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ON THE EXISTENCE OF CONJUGATE POINTS FOR LINEAR DIFFERENTIAL SYSTEMS

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1. Introduction

The principal aim of the paper is to study sufficient conditions for the existence of conjugate points of solutions of linear differential systems

$$y' = B(x)z, \qquad z' = -C(x)y, \tag{1.1}$$

where B, C are symmetric $n \times n$ matrices (i.e. $B^T = B$, $C^T = C$) of real-valued continuous functions and B is nonnegative definite.

A similar problem has been recently studied for various differential equations. For example, the differential equation

$$y'' + p(x)y = 0, \tag{1.2}$$

$p(x)$ being a real-valued continuous function, is conjugate on $R = (-\infty, \infty)$ (i.e. there exists a nontrivial solution of (1.2) vanishing in two distinct points of R) if

$$\liminf_{\substack{t \to -\infty \\ z \to \infty}} \int_t^z p(x)\, \mathrm{d}x > 0, \tag{1.3}$$

see Tipler [19]. The self-adjoint differential equation of the fourth order

$$(r(x)y'')'' - p(x)y = 0, \tag{1.4}$$

where $r(x)$, $p(x)$ are real-valued functions, $r(x) \in C^2(R)$, $r(x) > 0$, $p(x) \in C^0(R)$, is conjugate on R (i.e. there exist $x_1, x_2 \in R$ and a nontrivial solution $y(x)$ of (1.4) for which $y(x_1) = y'(x_1) = 0 = y(x_2) = y'(x_2)$) if either

(i) $\int_0^\infty x^2 r^{-1}(x)\, \mathrm{d}x = \infty$, $\int_{-\infty}^0 x^2 r^{-1}(x)\, \mathrm{d}x = \infty$ and $\liminf\limits_{\substack{t \to -\infty \\ z \to \infty}} \int_t^z p(x)\, \mathrm{d}x > 0$

or

(ii) $\int_0^\infty r^{-1}(x)\,dx = \infty$, $\int_{-\infty}^0 r^{-1}(x)\,dx = \infty$ and there exists $x_0 \in R$ such that $\liminf_{\substack{t \to -\infty \\ z \to \infty}} \int_t^z p(x)(x - x_0)^2\,dx > 0$, see [13].

In this paper we shall show that similar conditions are also sufficient for the existence of conjugate points relative to (1.1). In addition, we shall also discuss a certain duality between integral criteria for oscillation at infinity and integral criteria for the existence of conjugate points relative to (1.1) and the differential system $y'' + P(x)y = 0$, where $P(x)$ is a symmetric matrix of real-valued continuous functions.

The principal method we use is the variation principle of Courant applied to the quadratic functional corresponding to (1.1).

Matrix notation is used. E and 0 denote the identity and the zero matrix of any dimension. If we need to emphasize that E and 0 are $k \times k$ matrices, we shall denote them E_k and 0_k. If A is a symmetric matrix, $l_1(A) \leq l_2(A) \leq \ldots \leq l_n(A)$ denote the eigenvalues of A ordered by size.

2. Preliminary results

First recall some properties of solutions of differential equation (1.2). Let this equation be disconjugate on an interval $I = (a, b)$, i. e. there exist no two distinct points of I which are conjugate relative to (1.2). Then there exists a unique (up to a multiple by a nonzero real constant) solution $y_b(x)$ of (1.2) such that $\lim_{x \to b-} y_b(x)/y(x) = 0$ for every solution $y(x)$ of (1.2) which is linearly independent of $y_b(x)$. The solution $y_b(x)$ is said to be the *principal solution at b*. The principal solution of (1.2) at a is defined analogously, see, e.g. [7, p. 350]. The equation (1.2) whose principal solutions at a and b are linearly dependent, i. e. $y_a(x) = k \,.\, y_b(x)$, $k \neq 0$, is said to be *special on I*, see [5, p. 22].

The result of Tipler [17] can be seen in the following way. The equation

$$y'' = 0 \tag{2.1}$$

is disconjugate and special on R ($y_{-\infty}(x) = 1 = y_\infty(x)$) and consider equation (1.2) as a perturbation of (2.1). Then every perturbation of (2.1) by a function $p(x)$ which is positive on the average on R (i. e. (1.3) holds) makes the perturbed equation possess a nontrivial solution having at least two distinct zeros. Using the tranformation theory of self-adjoint linear differential equations of the second order, see, e.g., [1], one can reformulate the Tipler result in the following way:

Consider two equations

$$(r(x)y')' + p_i(x)y = 0, \qquad i = 1, 2, \tag{2.2$_i$}$$

$r(x) > 0$ on $I = (a, b)$. Let equation $(2.2)_1$ be disconjugate and special on I and denote by $y_0(x)$ the only (up to a multiple by a nonzero real constant) solution of $(2.2)_1$ which is nonzero on I (i. e. $y_0(x) = y_a(x) = y_b(x)$). If

$$\liminf_{\substack{t \to a+ \\ z \to b-}} \int_t^z y_0^2(x)(p_2(x) - p_1(x))\,\mathrm{d}x > 0,$$

then equation $(2.2)_2$ is conjugate on I.

Now, let us turn to the differential system (1.1) and recall some of its properties. Simultaneously with (1.1) we shall consider the matrix system

$$Y' = B(x)Z, \qquad Z' = -C(x)Y, \tag{1.1$_M$}$$

where Y, Z are $n \times n$ matrices

Let (Y_i, Z_i), $i = 1, 2$, be solutions of $(1.1)_M$; then

$$Y_1^T(x)Z_2(x) - Z_1^T(x)Y_2(x) = K, \tag{2.3}$$

where K is a constant $n \times n$ matrix. A solution (Y, Z) of $(1.1)_M$ is said to be *self-conjugate* if $Y^T(x)Z(x) - Z^T(x)Y(x) = 0$. The system (1.1) or $(1.1)_M$ is said to be *identically normal* on an interval I whenever the only solution (y, z) of (1.1) for which $y(x) \equiv 0$ on any nondegenerate subinterval of I is the trivial solution $(y, z) = (0, 0)$. Two distinct points $x_1, x_2 \in R$ are conjugate relative to (1.1) if there exists a nontrivial solution (y, z) of (1.1) such that $y(x_1) = 0 = y(x_2)$. The system (1.1) is said to be disconjugate on I if there exist no two distinct points of I which are conjugate relative to (1.1). Two solutions (Y_i, Z_i), $i = 1, 2$, of $(1.1)_M$ are said to be *linearly independent* if every solution (Y, Z) of $(1.1)_M$ can be expressed in the form $(Y, Z) = (Y_1M + Y_2N, Z_1M + Z_2N)$, where M, N are constant $n \times n$ matrices. If the solutions (Y_i, Z_i) are self-conjugate, one can show, see, e.g. [16, p. 308], that these solutions are linearly independent if and only if the constant matrix $Y_1^T(x)Z_2(x) - Z_1^T(x)Y_2(x)$ is nonsingular.

Let the system $(1.1)_M$ be disconjugate and identically normal on I. There exists a unique (up to a right multiple by a constant nonsingular $n \times n$ matrix) self-conjugate solution (Y_b, Z_b) of $(1.1)_M$ such that $Y_b(x)$ is nonsingular and $\lim_{x \to b-} Y^{-1}(x)Y_b(x) = 0$ for every self-conjugate solution (Y, Z) of $(1.1)_M$ which is linearly independent on (Y_b, Z_b). The solution (Y_b, Z_b) is said to be the *principal solution of* $(1.1)_M$ *at* b. For this solution we have $\lim_{x \to b-} l_1\left(\int^x Y^{-1}(s) \,.\, B(s)Y^{T-1}(s)\,\mathrm{d}s\right) = \infty$. The principal solution of $(1.1)_M$ at a is defined analogously, see, e.g., [16, p. 341]. System (1.1) or $(1.1)_M$ is said to be *k-general on I* if the rank of the matrix

$$\begin{pmatrix} Y_a(x) & Y_b(x) \\ Z_a(x) & Z_b(x) \end{pmatrix}$$

equals $n + k$ for every $x \in I$, see [6]. One can show that (1.1) is k-general on I if and only if the rank of the constant matrix $Y_a^T(x) Z_b(x) - Z_a^T(x) Y_b(x)$ equals k.

3. Conjugate points

In this section we use the variation principle of Courant (see, e.g., [18, p. 208] or [16, p. 337]) in order to prove a sufficient condition for the existence of conjugate points relative to the system (1.1). Before proving this result we make one auxiliary statement.

Lemma 1. *Let the differential system*

$$y' = B(x) z, \qquad z' = 0 \tag{3.1}$$

be k-general on $I = (a, b)$, $k \in \{0, \dots, n\}$, *and let* $\lim_{x \to b-} l_1 \left(\int_c^x B(s)\, \mathrm{d}s \right) = \infty$ *for some (and hence for every)* $c \in I$. *Then there exists an* $(n-k)$*-dimensional linear space* $V_{n-k} \subset R^n$ *such that* $\lim_{x \to a+} \int_x^c u^T B(s) u \, \mathrm{d}s = \infty$ for every $0 \neq u \in V_{n-k}$.

Proof. Since $\lim_{x \to b-} l_1 \left(\int_c^x B(s)\, \mathrm{d}s \right) = \infty$, $(Y_b, Z_b) = (E, 0)$ is the principal solution of (1.3) at b. This system is k-general on I, hence the rank of the matrix $K = Y_a^T Z_b - Z_a^T Y_b$ equals k and without loss of generality we can suppose that $K = \operatorname{diag}\{E_k, 0_{n-k}\}$, i.e., $Z_a = K = \operatorname{diag}\{E_k, 0_{n-k}\}$. Write the matrix $B(x)$ in the form

$$B(x) = \begin{pmatrix} B_1(x) & B_2(x) \\ B_2^T(x) & B_3(x) \end{pmatrix},$$

where B_1, B_2, B_3 are $k \times k$, $k \times (n-k)$, and $(n-k) \times (n-k)$ matrices, respectively. Hence, $Y_a' = B(x) Z_a = \begin{pmatrix} B_1 & 0 \\ B_2^T & 0 \end{pmatrix}$, thus

$$Y_a(x) = \begin{pmatrix} \int_c^x B_1(s)\, \mathrm{d}s + D_1 & D_2 \\ \int_c^x B_2^T(s)\, \mathrm{d}s + D_3 & D_4 \end{pmatrix}, \qquad c \in I,$$

where D_i, $i = 1, \dots, 4$, are constant matrices. The fact that the principal solution (Y_a, Z_a) is self-conjugate gives $D_1 = D_1^T$, $D_2 = 0$ and the nonsingularity of $Y_a(x)$ implies that D_4 is also nonsingular. Moreover, there is no loss of generality in assuming $D_4 = E_{n-k}$ and $D_3 = 0$. Let

$$Y = \begin{pmatrix} E_k & \int_c^x B_2(s)\,\mathrm{d}s \\ 0 & \int_c^x B_3(s)\,\mathrm{d}s \end{pmatrix}, \qquad Z = \begin{pmatrix} 0_k & 0 \\ 0 & E_{n-k} \end{pmatrix}.$$

Then (Y, Z) is a self-conjugate solution of (3.1) and $Y_a^T Z - Z_a^T Y$ is nonsingular, hence the solutions (Y_a, Z_a), (Y, Z) are linearly independent.
It implies $\lim_{x \to a+} Y^{-1}(x)\, Y_a(x) = 0$, i.e.

$$\lim_{x \to a+} \begin{pmatrix} E_k & \int_c^x B_2(s) \\ 0 & \int_c^x B_3(s) \end{pmatrix}^{-1} \begin{pmatrix} \int_c^x B_1(s) + D_1 & 0 \\ \int_c^x B_2^T(s) + D_3 & E_{n-k} \end{pmatrix} =$$

$$= \lim_{x \to a+} \begin{pmatrix} E_k & -\int_c^x B_2(s)\left(\int_c^x B_3(s)\right)^{-1} \\ 0 & \left(\int_c^x B_3(s)\right)^{-1} \end{pmatrix} \begin{pmatrix} \int_c^x B_1(s) + D_1 & 0 \\ \int_c^x B_2^T(s) & E_{n-k} \end{pmatrix} =$$

$$= \lim_{x \to a+} \begin{pmatrix} \int_c^x B_1 + D_1 - \int_c^x B_2 \left(\int_c^x B_3\right)^{-1} \int_c^x B_2^T & -\int_c^x B_2^T \left(\int_c^x B_3\right)^{-1} \\ -\left(\int_c^x B_3\right)^{-1} \int_c^x B_2 & \left(\int_c^x B_3\right)^{-1} \end{pmatrix} =$$

$$= 0, \text{ i.e. } \lim_{x \to a+} \left(\int_c^x B_3(s)\right)^{-1} = 0, \text{ hence } \lim_{x \to a+} l_n\left(\left(\int_x^c B_3(s)\right)^{-1}\right) =$$

$= 0$ and thus $\lim_{x \to a+} l_1\left(\int_x^c B_3(s)\right) = \infty$. Taking $V_{n-k} = \mathrm{Lin}\{e_{k+1}, \dots, e_n\}$, $\{e_j\}_{j=1}^n$ being the canonical base of R^n, we have the required statement.

Theorem 1. *Let the system* (3.1) *be k-general on* $I = (a, b)$, $0 \le k \le n - 1$, $\lim_{\to} l_1\left(\int_{b-}^x B(s)\,\mathrm{d}s\right) = \infty$ *and let there exist* $a_1, b_1 \in I$ *such that the matrix* $C(x)$ *is nonnegative definite for* $x \in (a, a_1) \cup (b_1, b)$. *If there exists a* $(k + 1)$*-dimensional* *linear space* $V_{k+1} \subset R^n$ *such that*

$$\liminf_{\substack{t \to a+ \\ z \to b-}} \int_t^z w^T C(x) w \, \mathrm{d}x > 0$$

for every $0 \neq w \in V_{k+1}$, *then there exists a pair of conjugate points relative to* (1.1).

Proof. We shall construct a pair of vector functions $(u(x), v(x))$ such that $u(x)$ has compact support in (a, b) and the quadratic functional corresponding to (1.1)

$$I(u, v; a, b) = \int_a^b (v^T(x) B(x) v(x) - u^T(x) C(x) u(x)) \, \mathrm{d}x,$$

$$u' = B(x) v \text{ on } I,$$

is negative. Then one can find by means of the Courant variation principle a solution (y, z) of (1.1) such that $y(x)$ vanishes in two distinct points of (a, b). This technique is similar to that used in [13] and [15].

Denote

$$c = \min_{\substack{z \in V_{k+1} \\ z^T z = 1}} \left\{ \liminf_{\substack{t \to a+ \\ x \to b-}} \int_t^x z^T C(s) z \, \mathrm{d}s \right\}$$

and let $\varepsilon > 0$. There exist $x_2, x_3 \in I$ such that $C(x)$ is nonnegative definite for $x \in (a, x_2) \cup (x_3, b)$ and $\int_{t_1}^{t_2} z^T C(x) z \, \mathrm{d}x > c - \varepsilon$, for every unit vector $z \in V_{k+1}$, whenever $t_1 < x_2$, $t_2 > x_3$. According to Lemma 1 there exists an $(n-k)$-dimensional linear space $V_{n-k} \subset R^n$ such that $\lim_{x \to a+} \int_x^{x_2} w^T B(x) w \, \mathrm{d}x = \infty$ for every $0 \neq w \in V_{n-k}$, hence by the Courant-Fischer min-max principle, see, e.g., [3, p. 137], $\lim_{x \to a+} l_{k+j}\left(\int_x^{x_2} B(s) \, \mathrm{d}s\right) = \infty, j = 1, \ldots, n-k$. Choose $x_1 \in (a, x_2)$ such that $l_{k+1}\left(\varepsilon \int_{x_1}^{x_2} B(x) \, \mathrm{d}x\right) = 1$ and let $w_j, j = 1, \ldots n-k$, be the unit eigenvectors corresponding to $l_{k+j}\left(\varepsilon \int_{x_1}^{x_2} B(x) \, \mathrm{d}x\right)$. At least one of the vectors w_j belongs to V_{k+1}, denote this vector by w_0 and the corresponding eigenvalue of $\varepsilon \int_{x_1}^{x_2} B(x) \, \mathrm{d}x$ by d_0.

Further, since $\lim_{x\to b-} l_1\left(\int_{x_3}^{x} B(s)\,\mathrm{d}s\right) = \infty$, we have $\lim_{x\to b-} l_n\left(\left(\int_{x_3}^{x} B(s)\,\mathrm{d}s\right)^{-1}\right) = 0$ and hence there exists $x_4 \in (x_3, b)$ such that $w_0^T\left(\int_{x_3}^{x_4} B(x)\,\mathrm{d}x\right)^{-1} w_0 = \varepsilon$. Note that the identical normality of (1.1) implies that for every $t_1, t_2 \in I$, $t_1 < t_2$, the matrix $\int_{t_1}^{t_2} B(x)\,\mathrm{d}x$ is nonsingular. In fact, since $B(x)$ is nonnegative definite, $\int_{t_1}^{t_2} B(x)\,\mathrm{d}x z_0 = 0$, z_0 being a nonzero vector, implies $z_0^T B(x) z_0 = 0$ on (t_1, t_2), i.e. $B(x) z_0 = 0$. Then $(0, z_0)$ is a solution of (1.1) on (t_1, t_2), which contradicts the identical normality of (1.1).

Now, define a pair of functions

$$(u, v) = \begin{cases} (0, 0), \ x \in (a, x_1], \\ \left(\varepsilon d_0^{-1}\int_{x_1}^{x} B(s)\,\mathrm{d}s w_0,\ \varepsilon d_0^{-1} w_0\right), \ x \in (x_1, x_2], \\ (w_0, 0), \ x \in (x_2, x_3], \\ \left(\int_{x}^{x_4} B(s)\,\mathrm{d}s\left(\int_{x_3}^{x_4} B(x)\,\mathrm{d}x\right)^{-1} w_0,\ -\left(\int_{x_3}^{x_4} B(x)\,\mathrm{d}x\right)^{-1} w_0\right), \ x \in (x_3, x_4], \\ (0, 0), \ x \in (x_4, b). \end{cases}$$

Then $u' = B(x)v$ on I, $\operatorname{supp} u(x) \subset I$ and $I(u, v; a, b) = I(u, v; x_1, x_4) =$

$$= \varepsilon^2 d_0^{-2} w_0^T \int_{x_1}^{x_2} B(x)\,\mathrm{d}x w_0 + w_0^T\left(\int_{x_3}^{x_4} B(x)\,\mathrm{d}x\right)^{-1}\int_{x_3}^{x_4} B(x)\,\mathrm{d}x\left(\int_{x_3}^{x_4} B(x)\,\mathrm{d}x\right)^{-1} w_0 -$$

$$- \int_{x_1}^{x_2} u^T(x)\ C(x)\ u(x)\ \mathrm{d}x \ - \ w_0^T \int_{x_2}^{x_3} C(x)\ \mathrm{d}x w_0 \ - \int_{x_3}^{x_4} u^T(x)\ C(x)\ u(x)\ \mathrm{d}x \le$$

$$\le \varepsilon d_0^{-1} w_0^T w_0 + w_0^T\left(\int_{x_3}^{x_4} B(x)\,\mathrm{d}x\right)^{-1} w_0 - c + \varepsilon \le \varepsilon d_0^{-1} + \varepsilon - c + \varepsilon - c + 3\varepsilon.$$

Consequently, taking ε sufficiently small $\left(< \frac{c}{3}\right)$, we have $I(u, v; a, b) < 0$, which was to be proved.

Remark 1. Comparing the statement of Theorem 1 with the Tipler result for equation (1.2), we see that the assumption concerning the definiteness of $C(x)$ near a and b has no analogy in the scalar case, but we were not successful in proving Theorem 1 without an assumption of a similar kind. In the next section we shall show that for certain special systems this assumption can be omitted.

Remark 2. Similarly as in the scalar case consider the system (1.1) as a perturbation of (3.1). Theorem 1 states that under certain assumptions the perturberation of the k-general system by a symmetric matrix which is positive definite on the average on a $(k+1)$-dimensional linear space makes the perturbed system conjugate. In Theorem 1 we have only considered linear perturbations, but the used method can be also extended to nonlinear perturbations, e.g., to the system

$$Y' = B(x)Z, \qquad Z' = -C(x, Y, Z),$$

where $C\colon R \times R^{n\times n} \times R^{n\times n} \to R^{n\times n}$ is a symmetric matrix. For some ideas concerning this method see [15].

4. The system $y'' + P(x)y = 0$

Consider the linear differential system of the second order

$$y'' = P(x)y = 0, \tag{4.1}$$

where $P(x)$ is a symmetric $n \times n$ matrix of real-valued continuous functions. This system can be rewritten in the form (1.1) ($y' = z$, $B(x) = E$, $C(x) = P(x)$), hence the definitions of conjugate points, disconjugacy, etc. for (1.1) hold also for system (4.1).

Corollary 1. *If there exists a unit vector $v \in R^n$ such that*

$$\liminf_{\substack{t\to-\infty\\ z\to\infty}} \int_t^z v^T P(x)\, v \,\mathrm{d}x = c > 0, \tag{4.2}$$

then there exists a pair of points which are conjugate relative to (4.1).

Proof. Similarly as in the proof of Theorem 1 let $x_2, x_3 \in R$ be such that $\int_{x_2}^{x_3} v^T P(x)\, v\,\mathrm{d}x > \frac{c}{2}$, and let $x_1 = x_2 - \frac{1}{\varepsilon}$, $x_4 = x_3 + \frac{1}{\varepsilon}$. Define a function

$$y(x) = \begin{cases} 0,\ x \in (-\infty, x_1], \\ \varepsilon(x - x_1)\, v,\ x \in (x_1, x_2], \\ v,\ x \in (x_2, x_3], \\ [1 - \varepsilon(x - x_4)]\, v,\ x \in (x_3, x_4], \\ 0,\ x \in (x_4, \infty). \end{cases}$$

We have $I(y; x_1, x_4) = \int_{x_1}^{x_4} (y^{T'}(x) y'(x) - y^T(x) P(x) y(x))\,\mathrm{d}x = \varepsilon^2 v^T v \int_{x_1}^{x_2} \mathrm{d}x + \varepsilon^2 v^T v \int_{x_3}^{x_4} \mathrm{d}x - \varepsilon^2 \int_{x_1}^{x_2} (x - x_1)^2 v^T P(x) v\,\mathrm{d}x - \int_{x_2}^{x_3} v^T P(x) v\,\mathrm{d}x - \int_{x_3}^{x_4} (1 - \varepsilon(x - x_4))^2 v^T P(x) v\,\mathrm{d}x = 2\varepsilon - \int_{\xi_1}^{\xi_2} v^T P(x) v\,\mathrm{d}x$, $\xi_1 \in [x_1, x_2]$, $\xi_2 \in [x_3, x_4]$, where the second mean value theorem of integral calculus has been used for computing the integrals $\int_{x_1}^{x_2} \varepsilon^2 (x - x_1)^2 v^T P(x) v\,\mathrm{d}x$ and $\int_{x_3}^{x_4} (1 - \varepsilon(x - x_4))^2 v^T P(x) v\,\mathrm{d}x$. Now, taking $\varepsilon < \frac{c}{4}$, we have $I(y; x_1, x_2) < 0$, hence there exists a pair of points which are conjugate relative to (4.1).

Remark 3. Corollary 1 can be also proved as a consequence of the above mentioned result of Tipler and the theorem of Hartman [8, Th. 1.1], which states that (4.1) is conjugate on an interval I if the scalar equation

$$y'' + q(P(x))y = 0$$

is conjugate on I, where q is a superadditive, superhomogeneous positive and normalized functional on the linear space of the $n \times n$ symmetric matrices (in Corollary 1, $q(P) = v^T P v$). Consequently, condition (4.2) can be replaced by the condition

$$\liminf_{\substack{t \to -\infty \\ z \to \infty}} \int_t^z q(P(x))\,\mathrm{d}x = c > 0\,,$$

where q is a functional with the above given properties.

Now, we shall discuss the following problem. In [9] Lewis and Hinton conjectured that the system (4.1) is oscillatory at infinity (i.e. there exists an arbitrarily large pair of conjugate points relative to this system) if

$$\lim_{x \to \infty} l_n\left(\int_1^x P(s)\,\mathrm{d}s\right) = \infty\,. \tag{4.3}$$

Kwong *et al.* [10] proved that the conjecture is true if $n = 2$, and Atkinson *et al.* [2] showed that this result is the consequence of the fact that (4.1) is oscillatory at infinity if at least $(n - 1)$ eigenvalues of the matrix $\int_1^x P(s)\,\mathrm{d}s$ tend to infinity as $x \to \infty$. Recently Byres *et al.* [4] and Kaper *et al.* [11] proved the conjecture for a general n without any additional assumptions.

There exists considerable duality between the integral criteria for the oscillation at infinity and the integral criteria for the existence of a pair of conjugate points for differential equations of various kinds.

For example, equation (1.2) is oscillatory at infinity if $\int^{\infty} p(x)\,\mathrm{d}x = \infty$ (see, e.g., [18, p. 45]), equation (1.4) is oscillatory at infinity if either

i) $\int^{\infty} r^{-1}(x)\,\mathrm{d}x = \infty$ and $\int^{\infty} (x - x_0)^2 p(x)\,\mathrm{d}x = \infty$ for some real x_0 or

ii) $\int^{\infty} x^2 r^{-1}(x)\,\mathrm{d}x = \infty$ and $\int^{\infty} p(x)\,\mathrm{d}x = \infty$ (see [13]).

Comparing these criteria with the criteria for the existence of conjugate points relative to (1.2) and (1.4) given in Sec. 1, we see that it suffices to replace the condition requiring certain integrals to be positive by the conditions requiring these integrals to diverge. A similar duality can be found in the oscillation theory of partial elliptic differential equations, see [17]. These examples lead us to the following conjecture, which is dual to the conjecture of Hinton and Lewis.

Conjecture. *Let*

$$\liminf_{\substack{t \to -\infty \\ z \to \infty}} l_n\left(\int_t^z P(x)\,\mathrm{d}x\right) > 0\,.$$

Then there exists a pair of points which are conjugate relative to (4.1).

Concerning the system (1.1), the following statement is dual to Theorem 1.

Theorem 2. *Let* (Y_R, Z_R) *be the principal solution of* (3.1) *at* ∞ *and suppose that the matrix* $C(x)$ *is nonnegative definite for large* x. *If there exists an n-dimensional constant vector* w *such that*

$$\int^{\infty} w^T Y_R^T(x)\, C(x)\, Y_R(x)\, w\,\mathrm{d}x = \infty\,, \tag{4.3}$$

then the system (1.1) *is oscillatory at infinity.*

Proof. To show that there exists an arbitrarily large pair of conjugate points relative to (1.1) (i.e. that (1.1) is oscillatory at ∞), it suffices to prove that for every $x_0 \in R$ there exists $x_1 > x_0$ and a pair of n-dimensional vector-valued functions $(y(x), z(x))$ with the properties:

i) $y(x)$, $z(x)$ are piecewise of the class C_1 and C^0, respectively,

ii) $y(x_0) = 0 = y(x_1)$,

iii) $y' = B(x) z$ for $x \in (x_0, x_1)$,

iv) $\int_{x_0}^{x_1} [z^T(x) B(x) z(x) - y^T(x) C(x) y(x)]\,\mathrm{d}x \le 0\,.$

The transformation

$$y = Y_R(x)\, u\,, \qquad z = Z_R(x)\, u + Y_R^{T-1}(x)\, v$$

tranforms the system (1.1) into the system

$$u' = \bar{B}(x), \qquad v' = \bar{C}(x)u,$$

where

$$\bar{B}(x) = Y_R^{-1}(x)\,B(x)\,Y_R^{T-1}(x),$$
$$\bar{C}(x) = Y_R^T(x)\,C(x)\,Y_R(x),$$

see, e.g., [1].

Let $x_0 \in R$ be arbitrary. Denote $t_1 = x_0 + 1$ and $d = w^T\left(\int_{x_0}^{t_1} \bar{B}(x)\,\mathrm{d}x\right)^{-1} w + {} + w^T\left(\int_{x_0}^{t_1} \bar{B}(x)\,\mathrm{d}x\right)^{-1} \int_{x_0}^{t_1} \bar{C}(x)\,\mathrm{d}x \left(\int_{x_0}^{t_1} \bar{B}(x)\,\mathrm{d}x\right)^{-1} w$. Since (4.4) holds, there exists $t_2 > t_1$ such that $\int_{t_1}^{t_2} w^T\bar{C}(x)\,w\,\mathrm{d}x > 2d$. Further, since (Y_R, Z_R) is the principal solution of (3.1),

$$\lim_{x\to\infty} l_1\left(\left(\int_{t_2}^{x} Y_R^{-1}(s)\;B(s)\;Y_R^{T-1}(s)\,\mathrm{d}s\right)\right) = \lim_{x\to\infty} l_1\left(\left(\int_{t_2}^{x} \bar{B}(s)\,\mathrm{d}s\right)\right) = \infty,$$

we have $\lim_{x\to\infty} w^T\left(\int_{t_2}^{x} \bar{B}(s)\,\mathrm{d}s\right)^{-1} w = 0$, i.e. there exists $x_1 > t_2$ such that $w^T\left(\int_{t_2}^{x_1} \bar{B}(s)\,\mathrm{d}s\right)^{-1} < d$.

Now, define

$$(u, v) = \begin{cases} (0, 0),\ x \le x_0, \\ \left(\int_{x_0}^{x} \bar{B}(s)\,\mathrm{d}s\left(\int_{x_0}^{t_1} \bar{B}(x)\,\mathrm{d}x\right)^{-1} w,\ \left(\int_{x_0}^{t_1} \bar{B}(x)\,\mathrm{d}x\right)^{-1} w\right),\ x \in [x_0, t_1], \\ (w, 0),\ x \in [t_1, t_2], \\ \left(\int_{x}^{x_1} \bar{B}(s)\,\mathrm{d}s\left(\int_{t_2}^{x_1} \bar{B}(x)\,\mathrm{d}x\right)^{-1} w,\ -\left(\int_{t_2}^{x_1} \bar{B}(x)\,\mathrm{d}x\right)^{-1} w\right),\ x \in [t_2, x_1], \\ (0, 0),\ x \ge x_1. \end{cases}$$

We have $\int_{x_0}^{t_1} [v^T(x)\,\bar{B}(x)\,v(x) - u^T(x)\,\bar{C}(x)\,u(x)]\,\mathrm{d}x = \int_{x_0}^{t_1} v^T(x)\,\bar{B}(x)\,v(x)\,\mathrm{d}x + {} + \int_{t_2}^{x_1} v^T(x)\,\bar{B}(x)\,v(x)\,\mathrm{d}x - \int_{x_0}^{t_1} u^T(x)\,\bar{C}(x)\,u(x)\,\mathrm{d}x - \int_{t_1}^{t_2} w^T\bar{C}(x)\,w\,\mathrm{d}x \;-$

$$- \int_{t_2}^{x_1} u^T(x)\, \bar{C}(x)\, u(x)\, \mathrm{d}x \le w^T \left[\left(\int_{x_0}^{t_1} \bar{B}(x)\, \mathrm{d}x \right)^{-1} - \int_{x_0}^{t_1} \bar{C}(x)\, \mathrm{d}x \right] w +$$

$$+ w^T \left(\int_{t_2}^{x_1} \bar{B}(x)\, \mathrm{d}x \right)^{-1} w - 2d - w^T \int_{t_2}^{x_1} \bar{C}(x)\, \mathrm{d}x w \le d + d - 2d \le 0. \; u' = \bar{B}(x) v$$

for $x \in [x_0, x_1]$ and $u(x_0) = 0 = u(x_1)$. Hence $y' = Y_R' u + Y_R u' = BZ_R u + Y_R \bar{B} v = BZ_R u + Y_R Y_R^{-1} B Y_R^{T-1} (Y_R^T z - Y_R^T Z_R u) = Bz$, $y(x_0) = 0 = y(x_1)$, $y(x)$, $z(x)$ are piecewise of the class C^1 and C^0, respectively (since so are $u(x)$ and $v(x)$) and $\int_{x_0}^{x_1} [z^T(x)\, B(x)\, z(x)\, \mathrm{d}x - y^T(x)\, C(x)\, y(x)]\, \mathrm{d}x = z^T(x)\, y(x)|_{x_0}^{x_1} + \int_{x_0}^{x_1} [v^T(x) \bar{B}(x) v(x) - u^T(x) \bar{C}(x) u(x)]\, \mathrm{d}x \le 0$. This completes the proof.

Remark 4. Note that the condition concerning nonnegative definiteness of the matrix $C(x)$ for large x can be omitted in some particular cases. For example, the Leighton-type oscillation criterion for the equation $(-1)^n (p(x) y^{(n)})^{(n)} + q(x) y = 0$ (which can be rewritten in the form (1.1) and $C(x)$ is nonnegative definite iff $q(x) \le 0$, see [1]) proved recently by Müller—Pfeiffer [14] is the special case of Theorem 2, but it needs no sign restriction on the function $q(x)$.

REFERENCES

[1] AHLBRANDT, C. D.—HINTON, D. B.—LEWIS, R. T.: The effect of variable change on oscillation and disconjugacy criteria with application to spectral theory and asymptotic theory. J. Math. Anal. Appl. 81, 1981, 234—277.

[2] ATKINSON, F. V.—KAPER, H. G.—KWONG, M. K.: An oscillation criterion for second order differential systems. Proc. Amer. Math. Soc. 96, 1985, 91—96.

[3] BELLMAN, R.: Vvedenie v teoriu matric, Nauka, Moskva 1976.

[4] BYRES, R.—HARRIS, B. J.—KWONG, M. K.: Weighted means and oscillation conditions for second order matrix differential equations. J. Diff. Equations 61, 1986, 164—177.

[5] BORŮVKA, O.: Linear Differentialtransformationen 2. Ordnung. VEB Deutscher Verlag der Wissenschaften, Berlin 1967.

[6] DOŠLÝ, O.: Riccati matrix differential equation and classification of disconjugate differential systems. Arch. Math. 23, 1987, 231—242.

[7] HARTMAN, P. Ordinary Differential Equations. John Wiley, New York, London, Sydney 1964.

[8] HARTMAN, P.: Oscillation criteria for self-adjoint second order differential systems and "principal sectional curvatures". J. Diff. Equations 34, 1979, 326—338.

[9] HINTON, D. B.—LEWIS, R. T.: Oscillation theory for generalized second order differential equations. Rocky Mountain J. Math. 10, 1980, 751—766.

[10] KAPER, H. G.—KWONG, M. K.: Oscillation of two-dimensional linear second-order differential systems. J. Diff. Equations 56, 1985, 195—205.

[11] KAPER, H. G.—KWONG, M. K.: Oscillation theory for linear second-order differential systems. Canadian Math. Soc. Conference Proceedings 8, 1987, 187—187.
[12] MORSE, M.: A generalization of the Sturm separation and comparison theorem in n-space. Math. Ann. 103, 1930, 52—69.
[13] MÜLLER-PFEIFFER, E.: Existence of conjugate points for second and fourth order differential equations. Proc. Roy. Soc. Edinburgh 89A, 1981, 281—291.
[14] MÜLLER-PFEIFFER, R.: An oscillation theorem for certain self-adjoint differential equations. Math. Nachr. 108, 1982. 79—92.
[15) NOUSSIAR, E. S.—SWANSON, C. A.: Oscillation criteria for differential systems. J. Math. Anal. Appl. 36, 1971, 575—580.
[16] REID, W. T.: Ordinary Differential Equations. John Wiley, New York 1971.
[17] SCHMINKE, V. W.: The lower spectrum of Schrödinger operator. Arch. Rational Mech. Anal. 75, 1981, 147—155.
[18] SWANSON, C. A.: Comparison and Oscillation Theory of Linear Differential Equations. Acad. Press, New York and London 1968.
[19] TIPLER, F. J.: General relativity and conjugate differential equations. J. Diff. Equations 30, 1978, 165—174.

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О СУЩЕСТВОВАНИИ СОПРЯЖЕННЫХ ТОЧЕК ДЛЯ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ СИСТЕМ

Ondřej Došlý

Резюме

В работе изучаются достаточные условия для существования сопряженных точек решений линейной дифференциальной системы $y' = B(x)z$, $z' = -C(x)y$, где $B(x)$, $C(x)$ — квадратные симметрические матрицы порядка n.