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Pavol Híc

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A CHARACTERIZATION OF $K_{r,s}$ — CLOSED GRAPHS

PAVOL HÍC

1. Introduction

A non-empty graph G containing a subraph H without isolated vertices is said to be H-closed if, whenever F is a subgraph of G without isolated vertices that is isomorphic to a subgraph of H, then F can be extended to a subgraph of G isomorphic to H.

H-closed graphs were introduced by Tomasta and Tomová [8], where also a characterization of H-closed graphs for H to be connected regular of degree $r \geq 2$ and H to be a cycle with one special chord, the so-called triangle chord, as well as H to be a cycle with two special triangle chords is given.

A characterization of H — closed graphs for H to be a star and H to be a cycle was given by Chartrand, Oelerman and Ruiz [2], but in terms of randomly H graphs.

A characterization of H-closed graphs for H to be a matching was given by Sumner [7]. Analogical questions were studied, for example, in [1, 3, 5, 6].

We prefer the term H-closed graph instead of the term randomly H graph.

In this paper is given a characterization of $K_{r,s}$ — closed graphs for arbitrary finite r, s.

2. Notations and preliminary results

We use the general notation and terminology of Harary [4].

In order to avoid a situtation where only a complete graph would be H-closed, require in the definition of H-closed graphs that H and F be without isolated vertices (see also [2]).

So all the graphs considered in this paper are simple undirected without isolated vertices. The distance between the vertices $u, v \in V(G)$ is denoted by $\varrho(u, v)$. Let H be a subgraph of G and $v \in V(G) - V(H)$, then $\varrho(v, H) = \min\{\varrho(v, u) | u \in V(H)\}$.

The family of all H-closed graphs will be denoted by $\sigma(H)$ and the family n-vertex H-closed graphs by $\sigma_n(H)$.

Obviously, every graph G is K_2-closed and also every graph G is G-closed. Further, K_n is H-closed for every $H \subset K_n$.

Lemma A (see Tomasta and Tomová [8, Lemma 1]).

(i) *If* $G \in \sigma(H)$, *then* $\sigma(G) \subset \sigma(H)$.

(ii) *If* $G \in \sigma_n(H)$, *then* $\sigma_n(G) \subset \sigma_n(H)$.

Lemma B (see Tomasta and Tomová [8, Proposition 1]). *Closeness Criterion*: *G is H — closed if and oly if for every minimal system $S = \{x_{e_1}, x_{e_2} \ldots, x_{e_k}\}$ of boolean variables for which the boolean expression*

$$W = \prod_{H \subset G} \sum_{e \in E(G) - E(H)} x_e$$

is true, $F_S \nsubseteq H$, where F_S is the graph consisting of the edges $\{e_1, e_2, \ldots e_k\}$ corresponding to the variables in S.

Lemma C (see Tomasta and Tomová [8, Lemma 2]).

Let H be a connected graph on at least four vertices different from a star. Then the H-closed graph is connected.

Lemma 1. *Let G be a $K_{r,s}$-closed graph and $|V(G)| > r + s$, $r \geq 2$, $s \geq 2$. Let $H \subseteq G$ and $H \cong K_{r,s}$. Then, for an arbitrary vertex $v \in V(G) - V(H)$, $\varrho(v, H) = 1$.*

Proof. Let v be any vertex of $V(G) - V(H)$. The existence of such a vertex is ensured because of $|V(G)| > r + s$. From $H \cong K_{r,s}$ it follows that $V(H) = A \cup B$, $|A| = r$, $|B| = s$. Let $A = \{v_1, v_2, \ldots, v_r\}$, $B = \{u_1, u_2, \ldots, u_s\}$. Let $P = [x_0, x_1, \ldots, x_t]$ be any shortest (v, H) — path of G and let t denote the length of P. We can suppose that $x_0 = u_1$, $x_t = v$. Such a path eists because G is connected by Lemma C.

Form the graph H' as follows: delete from H the vertex v_1 and add the adge $(x_1 u_1)$. Obviously, H' is a subgraph of $K_{r,s}$ and thus it can be extended to $K_{r,s}$ in G. However, the only possibility to extend H' to $K_{r,s}$ is the adding of edges (x_1, u_i) for every $i = 2, 3, \ldots, s$. Now, we have the graph F with the following properties:

1. $H' \subset F \subset G$.
2. $F \cong K_{r,s}$.
3. $V(F) = A_F \cup B_F$, $A_F = \{x_1, v_2, \ldots, v_r\}$, $B_F = \{u_1, u_2, \ldots, u_s\}$. Similarly, we can form the graph F' as follows: delete from F the vertex u_1 and add the edge (x_2, x_1). F' is a subgraph $K_{r,s}$ and the only possibility to extend it to $K_{r,s}$ is the adding of edges (x_2, v_i) for $i = 2, 3, \ldots, r$. Hence, $\varrho(x_t, v_i) = \varrho(x_t, H) = t - 1$, which is a contradiction to the assumption.

Q.E.D.

Lemma 2. *Let G be a $K_{r,s}$-closed graph and $|V(G)| \geq r + s$, $r \geq 2$, $s \geq 2$. Then $V(G) = A \cup B$, $|A| \geq r$, $|B| \geq s$ and every vertex of A is joined to every vertex of B.*

Proof. It is obvious if $|V(G)| = r + s$. Now, let $|V(G)| > r + s$ and $H \cong K_{r,s}$, $H \subset G$. $V(H) = A' \cup B'$. $A' = \{v_1, v_2, ..., v_r\}$, $B' = \{u_1, u_2, ..., u_s\}$. Lemma 1 implies $\varrho(v, H) = 1$ for an arbitrary vertex $v \in V(G) - V(H)$. Now, v can be tabled to $A \supseteq A'$ (if v is joined to every u_i) or to $B \supseteq B'$ (if v is joined to every v_i).

The assumption that G is $K_{r,s}$-closed implies the joining of any vertex of A to any vertex of B.

Lemma 3. *Let G be a $K_{r,s}$-closed graph and $V(G) = A \cup B$ (by Lemma 2). If there exists an adge in A [B], then G is a complete graph.*

Proof. By assumption $V(G) = A \cup B$. Let $A = \{v_1, v_2, ..., v_{r'}\}$. $B = \{u_1, u_2, ..., u_{s'}\}$, $r' \geq r$, $s' \geq s$. Let e be any edge joining two vertices of A, for example $e = (v_1 v_2)$. Then the subgraph $G' \subseteq G$ which is given in Fig. 1 can be extended to $K_{r,s}$ only by adding exactly all edges from u_1 to u_i, $i = 2, 3, ..., s$ and from v_1 to v_i, $i = 3, 4, ..., r$. As G was chosen arbitrarily G is a complete graph.

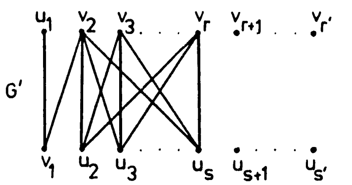

Fig. 1

Lemma 4. *The graph $K_{2,n+1}$ is $K_{2,n}$-closed for any $n \in N$.*

Proof. Apply the Lemma B. The graph $K_{2,n+1}$ contains exactly $n+1$ graphs isomorphic to $K_{2,n}$. Thus the boolean expression W has the form:

$$W = (e_1 \vee f_1) \wedge (e_2 \vee f_2) \wedge \dots \wedge (e_{n+1} \vee f_{n+1}) =$$

$$= \bigvee_{L \subset \{1,2,\dots,n+1\}} \left[\left(\bigwedge_{i \in L} e_i \right) \wedge \left(\bigwedge_{j \in \{1,2,\dots,n+1\} - L} f_j \right) \right] \tag{1}$$

Now, let S be a minimal system for which W is true. From (1) it follows that the corresponding graph F_s contains $n+1$ vertices which belong to the comon part of $K_{2,n+1}$. It implies that $F_s \not\subseteq K_{2,n}$. Hence, by Lemma B, $K_{2,n+1}$ is $K_{2,n}$-closed.

Q.E.D.

3. Main results

Theorem 1. *A graph G is $K_{1,s}$-closed if and only if*

(i) $s = 1$ *and G is an arbitrary graph,*

(ii) $s = 2$ *and* G *is a graph in which no component is isomorphic to* K_2,
(iii) $s \geq 3$ *and no component of* G *is isomorphic to* K_2 *and every vertex of* G *has degree* 1 *or at least* s.

The proof (i) is obvious and for the proofs of (ii) and (iii) see [2, propositions 1 and 2].

Teorem 2. *A graph* G *is* $K_{2,s}$*-closed if and only if*

(i) $s = 2$ *and* $G = K_p$ *with* $p \geq 4$ *or* $G = K_{m,n}$ *with* $2 \leq m \leq n$,
(ii) $s = 3$ *and* $G = K_p$ *with* $p \geq 5$ *or* $G = K_{m,n}$ *with* $m \geq 2$, $n \geq 3$,
(iii) $s \geq 4$ *and* $G = K_p$ *with* $p \geq s + 2$ *or* $G = K_{2,n}$ *with* $n \geq s$.

Proof.
(i) [2, Proposition 4 (ii)],
(ii) All subgraphs of $K_{2,3}$ are given in Fig. 2.

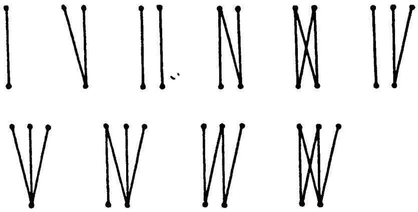

Fig. 2

It is easy to verify that K_p, $p \geq 5$ or $K_{r,s}$, $r \geq 2$, $s \geq 3$ are $K_{2,3}$-closed. Now, let us assume conversely that G is $K_{2,3}$-closed.
By Lemmas 1 and 2 $V(G) = A \cup B$ and every vertex of A is joined to every vertex of B. Hence, $G = K_{r,s}$. If there exists an edge between the vertices of A or B, then Lemma 3 implies that G is a complete graph.

Q.E.D

(iii) Obviously, $K_p, p \geq s + 2$ is $K_{2,s}$-closed. Now, we give the proof that $K_{2,n}$ is $K_{2,s}$-closed for any $n \geq s$. $K_{2,n+1}$ is $K_{2,n}$-closed for any $n \in N$ by Lemma 4. Now, using Lemma A, for arbitrary $n \geq s$:

$$\sigma(K_{2,n}) \subset \sigma(K_{2,n-1}) \subset \ldots \subset \sigma(K_{2,s+1}) \subset \sigma(K_{2,s}).$$

Hence, $K_{2,n}$ is $K_{2,s}$-closed.

Coversely, we assume that G is $K_{2,s}$-closed.
(1) *For every* $v \in V(G)$, $\deg(v) = 2$ *or* $\deg(v) \geq s$.
Proof of (1). Suppose, on the contrary, that $\deg(v) = r$ with $2 < r < s$. Denote by $\Gamma(v)$ the neighbourhood of the vertex v. Then the subgraph H containing v and $\Gamma(v)$ with edges between v and $\Gamma(v)$ is isomorphic to a subgraph of $K_{2,s}$ and cannot be extended to $K_{2,s}$ in G.
(2) *For any vertices* v, w *of degree two* $\Gamma(v) = \Gamma(w)$.

Proof of (2). If it is not true, then form the subgraph H containing v, w, $\Gamma(v)$ and $x \in \Gamma(w)$ together with the edges between v, $\Gamma(v)$ and the edge (x, w). Obviously, H cannot be extended to $K_{2,s}$ in G.

(3) *If there are vertices of degree two, then there are exactly two vertices of degree at least s.*

Proof of (3). Let v_1, v_2, v_3 be the vertices of degree at least s. From (2) it follows that one of them, say v_3, does not belong to the common neighbourhoods of the vertices of degree two. Now, form the graph H as follows: $V(H) = \{v_1, v_2, v_3, u_1, u_2\}$ when u_1 is a vertex of degree two and $E(H) = \{(v_1, u_1), (v_2, u_1), (v_3, u_2)\}$. It is impossible to extend H to $K_{2,s}$ in G.

(4) *If any vertex of G has a degree at least s, then G is a complete graph.*

Proof of (4). Suppose, on the contrary, that $G \neq K_P$. Then there are vertices $v, w \in V(G)$ and $(v, w) \notin E(G)$. By assumption, we can form the following subgraph H given in Figure 3.

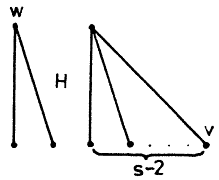

Fig. 3

Obviously, H is a subgraph of $K_{2,s}$ and thus it can be extended to $K_{2,s}$ in G, but it implies the existence of edge (w, v). Hence, G is a complete graph.

Combining (1), (2), (3) and (4) we obtain the statement that $G = K_p$, $p \geq s + 2$ or $G = K_{2,n}$, $n \geq s$. This completes the proof.

Q.E.D.

Theorem 3. *A graph G is $K_{r,s}$-closed with $r \geq 3$, $s \geq 3$ if and only if*

(i) $s = r$ *and* $G = K_{r,r}$ *or* $G = K_p$, $p \geq 2r$,

(ii) $s = r + 1$ *and* $G = K_{r,r+1}$ *or* $G = K_{r+1,r+1}$ *or* $G = K_p$, *with* $p \geq 2r + 1$,

(iii) $s \geq r + 2$ *and* $G = K_{r,s}$ *or* $G = K_p$ *with* $p \geq r + s$.

Proof. (i) It is obvious that $K_{r,r}$ and K_p with $p \geq 2r$ are $K_{r,r}$-closed. Thus we assume conversely that G is $K_{r,r}$-closed. If $|V(G)| = 2r$ and $G \neq K_{r,r}$, then there exists an edge joining vertices of the same part. Hence, G is a complete graph by Lemma 3. Let $|V(G)| > 2r$. Then, G is a complete graph by [8, Theorem 1].

(ii) Obviously, $K_{r,r+1}$ and K_p with $p \geq 2r + 1$ are $K_{r,r+1}$-closed. It is sufficient to prove that $K_{r+1,r+1}$ is $K_{r,r+1}$-closed. We apply the Closeness Criterion. The graph $K_{r+1,r+1}$ contains exactly $2(r + 1)$ graphs isomorphic to $K_{r,r+1}$. Every of them is $K_{r+1,r+1} - v$. Thus the boolean expression W has the form:

$$W = \left\{ \bigwedge_{i=1}^{r+1} \left[\bigvee_{j=1}^{r+1} (v_i, u_j) \right] \right\} \wedge \left\{ \bigwedge_{i=1}^{r+1} \left[\bigvee_{j=1}^{r+1} (u_i, v_j) \right] \right\}.$$

It can be verified that any minimal system S for which W is true contains a set of odges that covers all vertices of $K_{r+1,r+1}$. If, for example, a vertex u_k is not covered, then the expression $(u_k, v_1) \vee (u_k, v_2) \vee \ldots \vee (u_k, v_{r+1})$ is not true and hence W is not true, either. The corresponding graph F_s is not included in $K_{r,r+1}$, thus $K_{r+1,r+1}$ is $K_{r,r+1}$-closed.

Now, we assume conversely that G is $K_{r,r+1}$-closed. We shall consider the following cases:

Case 1. $|V(G)| = 2r + 1$. Then $G = K_{r,r+1}$ or K_{2r+1} because of Lemma 3.

Case 2. $|V(G)| = 2r + 2$. Then there exists a vertex $x \in V(G)$ which does not belong to $H \subset G$, $H \cong K_{r,r+1}$. By Lemma 2 $V(G) = A \cup B$.

We have two subcases:

(a) $|A| = |B| = r + 1$. Then $G = K_{r+1,r+1}$ or the existence of any edge between the vertices of A $[B]$, respectively, implies $G = K_{2r+2}$ by Lemma 3.

(b) $|A| = r$, $|B| = r + 2$. Consider F from Fig. 4.

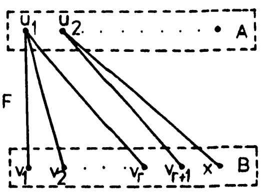

Fig. 4

It can be extended to $K_{r,r+1}$ only by adding the edges (u_1, u_2) and (x, v_i), (v_{r+1}, v_i) for $i = 1, 2, \ldots, r$. By Lemma 3 G is a complete graph.

Case 3. $|V(G)| \geq 2r + 3$. By Lemma 2 $V(G) = A \cup B$. We can always obtain the occurrence of a subgraph F such as in Fig. 4. Hence, G is a complete graph by Lemma 3.

(iii) Obviously $K_{r,s}$ and K_p $p \geq r + s$ are $K_{r,s}$-closed. Conversely, let G be $K_{r,s}$-closed. If $|V(G)| = r + s$, then $G = K_{r,s}$ or K_{r+s} by Lemma 3.

If $|V(G)| > r + s$, then by Lemma 2 $V(G) = A \cup B$. There is always at least one of the following subgraph G_1 $[G_2]$ from Fig. 5 [Fig. 6, respectively] in G. All of them can be extended to $K_{r,s}$ by adding edges in A or B. Hence, G is a complete graph because of Lemma 3. Thus the proof of Theorem is completed.

Q.E.D.

Remark. The graph $K_{2,s}$ $s \geq 2$ has no end vertex and it is not free (see [8]) but there exists no n_0 such that $\sigma_n(K_{2,s}) = K_n$ for every $n > n_0$. This is the answer to the Problem 1 of [8].

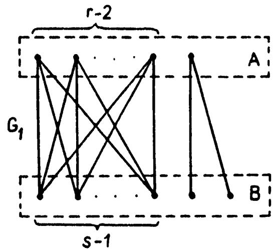

Fig. 5

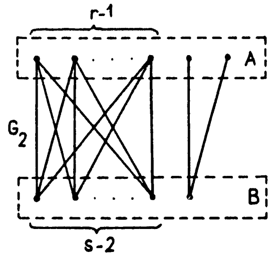

Fig. 6

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Strojnícka fakulta SVŠT
Hviezdoslavova 10
917 24 Trnava

ХАРАКТЕРИЗАЦИЯ $K_{r,s}$ — ЗАМКНУТЫХ ГРАФОВ

Pavol Híc

Резюме

Граф G называется H — замкнутым графом, если всякий подграф F графа G без изолированных вершин, который является изоморфным подграфу графа H, можно разширить на подграф графа G, изоморфный графу H.

Автор дает характеризацию $K_{r,s}$ — замкнутых графов для любых натуральных чисел r, s.