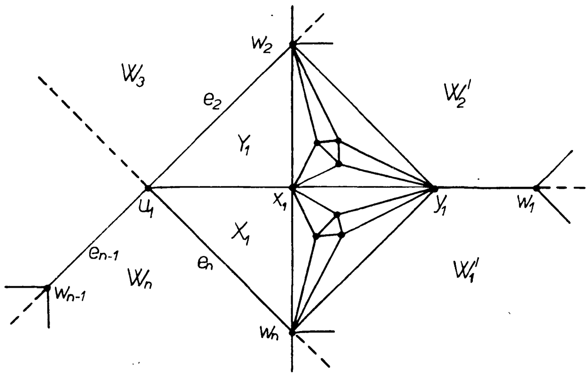

Fig. 27b

However in the complex D_n the distance from u_1 to w_n is three in this case (X_1 is a k-gon, $k \in \{4, 5\}$) and the shortest path joining u_1 and u_2 contains three edges appearing in the subdivision of e_n. Therefore the last member in the sequence of complexes $D, D_1, \ldots, D_n$ belongs to the class $M(m, k;\ 2, 0;\ g, d_0+2)$.

Theorem 5. *Let g be a non-negative and d a positive integer.*

a) If $(m, k) \in \{(3, 3), (3, 4), (4, 3), (5, 3)\}$, the class $M(m, k;\ 2, 0;\ g, d)$ is empty iff $(g, d) = (0, 1)$.

b) The class $M(3, 5;\ 2, 0;\ g, d)$ is empty iff $(g, d) \in \{(0, 1), (0, 3)\}$.

c) The class $M(4, 4;\ 2, 0;\ g, d)$ is empty iff $g < 2$.

The proof
in general will follow the pattern of the proof of Theorem 4. However in the case b) and partly in the case a) the induction will step on residue classes mod 2 and not directly on all positive integers.

a) The emptiness of $M(m, k;\ 2, 0;\ 0, 1)$ is mentioned in Malkevitch [11].

The dual of a complex belonging to $M(m, k;\ 0, 2;\ g, \bar{0})$ is a complex belonging to $M(k, m;\ 2, 0;\ g, 1)$ and so we have secured the existence of complexes in $M(m, k;\ 2, 0;\ 1, 1)$ by Theorem 4a). In Fig. 28a (D_k^m has the same meaning as before) a complex from $M(m, k;\ 2, 0;\ 0, 2)$ is schematically represented; W_i are k-gonal faces, w_i are $2m$-valent vertices, the only two exceptional cells are the vertices u_1, u_2 whose valencies are $m+1$ and m^2-1, respectively, and the distance between them equals two (the shortest path is $P = \{u_1,\ u_1w_1,\ w_1,\ w_1u_2,\ u_2\}$). In Fig. 28b an example for $(m, k) = (3, 4)$ is depicted.

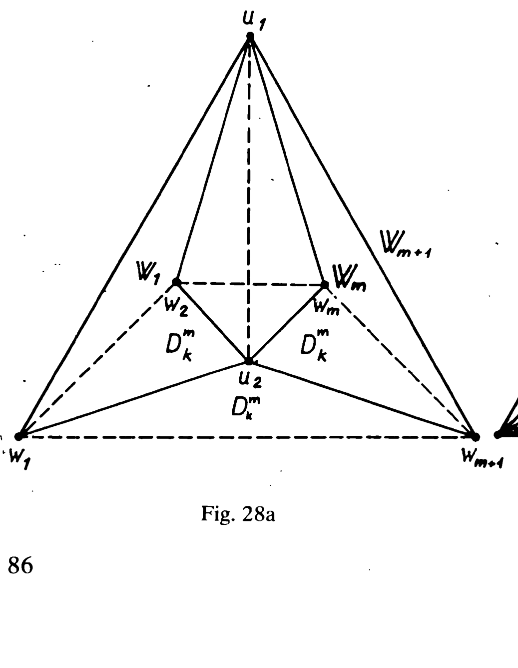

Fig. 28a

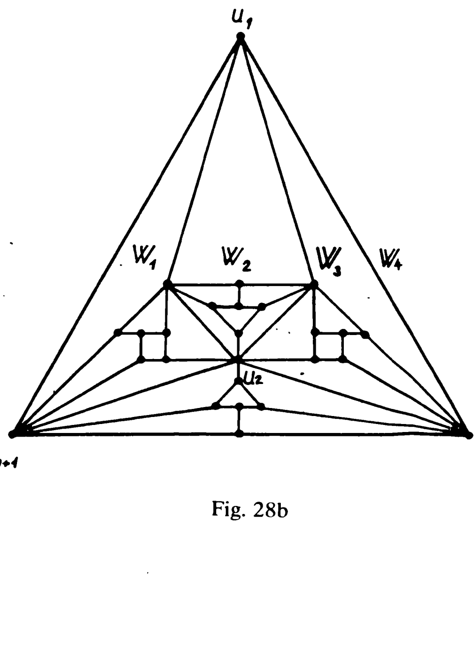

Fig. 28b

Fig. 29 represents a complex belonging to $M(3, 4;\ 2, 0;\ 0, 3)$. Now using Lemmas 2, 4 and 5 the assertion is proved.

b) For the emptiness of $M(3, 5;\ 2, 0;\ 0, 1)$ see Malkevitch [11]. To prove the emptiness of $M(3, 5;\ 2, 0;\ 0, 3)$ it is necessary to distinguish a large number of possible shapes of the path joining the exceptional vertices and the faces meeting it. We omit the proof here, it is contained in Horňák [6].

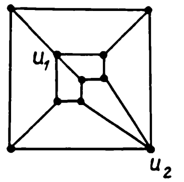

Fig. 29

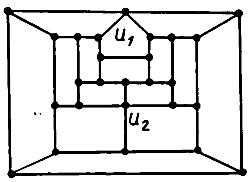

Fig. 30

From Theorem 4a) by duality there follows the existence of complexes from $M(3, 5;\ 2, 0;\ 1, 1)$ and by Lemmas 2 and 5 the existence of complexes from $M(3, 5;\ 2, 0;\ g, d)$ for every positive integer g and every odd positive integer d.

The construction used in Fig. 28a for $(m, k) = (3, 5)$ reaches a complex belonging to $M(3, 5;\ 2, 0;\ 0, 2)$. Fig. 30 represents a complex from $M(3, 5;\ 2, 0;\ 0, 5)$. To conclude the proof of our Theorem use Lemmas 2 and 5.

c) The proof follows by duality of the complexes described in the proof of Theorem 4c).

4. Remarks

a) In Theorems 1, 2, 3, 4 and 5 a very small portion of nearly regular decompositions of 2-manifolds of higher genus has been treated. While for the parameters m, k of the nearly regular cell-decompostitions from Euler's formula it follows that $m, k \leq 5$ for the sphere only, for manifolds of higher genus these parameters can be greater. We do not know of a general procedure allowing to decide whether, for given parameters m, k, g, d, the classes $M(m, k;\ 1, 0;\ g)$, $M(m, k;\ 0, 1;\ g)$, $M(m, k;\ \varepsilon_v, \varepsilon_f;\ g, d)$ and even $M(m, k;\ \varepsilon_v, \varepsilon_f;\ g)$ for $(\varepsilon_v, \varepsilon_f) \in \{(2, 0), (1, 1), (0, 2)\}$ are empty or not. We are unable to decide this problem even for maps which are not required to be cell-complexes. Our knowledge of different properties of cell-decompositions of 2-manifolds leads to the following working conjecture: To every sequence of non-negative integers $(m, k, \varepsilon_v, \varepsilon_f, d)$ with $\varepsilon_v + \varepsilon_f \leq 2$ there is a non-negative integer g_0 such that the class $M(m, k;\ \varepsilon_v, \varepsilon_f;\ g, d)$ is non-empty iff $g \geq g_0$.

b) Another problem not touched above is: If nearly regular complexes of a certain class $M(m, k;\ \varepsilon_v, \varepsilon_f;\ g)$ do exist, what are all possible numbers of edges incident with the exceptional cells? (Cf. Crowe [2] and Jendrol' [7] for the sphere.)

c) For $(m, k) \in \{(4, 5), (5, 4)\}$ a decomposition, with a unique exceptional cell, of the manifold of genus g such that all 2-cells are topological discs, does exist iff $g \geq 2$. However the decompositions of manifolds of genus ≥ 2 we know are not complexes. So this gap should be filled up.

d) Solutions of the mentioned *existence* questions concerning nearly regular cell-decompositions of 2-manifolds are only first steps in the study of these complexes. What can be told about the structure of their 1-skeletons? How many faces of certain kind and vertices of certain valency can they have? What can be stated about the valencies of their adjacent vertices (cf. Jucovič [9])?

e) Probably nothing has been published about nearly regular decompositions of 2-manifolds which are geometrical complexes. Finding general assertions concerning the existence of such complexes seems to be a difficult task.

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ПОЧТИ ПРАВИЛЬНЫЕ КЛЕТОЧНЫЕ РАЗЛОЖЕНИЯ ОРИЕНТИРУЕМЫХ ПОВЕРХНОСТЕЙ С≤2 ИСКЛЮЧИТЕЛЬНЫМИ КЛЕТКАМИ

Мирко Горняк—Эрнест Юцович

Резюме

Пусть $m, k>1$, ε_v, ε_f, $g\geqslant 0$ целые числа; символом $M(m, k; \varepsilon_v, \varepsilon_f; g)$ обозначим класс всех клеточных разложений связной ориентируемой поверхности рода g обладающих следующими свойствами: а) степени всех вершин (0-клетки) — множители числа m за исключением ε_v вершин; б) числа вершин всех граней (2-клетки) — множители k за исключением ε_f граней. Если $\varepsilon_v+\varepsilon_f=2$, то символ $M(m, k; \varepsilon_v, \varepsilon_f; g, d)$ обозначает подкласс класса $M(m, k; \varepsilon_v, \varepsilon_f; g)$, который содержит все разложения с исключительными клетками имеющими расстояние d (в смысле теории графов).

В статье доказываются следующие утверждения: Для $(m, k)\in\{(3, 4), (4, 3), (3, 5), (5, 3)\}$ а) классы $M(m, k; 0, 0; g)$ и $M(m, k; 0, 2; g, d)$ не пустые; б) классы $M(m, k; 1, 0; g)$, $M(m, k; 0, 1; f)$ и $M(m, k; 1, 1; g, d)$ пустые тогда и только тогда когда $g=0$.

Классы $M(4, 4; 0, 2; g, d)$ и $M(4, 4; 2, 0; g, d)$ для $d\geqslant 1$ пустые тогда и только тогда когда $g<2$.

Класс $M(4, 4; 0, 0; g)$ пустой тогда и только тогда когда $g=0$.

Класс $M(3, 3; 1, 1; g, d)$ пустой тогда и толско тогда когда $g=d=0$ а класс $M(3, 3; 0, 0; g)$ и $M(3, 3; 0, 2; g, d)$ не пустой.

Для $d>0$ и $(m, k)\in\{(3, 3), (3, 4), (4, 3), (5, 3)\}$ класс $M(m, k; 2, 0; g, d)$ пустой тогда и только тогда когда $(g, d)=(0, 1)$. Класс $M(3, 5; 2, 0; g, d)$ пустой тогда и только тогда когда $(g, d)\in\{(0, 1), (0, 3)\}$.

Аналогичные вопросы рассмотрены для разложений с двумя исключительными гранями расстояния 0.