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AN EXISTENCE THEOREM FOR A THIRD–ORDER THREE–POINT BOUNDARY VALUE PROBLEM WITHOUT GROWTH RESTRICTIONS

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ABSTRACT. In the paper there is proved an existence theorem for solutions u of the third-order nonlinear differential equation $u''' = f(t,u,u',u'')$ satisfying $u'(0) = u'(1) = u(\eta) = 0$, $0 \le \eta \le 1$ without growth restrictions on f.

1. Introduction

Rodriguez and Tineo [2] have proved an existence theorem for the Dirichlet problem $u'' = f(t,u,u')$, $u(0) = u(1) = 0$ without requiring growth restrictions on f under the assumption that f is continuous.

In this paper there are found some conditions for the existence of solutions of the third-order boundary value problem (BVP)

$$u''' = f(t,u,u',u''), \tag{1}$$

$$u'(0) = u'(1) = u(\eta) = 0, \quad 0 \le \eta \le 1, \tag{2}$$

where f satisfies only the local Carathéodory conditions on $(0,1) \times \mathbb{R}^3$. This problem models the static deflection of a three-layered elastic beam. Since the method used in this paper is very similar to that used by Rodriguez and Tineo [2], we also do not require any growth restrictions on f.

In [3] there is proved an existence theorem for BVP (1), (2) which requires a growth condition on f only in a neighbourhood of either 0 or 1.

2. Definitions and notations

Let $D' = \big((0,1) \times \mathbb{R}^3\big)$. We say that $f\colon D' \to \mathbb{R}$ satisfies the *local Carathéodory conditions on* $D'\big(f \in \mathrm{Car}_{\mathrm{loc}}(D')\big)$ if $f(\cdot\,,x,y,z)\colon (0,1) \to \mathbb{R}$ is measurable

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on $(0,1)$ for each $x, y, z \in \mathbb{R}$, $f(t,\cdot,\cdot,\cdot)\colon \mathbb{R}^3 \to \mathbb{R}$ is continuous for a.e. $t \in (0,1)$ and $\sup\{|f(t,x,y,z)|\colon |x|+|y|+|z| \le \rho\}$ is Lebesgue integrable for any $\rho \in (0,+\infty)$.

A function $u \in AC^2(0,1)$ satisfying (1) for a.e. $t \in (0,1)$ and fulfilling (2) will be called a solution of BVP (1), (2), where $AC^2(0,1) = \{x\colon\ x$ is a real function with one real argument and x'' is absolutely continuous on $[0,1]\}$.

$X = \{x \in C^2(0,1),\ x'(0) = x'(1) = x(\eta) = 0\}$, where $C^2(0,1) = \{x\colon\ x$ is a real function with one real argument and x'' is continuous$\}$.

In the whole paper we shall assume that $f \in \mathrm{Car}_{\mathrm{loc}}(D')$.

3. The main result

First we state a general existence theorem.

THEOREM 1. *Let $f^* \in \mathrm{Car}_{\mathrm{loc}}\big((0,1)\times\mathbb{R}^3\times(0,1)\big)$ and let there exist an open bounded set $D \subset X$ such that for any $\lambda \in (0,1)$ each solution $u_\lambda \in X$ of the equation*

$$u''' = \lambda f^*(t,u,u',u'',\lambda) \tag{3}$$

satisfies

$$u_\lambda \notin \delta D \qquad (\delta D \text{ is the boundary of } D)$$

and let $0 \in D$.

Then for any $\lambda \in [0,1]$ the equation (3) has at least one solution in $\mathrm{cl}\, D$ ($\mathrm{cl}\, D$ is the closure of D).

P r o o f. The theorem follows from Mawhin's continuation theorem [1, Theorem IV.1, p. 27].

LEMMA 1. *Let $u \in X$ and $c_1 \le u'' \le c_2$ for every $t \in [0,1]$, where $c_1, c_2 \in \mathbb{R}$, $c_1 < 0 < c_2$. Then the inequalities*

$$|u'(t)| < M \quad \text{and} \quad |u(t)| < ML \qquad \text{for every} \quad t \in [0,1], \tag{4}$$

where $M = c_1c_2(c_1-c_2)^{-1}$, $L = \max\{\eta, 1-\eta\}$, are valid.

P r o o f. From the equalities $u'(t) = \int_0^t u''(s)\,\mathrm{d}s$, $-u'(t) = \int_t^1 u''(s)\,\mathrm{d}s$ it follows that

$$\begin{aligned} c_1 t \le\ & u'(t) \le c_2 t, \\ c_2(1-t) \ge -\,& u'(t) \ge c_1(1-t) \qquad \text{for every} \quad t \in (0,1). \end{aligned}$$

Since u'' is continuous we obtain from the last two inequalities and from (2) the inequalities (4). The lemma is proved.

LEMMA 2. *Let there exist $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ such that $f(t,x,y,z) < 0$ for a.e. $t \in (0,1)$ and for every $x \in (-ML, ML)$, $y \in [-\varepsilon, \varepsilon)$, $z \in (-\varepsilon, \varepsilon)$. Let u be a solution of (1), (2) such that $u'(t) \geq -\varepsilon$, $c_1 \leq u''(t) \leq c_2$ for every $t \in (0,1)$, where $c_1, c_2 \in \mathbb{R}$ and $c_1 < 0 < c_2$. Then $u'(t) > 0$ for $t \in (0,1)$ and $u''(1) < 0 < u''(0)$.*

Proof. Let u be a solution of (1), (2) satisfying the assumptions of Lemma 2 and $u'(t_0) = 0$, where $t_0 \in [0,1)$. If $u''(t_0) = 0$, then there exists $\delta \in \mathbb{R}$, $\delta > 0$ such that $|u''(t)| < \varepsilon$ and $|u'(t)| < \varepsilon$ for $t \in (t_0, t_0 + \delta)$ and we obtain

$$\int_{t_0}^{t} f(s,u,u',u'')\,\mathrm{d}s = u''(t) < 0 \qquad \text{for} \quad t \in (t_0, t_0 + \delta).$$

Thus under the assumption that $u''(t_0) \leq 0$ there exists $t_1 \in (t_0, 1)$ such that $u'(t_1) < 0$, $\min\{u'(t),\ t_0 \leq t \leq 1\} = u'(t_1)$ and $u''(t_1) = 0$. Further there exists $\delta_1 \in \mathbb{R}$, $\delta_1 > 0$ such that $u'(t) \in [-\varepsilon, \varepsilon)$, $u''(t) \in (-\varepsilon, \varepsilon)$, for $t \in (t_1, t_1 + \delta)$ and by integrating (1) from t_1 to t, where $t \in (t_1, t_1 + \delta)$, we obtain $u''(t) < 0$ for $t \in (t_1, t_1 + \delta)$; but $u'(t_1) = \min\{u'(t),\ t_0 \leq t \leq 1\}$, and this contradiction proves that $u''(t_0) > 0$ if $t_0 \in [0,1)$ and $u'(t_0) = 0$. Since $u''(0) > 0$ there exists $t_2 \in (0,1]$ such that $u'(t) > 0$ for $t \in (0, t_2)$, $u'(t_2) = 0$, $u''(t_2) \leq 0$ and by (the part of) the proof above, $t_2 = 1$. If $u''(1) = 0$, then there exists $\delta_2 \in \mathbb{R}$, $\delta_2 > 0$ such that $u''(t) \in (-\varepsilon, \varepsilon)$, $u'(t) \in (-\varepsilon, \varepsilon)$ for $t \in (1 - \delta_2, 1)$ and by integrating (1) from t to 1 for $t \in (1 - \delta_2, 1)$ we obtain $-u''(t) < 0$ for $t \in (1 - \delta_2, 1)$. On the other hand $u'(t) > 0$ for $t \in (0,1)$ and this contradiction completes the proof of Lemma 2.

LEMMA 3. *Let there exist $c_1, c_2 \in \mathbb{R}$, $c_1 < 0 < c_2$ such that*

$$\liminf_{z \to c_1} f(t,x,y,z) > 0, \qquad \liminf_{z \to c_2} f(t,x,y,z) > 0$$

uniformly for $x \in (-ML, ML)$, $y \in [0, M)$, $t \in [0,1]$. Further let u be a solution of (1), (2), $u'(t) > 0$ for $t \in (0,1)$, $u''(1) < 0 < u''(0)$ and $c_1 \leq u''(t) \leq c_2$ for $t \in [0,1]$. Then $c_1 < u''(t) < c_2$ for $t \in [0,1]$.

Proof. Let us suppose that $u''(t_1) = c_2$, where $t_1 \in [0,1]$, then $t_1 < 1$ since $u''(1) < 0$. From the properties of f there follows the existence of $\delta \in \mathbb{R}$, $\delta > 0$ such that $f(t,u,u',u'') > 0$ for a.e. $t \in (t_1, t_1 + \delta)$. By integrating (1) from t_1 to t where $t \in (t_1, t_1 + \delta)$ we obtain $u''(t) > c_2$ for $t \in (t_1, t_1 + \delta)$ and this contradiction proves that $u''(t) < c_2$ for $t \in [0,1]$. Analogously $c_1 < u''(t)$ for $t \in [0,1]$ and the proof of Lemma 3 is complete.

THEOREM 2. *Let there exist* $c_1, c_2 \in \mathbb{R}$, $c_1 < 0 < c_2$ *such that*

$$\liminf_{z\to c_1} f(t,x,y,z) \ge 0, \qquad \liminf_{z\to c_2} f(t,x,y,z) \ge 0$$

uniformly for $(t,x,y) \in [0,1]\times(-ML,\, ML)\times[0,M)$. *Further let*

$$\limsup_{(y,z)\to(0,0)} f(t,x,y,z) \le 0$$

uniformly for $(t,x) \in [0,1]\times(-ML,\, ML)$. *Then* BVP (1), (2) *has a solution* u *satisfying*

$$-ML < u(t) < ML\,, \quad 0 \le u'(t) < M\,, \quad c_1 \le u''(t) \le c_2 \qquad \text{for} \quad t\in[0,1]\,.$$

Proof. By the Tietze-Urysohn lemma there exists a continuous function $g\colon \mathbb{R}\times\mathbb{R} \to [-1,1]$ such that $g(0,0) = -1$ and $g(y,c_i) = 1$ for $i = 1,2$ $y\in[0,M]$. Let us put

$$f_n(t,x,y,z) = f(t,x,y,z) + n^{-1}g(y,z) \qquad \text{for} \quad n\in\mathbb{N}\,.$$

Then we obtain that

$$\limsup_{(y,z)\to(0,0)} f_n(t,x,y,z) \le -n^{-1} \qquad (n\in\mathbb{N})$$

uniformly for $(t,x)\in[0,1]\times(-ML,\, ML)$ and

$$\liminf_{z\to c_1} f_n(t,x,y,z) \ge n^{-1}, \quad \liminf_{z\to c_2} f_n(t,x,y,z) \ge n^{-1} \qquad (n\in\mathbb{N})$$

uniformly for $(t,x,y)\in[0,1]\times(-ML,\, ML)\times[0,M)$. For every fixed $n\in\mathbb{N}$ there exists $\varepsilon_n\in\mathbb{R}$, $1>\varepsilon_n>0$ such that $f_n(t,x,y,z)<0$ for a.e. $t\in(0,1)$ and for every $x\in(-ML,ML)$, $y\in[-\varepsilon_n,\varepsilon_n)$, $z\in(-\varepsilon_n,\varepsilon_n)$. Put $U_n=\{x\in X: -ML < x(t) < ML,\ -\varepsilon_n < x'(t) < M,\ c_1 < x''(t) < c_2, \ \text{for} \ t\in[0,1]\}$. From Lemmas 1 – 3 it follows that BVP

$$u''' = \lambda f_n(t,u,u',u'')\,,$$

with conditions (2) has no solutions in δU_n for $\lambda > 0$. By Theorem 1 BVP

$$u''' = f_n(t,u,u',u'') \tag{5}$$

with conditions (2) has a solution $u_n \in \operatorname{cl} U_n$. It can be easily seen that the sequences $(u_n)_{n=1}^{\infty}$, $(u'_n)_{n=1}^{\infty}$ are uniformly bounded and equi-continuous on $[0,1]$ and that the sequence $(u''_n)_{n=1}^{\infty}$ is uniformly bounded on $[0,1]$. From (4) and by the theory of the Lebesgue integral we get that the sequence $(u''_n)_{n=1}^{\infty}$ is equi-continuous on $[0,1]$. By the Arzela-Ascoli lemma without loss of generality, we may suppose that all the three sequences are uniformly converging on $[0,1]$. By the Lebesgue theorem and by (5) the function $u(t) = \lim_{n\to\infty} u_n(t)$ on $[0,1]$ is a solution of (1), (2) and fulfils the assertion of Theorem 2. The theorem is proved.

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