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A CLASS OF POLYNOMIALS

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ABSTRACT. We characterize the polynomials $\varphi(x) \in \mathbf{Z}[x]$ such that for any $f(x) \in \mathbf{Z}[x]$ from inclusion $\{f(a);\ a = k, k+1, \dots\} \subset \{\varphi(b);\ b = 0, \pm 1, \pm 2, \dots\}$ follows $f(x) = \varphi(h(x))$ for some $h(x) \in \mathbf{Z}[x]$.

Call a polynomial $\varphi(x)$ *good* if it has the following property:

For every polynomial $f(x) \in \mathbb{Z}[x]$ such that for every sufficiently large integer $a \in \mathbb{Z}$ there is $b \in \mathbb{Z}$ such that $f(a) = \varphi(b)$ there is a polynomial $h(x) \in \mathbb{Z}[x]$ such that $f(x) = \varphi(h(x))$.

I. Korec suggested to study good polynomials in connection with his results concerning palindromic squares in [1].

In this note we prove the following criterion:

Theorem. *A polynomial $\varphi \in \mathbb{Z}[x]$ is good if and only if $\varphi\left(\frac{x}{m}\right) \notin \mathbb{Z}[x]$ for all $m > 1$.*

To prove this result we need the

Lemma. *Let for a polynomial F with algebraic coefficients $C(F)$ denote the content of F, i.e. the ideal generated by the coefficients of F. If $p \in \mathbb{Z}[x]$, $q \in \mathbf{Q}[x]$ and $p(0) = 0$, then*

$$C(q(p)) \mid C(q)C(p)^{\deg q}.$$

Proof. We have

$$q(x) = q_0 \prod_{i=1}^{\deg q} (x - \varrho_i),$$

and by the generalized Gauss lemma

$$C(q) = (q_0) \prod_{i=1}^{\deg q} C(x - \varrho_i) = (q_0) \prod_{i=1}^{\deg q} (1, \varrho_i).$$

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Similarly

$$C(q(p)) = (q_0) \prod_{i=1}^{\deg q} C(p(x) - \varrho_i) = (q_0) \prod_{i=1}^{\deg q} (C(p), \varrho_i),$$

and since $C(p)$ is integral, the lemma follows. ■

Proof of the Theorem. We shall prove first that the condition is necessary. If for an $m > 1$ $\varphi\left(\frac{x}{m}\right) \in \mathbf{Z}[x]$, we have

$$f(x) = \varphi\left((m-1)!\binom{x}{m}\right) \in \mathbf{Z}[x].$$

Also for every $x^* \in \mathbf{Z}$ there exists a $y^* \in \mathbf{Z}$ such that

$$f(x^*) = \varphi(y^*).$$

If, however, we had $f(x) = \varphi(g(x))$, $g \in \mathbf{Z}[x]$, it would follow that

$$\varphi\left((m-1)!\binom{x}{m}\right) = \varphi(g(x)),$$

which gives a contradiction, since the leading coefficient of the left hand side is smaller than the leading coefficient of the right hand side.

In order to prove that the condition is sufficient, let a be the leading coefficient of φ and assume, that for an $f \in \mathbf{Z}[x]$ we have $f(x^*) = \varphi(y^*)$ for every $x^* \in \mathbf{Z}$, $x \geq K$ and a suitable $y^* \in \mathbf{Z}$. Let

$$\varphi(y) - f(x) = \prod_{i=1}^{n} F_i(x, y), \tag{1}$$

where the polynomials $F_i \in \mathbf{Z}[x, y]$ are irreducible and F_i viewed as a polynomial in y has the leading coefficient $a_i(x)$. Clearly

$$a = \prod_{i=1}^{n} a_i(x),$$

hence $a_i(x) \in \mathbf{Z}$ for all $i \leq n$. Without loss of generality we may assume that

$$F_i(y) = a_i y - h_i(x) \qquad \text{for} \quad i \leq m,$$
$$\deg_y F_i > 1 \qquad \text{for} \quad i > m.$$

By Hilbert's irreducibility theorem there exists an integer t^* such that $at^* \geq K$, $F_i(at^*, y)$ is irreducible for all $i > m$ and hence

$$F_i(at^*, y) = 0$$

has no rational root. Since by the assumption

$$\varphi(y^*) - f(at^*) = 0 \qquad \text{for a} \quad y^* \in \mathbb{Z},$$

by (1) there is a $j \leq m$ such that

$$F_j(at^*, y^*) = 0,$$

which gives

$$a_j y^* - h_j(at^*) = 0,$$

and since $a_j | a$

$$h_j(0) \equiv h_j(at^*) \equiv 0 \pmod{a_j}. \tag{2}$$

Let

$$C(h_j(x) - h_j(0)) = (c),$$

and take in the lemma

$$p(x) = \frac{h_j(x) - h_j(0)}{(c, a_j)}, \qquad q(x) = \varphi\left(\frac{x}{a_j/(c, a_j)} + \frac{h_j(0)}{a_j}\right).$$

We obtain

$$C(q(p)) \mid C(q)C(p)^{\deg q} = C(q) \cdot \left(\frac{c}{(c, a_j)}\right)^{\deg q},$$

and since by (1) $q(p) = f \in \mathbb{Z}[x]$

$$C(q) \cdot \left(\frac{c}{(c, a_j)}\right)^{\deg q} \subset \mathbb{Z}.$$

However by (2)

$$C(q) \cdot \left(\frac{a_j}{(c, a_j)}\right)^{\deg q} \subset \mathbb{Z},$$

and since

$$\left(\frac{c}{(c, a_j)}, \frac{a_j}{(c, a_j)}\right) = 1$$

the two inclusion give

$$C(q) \subset \mathbb{Z};$$

$$q \in \mathbb{Z}[x], \qquad \varphi\left(\frac{x}{a_j/(c,a_j)}\right) = q(x - h_j(0)(c,a_j)) \in \mathbb{Z}[x] .$$

By the condition on φ:

$$|a_j|/(c,a_j) = 1 ,$$

hence $a_j | c$ and by (2)

$$\frac{h_j(x)}{a_j} = \frac{h_j(x) - h_j(0)}{a_j} + \frac{h_j(0)}{a_j} \in \mathbb{Z}[x] .$$

Since by (1)

$$f(x) = \varphi\left(\frac{h_j(x)}{a_j}\right) ,$$

the proof is complete. ■

REFERENCES

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