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STRUCTURE MORPHISMS OF PROLONGATION FUNCTORS

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In differential geometry one meets frequently with such a prolongation procedure that can be interpreted as a functor p of a category $\mathscr{C}$ into a category $\mathscr{D}$, i.e. every $\mathscr{C}$-morphism $f\colon M \to N$ is prolonged into a $\mathscr{D}$-morphism $pf\colon pM \to pN$. As a rule, there exists another functor $q\colon \mathscr{D} \to \mathscr{C}$ satisfying $q \circ p = id_{\mathscr{C}}$, which represents a back projection related with the prolongation procedure p. Some general properties of prolongation functors are studied in [11], [13], [15], [16]. In many cases it is interesting to find certain necessary and sufficient conditions for an arbitrary $\mathscr{D}$-morphism $F\colon pM \to pN$ over a $\mathscr{C}$-morphism $f = qF\colon M \to N$ to be the prolongation pf of f. We first give two simple examples of this kind.

Let $\mathscr{M}$ be the category of differentiable manifolds and mappings, $\mathscr{M}_0 \subset \mathscr{M}$ the subcategory of all diffeomorphisms, $\mathscr{FM}$ the category of fibered manifolds, $B\colon \mathscr{FM} \to \mathscr{M}$ the base functor and $\mathscr{FM}_0 \subset \mathscr{FM}$ the subcategory $B^{-1}(\mathscr{M}_0)$, i.e. an $\mathscr{FM}$-morphism belongs to $\mathscr{FM}_0$ iff the induced base mapping is a diffeomorphism.

Example 1. If we assign to every manifold M the fibered manifold H^1M of all linear frames on M and to every diffeomorphism $f\colon M \to \bar{M}$ the induced map $H^1f\colon H^1M \to H^1\bar{M}$, we get a functor $H^1\colon \mathscr{M}_0 \to \mathscr{FM}_0$. Consider the canonical $\mathbf{R}^n$-valued form $\Theta H^1M\colon TH^1M \to \mathbf{R}^n$, $n = \dim M$. The following assertion is classical. If $F\colon H^1M \to H^1\bar{M}$ is an $\mathscr{FM}_0$-morphism over $f = BF$, then $F = H^1f$ iff $\Theta H^1M = (\Theta H^1\bar{M}) \circ TF$. (Kobayashi and Nomizu, [7], p. 226, assume that F is an $\mathscr{FM}$-isomorphism, but their proof holds even for our stronger assertion.)

Example 2. Consider a functor J^1 transforming a fibered manifold Y into its first jet prolongation J^1Y and an $\mathscr{FM}_0$-morphism $f\colon Y \to \bar{Y}$ into the induced map $J^1f\colon J^1Y \to J^1\bar{Y}$, $j^1_x\sigma \mapsto j^1_{Bf(x)}(f \circ \sigma \circ (Bf)^{-1})$, where σ is a local section of Y. Further, denote by $VY \subset TY$ the vertical tangent bundle of a fibered manifold Y and by Vf the restriction $Tf \mid VY$ of the tangent map of an $\mathscr{FM}$-morphism $f\colon Y \to \bar{Y}$. Hence V is a functor of $\mathscr{FM}$ into the category $\mathscr{VB}$ of differentiable vector bundles. Let $\beta\colon J^1Y \to Y$ be the target jet projection. The so-called structure form $\Theta J^1Y\colon TJ^1Y \to VY$ maps a vector $A \in T_UJ^1Y$ into the projection of $T\beta(A) \in T_uY$ into

$V_u Y$ in the direction of the horizontal subspace hU of $T_u Y$ corresponding to 1-jet U, $u = \beta U$, [4], [5].

Proposition 1. *A mapping $F\colon J^1 Y \to J^1 \bar{Y}$ over an $\mathcal{FM}_0$-morphism $f\colon Y \to \bar{Y}$ is of the form $F = J^1 f$ iff the following diagram commutes*

$$
(1) \qquad
\begin{array}{ccc}
VY & \xrightarrow{\;Vf\;} & V\bar{Y} \\
{\scriptstyle \Theta J^1 Y}\big\uparrow & & \big\uparrow{\scriptstyle \Theta J^1 \bar{Y}} \\
TJ^1 Y & \xrightarrow{\;TF\;} & TJ^1 \bar{Y}
\end{array}
$$

Proof. Denote by $\Theta_U\colon T_u Y \to V_u Y$ the projection determined by hU, so that $hU = \ker \Theta_U$. For every $U \in J^1 Y$, we have such a situation

$$
\begin{array}{ccccccccc}
0 & \to & hU & \to & T_u Y & \xrightarrow{\;\Theta_U\;} & V_u Y & \to & 0 \\
 & & & & \big\downarrow{\scriptstyle T_u f} & & & & \\
0 & \to & hFU & \to & T_{fu} \bar{Y} & \xrightarrow{\;\Theta_{FU}\;} & V_{fu} \bar{Y} & \to & 0
\end{array}
$$

where both rows are exact sequences of vector spaces. Obviously, $V_u f\colon V_u Y \to V_{fu} \bar{Y}$ commutes in this diagram iff $T_u f$ maps hU into hFU. The latter condition means $FU = (J^1 f)(U)$, QED.

We shall now specify the category in which the values of the functor J^1 lie. We define a 2-fibered manifold as a quintuple $Z \xrightarrow{\mu} Y \xrightarrow{\lambda} X$, where Z, Y, X are manifolds and μ, λ are surjective submersions; Z is called the total space, $MZ = Y$ (or the fibered manifold $Y \xrightarrow{\lambda} X$) is the middle space and $BZ = Y$ is the base. A 2-fibered morphism is a triple $(f, Mf, Bf)\colon (Z, Y, X) \to (\bar{Z}, \bar{Y}, \bar{X})$ such that the following diagram commutes

$$
\begin{array}{ccccc}
Z & \xrightarrow{\;\mu\;} & Y & \xrightarrow{\;\lambda\;} & X \\
\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle Mf} & & \big\downarrow{\scriptstyle Bf} \\
\bar{Z} & \xrightarrow{\;\bar{\mu}\;} & \bar{Y} & \xrightarrow{\;\bar{\lambda}\;} & \bar{X}
\end{array}
$$

We get a category $2\mathcal{FM}$ and two functors $M\colon 2\mathcal{FM} \to \mathcal{FM}$ and $B\colon 2\mathcal{FM} \to \mathcal{M}$. Let $2\mathcal{FM}_0$ be the subcategory of all $2\mathcal{FM}$-morphisms f satisfying $Bf \in \mathcal{M}_0$. Clearly, J^1 is a functor $\mathcal{FM}_0 \to 2\mathcal{FM}_0$, the back projection being the middle projection M, and Proposition 1 gives a necessary and sufficient condition for a $2\mathcal{FM}_0$-morphism $F\colon J^1 Y \to J^1 \bar{Y}$ over $f = MF\colon Y \to \bar{Y}$ to be of the form $F = J^1 f$.

1. Structure morphisms

The above examples (as well as those given below) can be expressed in terms of the following general scheme. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, p a (prolongation) functor $p: \mathscr{C} \to \mathscr{D}$ and q a (back projection) functor $q: \mathscr{D} \to \mathscr{C}$ satisfying $q \circ p = id_{\mathscr{C}}$. Consider an auxiliary category $\mathscr{E}$ and two auxiliary functors φ: $\mathscr{C} \to \mathscr{E}$ and ψ: $\mathscr{D} \to \mathscr{E}$. We shall say that $\mathscr{E}$-morphisms ΘpM: $\psi(pM) \to \varphi M$ defined for all $M \in Ob\mathscr{C}$ are structure morphisms of functor p with respect to the pair (φ, ψ) if the following two conditions are equivalent for every M, $\bar{M} \in Ob\mathscr{C}$:

a) a $\mathscr{D}$-morphism F: $pM \to p\bar{M}$ over $f = qF$: $M \to \bar{M}$ is of the form $F = pf$,

b) the diagram

$$
\begin{array}{ccc}
\varphi M & \xrightarrow{\varphi f} & \varphi \bar{M} \\
{\scriptstyle \Theta pM}\uparrow & & \uparrow{\scriptstyle \Theta p\bar{M}} \\
\psi(pM) & \xrightarrow{\psi F} & \psi(p\bar{M})
\end{array}
\tag{2}
$$

is commutative.

Clearly it is useful to study geometrically interesting structure morphisms only, as id_{pM} are trivial structure morphisms of any functor p with respect to the pair $(p, id_{\mathscr{D}})$. In differential geometry, $\mathscr{C}$ is usually (i.e. in all examples we know) a subcategory of $\mathscr{M}$, $\mathscr{D}$ is a subcategory of $\mathscr{F}\mathscr{M}$, q is the base functor and the whole category $\mathscr{M}$ can be mostly taken as $\mathscr{E}$. (On the other hand, the specification of $\mathscr{D}$ is significant, since it determines for how large a class of mappings the commutativity of (2) implies $F = pf$.)

In Example 1, we have $\psi = T$, while φ is a trivial functor I defined by $IM = \mathbf{R}^{\dim M}$, $If = id$. We now present some further examples.

Example 3. We first discuss two special cases of Proposition 1.

a) Consider the restriction $J^1|\mathscr{V}\mathscr{B}_0$, where $\mathscr{V}\mathscr{B}_0 = \mathscr{V}\mathscr{B} \cap \mathscr{F}\mathscr{M}_0$. For every vector bundle E, there is a canonical projection τ: $VE \to E$ translating any vertical vector into the tangent space at the zero vector. This projection transforms ΘJ^1E into a mapping $\tilde{\Theta} J^1E$: $TJ^1E \to E$. By Proposition 1, we deduce

Corollary 1. *$\tilde{\Theta} J^1E$ are structure morphisms of $J^1|\mathscr{V}\mathscr{B}_0$ with respect to $(id_{\mathscr{V}\mathscr{B}_0}, T)$.*

b) Let $\mathscr{P}\mathscr{B}$ be the category of principal fiber bundles and their homomorphisms and $\mathscr{P}\mathscr{B}_0 = \mathscr{P}\mathscr{B} \cap \mathscr{F}\mathscr{M}_0$. We have a functor l of $\mathscr{P}\mathscr{B}$ into the category of Lie algebras that assigns to every principal fiber bundle $P(X, G)$ the Lie algebra $lP = \mathfrak{G}$ of its structure group and to every $\mathscr{P}\mathscr{B}$-morphism f: $P \to \bar{P}$ the induced Lie algebra homomorphism lf: $lP \to l\bar{P}$. Every V_uP, $u \in P$, is canonically identified with lP. Using these identifications, we can modify ΘJ^1P into a mapping $\tilde{\Theta} J^1P$: $TJ^1P \to lP$. By Proposition 1 we find directly

Corollary 2. *$\tilde{\Theta} J^1P$ are structure morphisms of $J^1|\mathscr{P}\mathscr{B}_0$ with respect to (l, T).*

Example 4. Let J^r: $\mathscr{F}\mathscr{M}_0 \to 2\mathscr{F}\mathscr{M}_0$ be the functor transforming a fibered manifold $Y \to X$ into its r-th jet prolongation $J^rY \to Y \to X$ and an $\mathcal{FM}_0$-morphism $f\colon Y \to \bar{Y}$ into the induced map $J^rf\colon J^rY \to J^r\bar{Y}$. The canonical form $\Theta J^rY\colon TJ^rY \to VJ^{r-1}Y$ is defined by means of the inclusion $J^rY \subset J^1(J^{r-1}Y)$.

Proposition 2. *ΘJ^rY are structure morphisms of functor J^r with respect to (VJ^{r-1}, T), i.e. a $2\mathcal{FM}_0$-morphism $F\colon J^rY \to J^r\bar{Y}$ over $f = MF\colon Y \to \bar{Y}$ is of the form $F = J^rf$ iff $(VJ^{r-1}f)\circ(\Theta J^rY) = (\Theta J^r\bar{Y})\circ TF$.*

Proof. It is well known that an arbitrary mapping of a fibered manifold into another is locally an $\mathcal{FM}$-morphism iff its tangent map transforms the vertical tangent bundle of the first fibered manifold into the vertical tangent bundle of the second one. By the definition of ΘJ^rY this form maps every $(J^rY \to J^{r-1}Y)$-vertical vector into zero. Conversely, if A is an $(J^rY \to X)$-vertical vector on J^rY satisfying $(\Theta J^rY)(A) = 0$, then A is $(J^rY \to J^{r-1}Y)$-vertical. Thus, if TF commutes with the canonical forms as assumed in Proposition 2, then TF maps the vertical tangent bundle of $J^rY \to J^{r-1}Y$ into the vertical tangent bundle of $J^r\bar{Y} \to J^{r-1}\bar{Y}$. Hence there exists a local map F_0 of $J^{r-1}Y$ into $J^{r-1}\bar{Y}$ under F and we can use induction. For $r = 1$, Proposition 2 coincides with Proposition 1; assume that it holds for $r-1$. Since the jet projections commute with the canonical forms on the successive jet prolongations, F_0 satisfies the assumptions of Proposition 2. Hence F_0 is locally of the form $J^{r-1}f$ by the induction hypothesis. By Proposition 1 we now deduce that F is the restriction of $J^1(J^{r-1}f)$ to J^rY, but the latter restriction coincides with J^rf, QED.

Example 5. In [8] we introduced the r-th prolongation $W^rP(X, G^r_n)$ of any principal fiber bundle $P(X, G)$, $n = \dim X$. Any $\mathcal{PB}_0$-morphism $f\colon P \to \bar{P}$ induces naturally a map $W^rf\colon W^rP \to W^r\bar{P}$, so that W^r is a functor of $\mathcal{PB}_0$. We first discuss the case $r = 1$. Every $U \in W^1P$, $\beta U = u \in P$, determines a linear isomorphism $\tilde{U}\colon \mathbf{R}^n \oplus lP \to T_uP$, [8]. The canonical form of W^1P is a mapping $\Theta W^1P\colon TW^1P \to \mathbf{R}^n \oplus lP$, $A \mapsto \tilde{U}^{-1}(T\beta(A))$ for all $A \in T_UW^1P$.

Proposition 3. *ΘW^1P are structure morphisms of W^1 with respect to $(I \oplus l, T)$, i.e. a $2\mathcal{FM}_0$-morphism $F\colon W^1P \to W^1\bar{P}$ over a $\mathcal{PB}_0$-morphism $f = MF\colon P \to \bar{P}$ is of the form $F = W^1f$ iff $(I \oplus l)\circ(\Theta W^1P) = (\Theta W^1\bar{P})\circ TF$.*

Proof. A linear isomorphism $\mathbf{R}^n \oplus lP \to T_uP$ will be said to be admissible if its restriction to lP is the canonical identification of lP and V_uP. From the decomposition $W^1P = H^1X \oplus J^1P$, [8], we deduce that the admissible isomorphisms are in a one-to-one correspondence with the elements of W^1_uP. Then we easily verify that the diagram

$$\begin{array}{ccc}
\mathbf{R}^n \oplus lP & \xrightarrow{\ \tilde{U}\ } & T_uP \\
{\scriptstyle id \oplus lf}\downarrow & & \downarrow{\scriptstyle T_uf} \\
\mathbf{R}^n \oplus l\bar{P} & \xrightarrow{\ \tilde{V}\ } & T_{fu}\bar{P}
\end{array}$$

$U \in W^1_uP$, $V \in W^1_{fu}\bar{P}$, commutes iff $V = (W^1f)(U)$, QED.

For $r>1$, the inclusion $W^rP \subset W^1(W^{r-1}P)$ defines the canonical form ΘW^rP: $TW^rP \to \mathbf{R}^n \oplus \mathfrak{G}_n^{r-1}$, where $\mathfrak{G}_n^{r-1}$ is the Lie algebra of the structure group of $W^{r-1}P$. Every Lie algebra homomorphism $m: \mathfrak{G} \to \bar{\mathfrak{G}}$ is prolonged into $m_n^{r-1}: \mathfrak{G}_n^{r-1} \to \bar{\mathfrak{G}}_n^{r-1}$. Denote by $I \oplus l^{r-1}$ the functor $P \mapsto \mathbf{R}^n \oplus (lP)_n^{r-1}$, $f \mapsto I \oplus (lf)_n^{r-1}$ for any $P \in Ob\mathcal{PB}$ and any $\mathcal{PB}_0$-morphism $f: P \to \bar{P}$. Using the same procedure as in the proof of Proposition 2, we deduce by Proposition 3

Corollary 3. *ΘW^rP are structure morphisms of W^r with respect to $(I \oplus l^{r-1}, T)$, i.e. a $2\mathcal{FM}_0$-morphism F: $W^rP \to W^r\bar{P}$ over a $\mathcal{PB}_0$-morphism $f = MF$: $P \to \bar{P}$ is of the form $F = W^rf$ iff $(I \oplus l^{r-1}) \circ (\Theta W^rP) = (\Theta W^r\bar{P}) \circ TF$.*

Example 6. The functor H^r assigns to a manifold M the fibered manifold H^rM of all r-th order frames on M and to a diffeomorphlism f: $M \to \bar{M}$ the induced map H^rf: $H^rM \to H^r\bar{M}$. Consider the canonical form ΘH^rM: $TH^rM \to \mathbf{R}^n \oplus \mathfrak{L}_n^{r-1}$, $n = \dim M$, $\mathfrak{L}_n^{r-1}$ = the Lie algebra of the structure group L_n^{r-1} of $H^{r-1}M$, [6]. Let I_r be the trivial functor $I_rM = \mathbf{R}^n \oplus \mathfrak{L}_n^{r-1}$, $I_rf = id$. Every manifold M can be interpreted as a trivial principal fiber bundle $M \times \{e\}$ with a one-element structure group $\{e\}$, which determines an injection $i: \mathcal{M} \to \mathcal{PB}$. Since $H^r = W^r \circ i$, see [8], we deduce by Corollary 3

Corollary 4. *ΘH^rM are structure morphisms of H^r with respect to (I_r, T), i.e. an $\mathcal{FM}_0$-morphism F: $H^rM \to H^r\bar{M}$ over f: $M \to \bar{M}$ is of the form $F = H^rf$ iff $\Theta H^rM = (\Theta H^r\bar{M}) \circ TF$.*

This is, in fact, the classical result by Guillemin and Sternberg, [6] (our assumptions are somewhat modified).

Example 7. We shall study the tangent functor $T: \mathcal{M} \to \mathcal{FM}$. We first recall that, given a manifold M, there are two natural projections $TTM \to TM$, namely the bundle projection π_1: $TTM \to TM$ and the tangent map $T\pi$: $TTM \to TM$ of the bundle projection π: $TM \to M$. Duc introduces the canonical form ΘTM: $TTM \to VTM$, [2], that maps a vector $A \in TTM$ into the vertical translation (defined in Example 3a)) of the vector $T\pi(A)$ into the point π_1A. By a simple evaluation in local coordinates we deduce that an $\mathcal{FM}$-morphism F: $TM \to T\bar{M}$ over f: $M \to \bar{M}$ is of the form $F = Tf$ iff $(VTf) \circ (\Theta TM) = (\Theta T\bar{M}) \circ TF$. (Theorem 10 on p. 378 in [2] states one part of this equivalence only.) In other words, ΘTM are structure morphisms of T with respect to (VT, T).

We give another simple construction of structure morphisms of T. Define $\bar{\Theta}TM$: $VTM \to TM$, $A \mapsto vA - \tau A$, where v: $VTM \to TM$ is the bundle projection and τ was defined in Example 3a. In local coordinates, one verifies easily that $\bar{\Theta}TM$ are structure morphisms of T with respect to (T, V).

Example 8. We find it worthwhile to present now an example of canonical mappings that are not structure morphisms. Consider the functor T^r that assigns to a manifold M the vector bundle π: $T^rM \to M$ of all the tangent vectors of order r on M and to a mapping f: $M \to \bar{M}$ the induced map T^rf: $T^rM \to T^r\bar{M}$. Using the well-known inclusion $TM \subset T^rM$, Duc constructs a canonical form $\varkappa$: $TT^rM \to VT^rM$, [2], p. 404, as follows. If $A \in TT^rM$, then $T\pi(A) \in TM \subset T^rM$ and $\varkappa A$ means the vertical translation of the latter vector to the point $\pi A \in T^rM$. These forms are not structure morphisms of T^r. However, we can define similarly a mapping ΘT^rM: $T^rT^rM \to VT^rM$ transforming a vector $A \in T^rT^rM$ into the vertical translation of $T^r\pi(A) \in T^rM$ to the point πA. Using local coordinates, we deduce that an $\mathcal{FM}$-morphism F: $T^rM \to T^r\bar{M}$ over f: $M \to \bar{M}$ is of the form $F = T^rf$ iff $(VT^rf) \circ (\Theta T^rM) = (\Theta T^r\bar{M}) \circ T^rF$, i.e. ΘT^rM are structure morphisms of T^r with respect to (VT^r, T^r).

Example 9. Another general type of a prolongation functor is the functor T^r_k: $\mathcal{M} \to \mathcal{FM}$ of k-dimensional velocities of the order r. We shall construct structure morphisms of T^2_1. Consider the jet projection π_1: $T^2_1M \to T^1_1M = TM$, the bundle projection π: $TT^2_1M \to T^2_1M$ and the injection i: $T^2_1M \to TTM$, $j^2_0\gamma \mapsto j^1_0(j^1_t\gamma)$, where γ is a curve on M. Define ΘT^2_1M: $TT^2_1M \to TTM$ by $A \mapsto T\pi_1(A) - i(\pi A)$. A simple evaluation shows that an $\mathcal{FM}$-morphism F: $T^2_1M \to T^2_1\bar{M}$ over f: $M \to \bar{M}$ is of the form $F = T^2_1f$ iff $(TTf) \circ (\Theta T^2_1M) = (\Theta T^2_1\bar{M}) \circ TF$, i.e. ΘT^2_1M are structure morphisms of T^2_1 with respect to (TT, T).

Example 10. Finally, we construct structure morphisms for the prolongations of differentiable groupoids in the sense of Ehresmann, [3]. To make some considerations more clear, we first introduce certain auxiliary categories. A double fibered manifold (an object of $\mathcal{F}2\mathcal{M}$) is a quadruple Y, $X = BX$, a, b such that $Y_a := Y \xrightarrow{a} X$ and $Y_b := Y \xrightarrow{b} X$ are fibered manifolds. If $\bar{Y}$, $\bar{X}$, $\bar{a}$, $\bar{b}$ is another double fibered manifold, then an $\mathcal{F}2\mathcal{M}$-morphism is a pair (f, Bf): $(Y, X) \to (\bar{Y}, \bar{X})$ satisfying $\bar{a} \circ f = Bf \circ a$ and $\bar{b} \circ f = Bf \circ b$. We have a base functor B: $\mathcal{F}2\mathcal{M} \to \mathcal{M}$ and we define $\mathcal{F}2\mathcal{M}_0 = B^{-1}(\mathcal{M}_0)$. Moreover, we introduce another category $2\mathcal{F}2\mathcal{M}$ whose objects are sixtuples $(Z\ Y, X, \pi, a, b)$ such that π: $Z \to Y$ is a surjective submersion and $(Y, X, a, b) \in Ob\mathcal{F}2\mathcal{M}$. In particular, $(Z, X, a\pi, b\pi)$ is also an object of $\mathcal{F}2\mathcal{M}$. Similarly as in Example 2, we write $Y = MZ$, $X = BZ$. $2\mathcal{F}2\mathcal{M}$-morphisms are triples (f, Mf, Bf): $(Z, Y, X) \to (\bar{Z}, \bar{Y}, \bar{X})$ commuting with the corresponding projections on both objects. We have two functors M: $2\mathcal{F}2\mathcal{M} \to \mathcal{F}2\mathcal{M}$, B: $2\mathcal{F}2\mathcal{M} \to \mathcal{M}$ and we define $2\mathcal{F}2\mathcal{M}_0 = B^{-1}(\mathcal{M}_0)$.

Let $\mathcal{DG}$ be the category of differentiable groupoids and their homomorphisms, [3], so that $\mathcal{DG}$ is a subcategory of $\mathcal{F}2\mathcal{M}$. Set $\mathcal{DG}_0 = \mathcal{DG} \cap \mathcal{F}2\mathcal{M}_0$. Denote by E^r the Ehresmann functor that assigns to a differentiable groupoid G over x its r-th prolongation E^rG, which is another groupoid over X, and to every $\mathcal{DG}_0$-morphism f: $G \to \bar{G}$ the induced map E^rf: $E^rG \to E^r\bar{G}$. We recall that E^rG is an open submanifold of J^rG_a of all elements A such that bA is an invertible r-jet of X into X and E^rf is the restriction of J^rf: $J^rG_a \to J^r\bar{G}_{\bar{a}}$ to E^rG. We first construct structure morphisms of E^1.

The Lie algebroid $LG \to X$ of a differentiable groupoid G is the vector bundle of all a-vertical tangent vectors at the units of G, [12], [14]. Every $f \in \mathcal{DG}$, $f: G \to \bar{G}$ induces a $\mathcal{VB}$-morphism $Lf: LG \to L\bar{G}$, so that L is a functor $\mathcal{DG} \to \mathcal{VB}$. Consider the structure form $\Theta J^1 G_a$: $TJ^1 G_a \to VG_a$ of Example 2. Every $g \in G$, $ag = x$, $bg = y$, determines a mapping R_g: $(G_a)_y \to (G_a)_x$, $\bar{g} \mapsto \bar{g} \cdot g$, the differential of which maps $V_{g^{-1}} G_a$ into $V_{e_x} G_a = (LG)_x$, e_x being the unit over x. In this way, $\Theta J^1 G_a$ is transformed into a mapping $TJ^1 G_a \to LG$, whose restriction to $E^1 G$ will be called the canonical form of $E^1 G$ and will be denoted by $\Theta E^1 G$: $TE^1 G \to LG$. As a direct consequence of Proposition 1, we obtain

Proposition 4. *$\Theta E^1 G$ are structure morphisms of E^1 with respect to (L, T), i.e. a $2\mathcal{F}2\mathcal{M}_0$-morphism F: $E^1 G \to E^1 \bar{G}$ over a $\mathcal{DG}_0$-morphism f: $G \to \bar{G}$ is of the form $F = E^1 f$ iff $(Lf) \circ (\Theta E^1 G) = (\Theta E^1 \bar{G}) \circ TF$.*

In higher orders, we use the inclusion $E^r G \subset E^1(E^{r-1} G)$, which gives a canonical form $\Theta E^r G$: $TE^r G \to L(E^{r-1} G)$. Similarly as in Example 4 we deduce by Proposition 4 that $\Theta E^r G$ are structure morphisms of E^r with respect to (LE^{r-1}, T).

We remark that there is another approach to structure morphisms of E^r developped by Bossard for the special case of the groupoid $\Pi^r M$ of all invertible r-jets of a manifold M into itself, [1]. Denote by $T_0 G \to X$ the vector bundle of all tangent vectors of G at the units and extend T_0 naturally into a functor of $\mathcal{DG}$ into $\mathcal{VB}$. Every $U \in E^1 G$ over $g = \beta U \in G$ determines a linear isomorphism $\tilde{U}$: $T_{e_x} G \to T_g G$, $x = ag$. Then we define $\bar{\Theta} E^1 G$: $TE^1 G \to T_0 G$, $A \mapsto \tilde{U}^{-1}(T\beta(A))$. Analogously to Bossard, [1], one deduces that $\bar{\Theta} E^1 G$ are structure morphisms of E^1 with respect to (T_0, T). In higher orders we define similarly $\bar{\Theta} E^r G$: $TE^r G \to T_0 E^{r-1} G$, which are structure morphisms of E^r with respect to $(T_0 E^{r-1}, T)$. It will be interesting to discuss the difference between the use of each of those forms in further investigations.

2. Prolongations of vector fields

The prolongations of vector fields with respect to certain functors are studied in [5], [9], [10], [16]. The prolongation procedure for vector fields is based on the use of flows. As flows are defined locally, we have to postulate some additional requirements in order to get a general theory.

Assume that p is a functor of subcategory $\mathcal{C} \subset \mathcal{M}$ into a subcategory $\mathcal{D} \subset \mathcal{FM}$ and the back projection is the base functor $\mathcal{FM} \to \mathcal{M}$. We denote by π_M: $pM \to M$ the bundle projection of fibered manifold pM. We shall say that p is a prolongation functor, if it satisfies the following locality and regularity conditions.

Locality condition. a) *If $M_1, M_2 \in Ob\mathcal{C}$ and U is an open subset in both M_1, M_2, then $\pi_{M_1}^{-1}(U) = \pi_{M_2}^{-1}(U)$ is the same fibered manifold, which will be denoted by pU.*

b) *If $f \in \mathcal{C}(M_1, N_1)$ and $g \in \mathcal{C}(M_2, N_2)$ satisfy $f|U = g|U$ and if there is an open* *subset V in both N_1, N_2 such that $f(U) \subset V$, then the restriction of both pf and pg to pU is the same mapping of pU into pV.*

A mapping f of a manifold U into a manifold V will be called a local $\mathscr{C}$-morphism if there are $M, N \in Ob\mathscr{C}$ and $\bar{f} \in \mathscr{C}(M, N)$ such that U or V is an open subset of M or N, respectively, and $\bar{f}|U = f$. By the locality condition, $pf := p\bar{f}|pU$ is a well-defined local $\mathscr{D}$-morphism of pU into pV.

Regularity condition (cf. [13]). *If M, $\bar{M}$, N are manifolds and $f: M \times N \to \bar{M}$ is a differentiable mapping such that $f(-, x): M \to \bar{M}$ is a local $\mathscr{C}$-morphism for all $x \in N$, then the mapping $pf: pM \times N \to p\bar{M}$ defined by $(pf)(-, x) = p(f(-, x))$ is also differentiable.*

A vector field on $M \in Ob\mathscr{C}$ will be called a $\mathscr{C}$-field if its flow is formed by local $\mathscr{C}$-morphisms. For example, a vector field on a fibered manifold is an $\mathscr{FM}$-field iff it is projectable. A vector field on a principal fiber bundle is a $\mathscr{PB}$-field iff it is right-invariant. Using flows, we deduce that, if ξ, η are $\mathscr{C}$-field on M and $k \in \mathbf{R}$, then $\xi + \eta$, $k\xi$ and $[\xi, \eta]$ are also $\mathscr{C}$-fields.

If ξ is a $\mathscr{C}$-field on M, then its flow ξ_t is prolonged into a flow $p\xi_t$ on pM that determines a $\mathscr{D}$-field $p\xi$ on pM called the prolongation of ξ with respect to p. From the geometric properties of flows, we deduce

Proposition 5. *a) For any $\mathscr{C}$-fields ξ, η on M and any $k \in \mathbf{R}$, we have*

$$p(\xi + \eta) = p\xi + p\eta, \quad p(k\xi) = kp\xi, \quad p([\xi, \eta]) = [p\xi, p\eta].$$

b) If $\mathscr{C}$-fields ξ on M and η on N are f-related, $f \in \mathscr{C}(M, N)$, then the $\mathscr{D}$-fields $p\xi$ on pM and $p\eta$ on pN are pf-related.

Consider now a $\mathscr{D}$-field η on pM, so that η is a projectable vector field on fibered manifold pM over a $\mathscr{C}$-field ξ on M. A natural question is: under what condition $\eta = p\xi$? To answer this question we can use structure morphisms of p. Assume that φ and ψ are prolongation functors and that $\Theta pM: \psi(pM) \to \varphi M$ is an $\mathscr{FM}$-morphism over $\pi_M: pM \to M$. Assume further that the structure morphisms have the following

Localization property. *If U_i is an open subset of $M_i \in Ob\mathscr{C}$, V_i is an open subset of pM_i over U_i and $F: V_1 \to V_2$ is a local $\mathscr{D}$-morphism over a local $\mathscr{C}$-morphism $f: U_1 \to U_2$, then the commutativity of the diagram*

$$\begin{array}{ccc} \varphi U_1 & \xrightarrow{\varphi f} & \varphi U_2 \\ {\scriptstyle \Theta V_1}\uparrow & & \uparrow{\scriptstyle \Theta V_2} \\ \psi V_1 & \xrightarrow{\psi F} & \psi V_2 \end{array}$$

implies $F = pf|V_1$, provided ΘV_i means the restriction of ΘpM_i to ψV_i, $i = 1, 2$.

We now recall the general concept of the Lie derivative $\mathscr{L}_{(\zeta_1, \zeta_2)}f$ of an arbitrary differentiable mapping $f\colon Q_1 \to Q_2$ with respect to a vector field ζ_1 on Q_1 and a vector field ζ_2 on Q_2, [17]. This is a mapping $Q_1 \to TQ_2$ formally determined by

(3) $$(\mathcal{L}_{(\zeta_1, \zeta_2)}f)(x) = Tf(\zeta_1(x)) - \zeta_2(f(x)).$$

In particular, ζ_1 and ζ_2 are f-related iff $\mathcal{L}_{(\zeta_1, \zeta_2)}f = 0$. In the special case of a vector space V and a V-valued 1-form on a manifold Q, $\omega\colon TQ \to V$, the standard Lie derivative $\mathcal{L}_\zeta \omega$ of ω with respect to a vector field ζ on Q is related to the above general concept by

(4) $$\mathcal{L}_\zeta \omega = \mathcal{L}_{(T\zeta, 0_V)}\omega,$$

provided 0_V means the zero vector field on V.

Proposition 6. *If η is a $\mathcal{D}$-field on pM over a $\mathcal{C}$-field ξ on M, then $\eta = p\xi$ iff*

(5) $$\mathcal{L}_{(\psi\eta, \varphi\xi)}\Theta pM = 0.$$

Proof. Since η is a projectable vector field over ξ, there exist, for every $y \in pM$, a neighbourhood V of y, an interval $I = (-\varepsilon, \varepsilon) \subset \mathbf{R}$ and a local flow $\eta_t\colon V \times I \to pM$ over a local flow $\xi_t\colon U \times I \to M$. Those local flows are prolonged into local flows $\psi\eta_t\colon \psi(V) \times I \to \psi pM$ and $\varphi\xi_t\colon \varphi(U) \times I \to \varphi M$. If $\mathcal{L}_{(\psi\eta, \varphi\xi)}\Theta pM = 0$, the latter flows are ΘpM-related, i.e. for every $t \in I$, we have a commutative diagram

$$\begin{array}{ccc} \varphi U & \xrightarrow{\ \varphi\xi_t\ } & \varphi U_t \\ {\scriptstyle \Theta V}\uparrow & & \uparrow{\scriptstyle \Theta V_t} \\ \psi V & \xrightarrow{\ \psi\eta_t\ } & \psi V_t \end{array}$$

where $U_t = \xi_t(U)$, $V_t = \eta_t(V)$. By the localization property, we obtain $\eta_t = p\xi_t | V$, which implies $\eta = p\xi$. The converse assertion is trivial, QED.

Proposition 6 can be applied directly to our Examples 2, 3a, 4, 7, 8, 9. For instance, for the functor J^1 of Example 2, we obtain

Corollary 5. *A vector field η on J^1Y over a projectable vector field ξ on Y is of the form $\eta = J^1\xi$ iff*

(6) $$\mathcal{L}_{(T_\eta, V\xi)}\Theta J^1 Y = 0.$$

This property of the prolonged vector fields on J^1Y is close to the approach of García to the prolongation of a vector field on Y to J^1Y, [4]. As $\Theta J^1 Y$ has simple coordinate expression, (6) can be also used for finding the coordinate expression of the field $J^1\xi$. The situation of Example 4 is quite similar.

In Examples 1, 3b, 5, 6, 10, the functor φ is not a prolongation functor. Nevertheless, a formula like (5) can be deduced for each of those examples by a simple additional consideration. In Example 1, construct another functor $\tilde{\varphi}$

transforming M into $\mathbf{R}^n \times M$ and $f\colon M \to \bar{M}$ into $id \times f\colon \mathbf{R}^n \times M \to \mathbf{R}^n \times \bar{M}$. The canonical form $\Theta H^1M\colon TH^1M \to \mathbf{R}^n$ can be naturally modified into a mapping $\tilde{\Theta}H^1M\colon TH^1M \to \mathbf{R}^n \times M$. Even $\tilde{\Theta}H^1M$ are structure morphisms of H^1 and the assumptions of Proposition 6 are now satisfied. One finds easily that, for any projectable vector field η on H^1M over ξ on M, $\mathscr{L}_{(T_\eta, \varphi\xi)}\tilde{\Theta}H^1M = 0$ iff $\mathscr{L}_{(T_\eta, 0_{\mathbf{R}^n})}\Theta H^1M = 0$. By (4), we deduce

Corollary 6. *A projectable vector field η on H^1M over a vector field ξ on M is of the form $\eta = H^1\xi$ iff $\mathscr{L}_\eta(\Theta H^1M) = 0$.*

This corollary clarifies some further aspects of an assertion by Kobayashi and Nomizu, [7], p. 229.

In Examples 3b, 5 and 6, we obtain similarly the following results.

Corollary 7. *A projectable vector field η on J^1P over a right-invariant vector field ξ on P is of the form $\eta = J^1\xi$ iff $\mathscr{L}_\eta(\tilde{\Theta}J^1P) = 0$.*

Corollary 8. *A projectable vector field η on W^rP over a right-invariant vector field ξ on P is of the form $\eta = W^r\xi$ iff $\mathscr{L}_\eta(\Theta W^rP) = 0$.*

Corollary 9. *A projectable vector field η on H^rM over a vector field ξ on M is of the form $\eta = H^r\xi$ iff $\mathscr{L}_\eta(\Theta H^rM) = 0$.*

In Example 10, if ξ is a $\mathscr{D}\mathscr{G}$-field on G, then ξ is a-projectable and determines a prolonged field $V\xi$ on VG_a. This field is tangent to $LG \subset VG_a$ and its restriction to LG will be denoted by $L\xi$. Considering the projection $b\colon G \to X$, we can construct the pull-back $b^{-1}LG$. This is a vector bundle over G naturally isomorphic to VG_a. The canonical form $\Theta E^1G\colon TE^1G \to LG$ was derived from a mapping $\tilde{\Theta}E^1G\colon TE^1G \to VG_a$ and these mappings are also structure morphisms of E^1. By Corollary 6, a projectable vector field η on E^1G over a $\mathscr{D}\mathscr{G}$-field ξ on G is of the form $\eta = E^1\xi$ iff $\mathscr{L}_{(T\eta, V\xi)}(\tilde{\Theta}E^1G) = 0$. Using standard pull-back manipulations, we deduce that the latter equation holds iff $\mathscr{L}_{(T\eta, L\xi)}\Theta E^1G = 0$. Hence we obtain

Corollary 10. *In the above situation, $\eta = E^1\xi$ iff $\mathscr{L}_{(T\eta, L\xi)}(\Theta E^1G) = 0$.*

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СТРУКТУРНЫЕ МОРФИЗМЫ ПРОДОЛЖАЮЩИХ ФУНКТОРОВ

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Резюме

Структурные морфизмы любого продолжающего функтора получаются как обобщение нескольких построений дифференциальной геометрии. Они дают общий критерий для того, чтобы любой морфизм между продолжениями двух объектов был продолжением нижележащего морфизма. Находятся основные свойства операции продолжения векторных полей по общему функтору и показывается, что аналогичное соотношение между векторными полями имеет место тогда и только тогда, когда производная Ли структурного морфизма обращается в нуль.