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ESTIMATION IN A SPECIAL STRUCTURE OF THE LINEAR MODEL

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ABSTRACT. The paper shows locally best linear-quadratic unbiased estimators in linear model, where the dispersions depend quadratically on mean value parameters. Investigated are the cases of full rank design matrix and of design matrix with one linear dependent row. Determined is also the existence of these estimators in all investigated cases and situations.

Introduction

If in the well-known regression model $(\widetilde{\mathbf{Y}}, \widetilde{\mathbf{X}}\boldsymbol{\beta}, \widetilde{\mathbf{\Sigma}})$ the dispersion matrix is of the form

$$\widetilde{\mathbf{\Sigma}} = \sigma^2 \widetilde{\mathbf{\Sigma}}(\boldsymbol{\beta}) = \sigma^2 \begin{pmatrix} (a+b|\mathbf{e}_1'\widetilde{\mathbf{X}}\boldsymbol{\beta}|)^2 & 0 & \dots & 0 \\ 0 & (a+b|\mathbf{e}_2'\widetilde{\mathbf{X}}\boldsymbol{\beta}|)^2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & (a+b|\mathbf{e}_n'\widetilde{\mathbf{X}}\boldsymbol{\beta}|)^2 \end{pmatrix},$$

we obtain a linear model of measurement with variances depending on the mean value parameters. In this model the result of observations is a realization of a random vector $\widetilde{\mathbf{Y}}_{n,1}$ with mean value $\mathcal{E}_{\boldsymbol{\beta}}(\widetilde{\mathbf{Y}}) = \widetilde{\mathbf{X}}\boldsymbol{\beta}$ ($\widetilde{\mathbf{X}}_{n,k}$ is a known design matrix, $\boldsymbol{\beta}_{k,1} \in \mathbb{R}^k$ is the vector of unknown parameters). The covariance matrix of the random vector $\widetilde{\mathbf{Y}}$ is $\widetilde{\mathbf{\Sigma}}$, where a, b and σ^2 are known positive constants, $\mathbf{e}_i'$ is the transpose of the i-th unity vector.

The motivation behind the model is based on the fact that a large class of measurement devices has its dispersion characteristic of the form $\sigma^2\big(a+b|\varphi|\big)^2$,

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where φ is the true measured value (see e.g. [1, p. 28], [6, p. 456, 914]) and the measurements are independent.

Locally best linear unbiased estimators and uniformly best linear unbiased estimators of linear functionals of parameter $\boldsymbol{\beta}$ in this model are investigated in [7].

The aim of the paper is to investigate the $\boldsymbol{\beta}_0$-locally best linear-quadratic unbiased estimators ($\boldsymbol{\beta}_0$-LBLQUE) of the elements of the covariance matrix in this model.

We investigate the $\boldsymbol{\beta}_0$-LBLQUE of the elements of the covariance matrix if the design matrix is of full rank in rows (in Section 2, Corollary 2.2).

Further we investigate this $\boldsymbol{\beta}_0$-LBLQUE in the case $R(\widetilde{\mathbf{X}}) = n - 1 \leqq k$ ($R(\widetilde{\mathbf{X}})$ is the rank of the matrix $\widetilde{\mathbf{X}}$). In this case we distinguish three situations:

(i) the linear dependent row is equal to another row multiplied by γ, where $\gamma \neq 0$ and $|\gamma| \neq 1$ (in Section 3, Theorem 3.10),

(ii) the linear dependent row is equal to another row multiplied by γ, where $|\gamma| = 1$ (in Section 4, Theorem 4.10),

(iii) the linear dependent row is a linear combination of two or more other rows (in Section 5, Theorem 5.4).

The existence of the $\boldsymbol{\beta}_0$-LBLQUE of the covariance matrix elements in situations (i), (ii) and (iii) is determined in Section 6.

With regard to the complicated problem we shall show the methodics of solution on relative simple cases, which enables us to generalization.

1. Preliminaries

Let us rearrange the rows in the matrix $\widetilde{\mathbf{X}}$ to obtain the matrix

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{R(\tilde{\mathbf{X}}),R(\tilde{\mathbf{X}})} \\ \mathbf{E} \end{pmatrix} \mathbf{X}_1 ,$$

where $\mathbf{X}_1$ is a matrix of order $R(\widetilde{\mathbf{X}}) \times k$ and $\mathbf{E} = \mathbf{X}_2\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}$ is of order $(n - R(\widetilde{\mathbf{X}})) \times R(\widetilde{\mathbf{X}})$.

In the same way we rearrange the coordinates of $\widetilde{\mathbf{Y}}$ and the rows of the matrix $\widetilde{\boldsymbol{\Sigma}}(\boldsymbol{\beta})$. We obtain the vector $\mathbf{Y}$ and the matrix

$$\boldsymbol{\Sigma}(\boldsymbol{\beta}) = \begin{pmatrix} \boldsymbol{\Sigma}_1(\boldsymbol{\beta}) & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\Sigma}_2(\boldsymbol{\beta}) \end{pmatrix},$$

where

$$\boldsymbol{\Sigma}_1(\boldsymbol{\beta}) = \begin{pmatrix} (a+b|\mathbf{e}_1'\mathbf{X}_1\boldsymbol{\beta}|)^2 & 0 & \dots & 0 \\ 0 & (a+b|\mathbf{e}_2'\mathbf{X}_1\boldsymbol{\beta}|)^2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & (a+b|\mathbf{e}_{R(\widetilde{\mathbf{X}})}'\mathbf{X}_1\boldsymbol{\beta}|)^2 \end{pmatrix}$$

and

$$\boldsymbol{\Sigma}_2(\boldsymbol{\beta}) = \begin{pmatrix} (a+b|\mathbf{e}_1'\mathbf{E}\mathbf{X}_1\boldsymbol{\beta}|)^2 & 0 & \dots & 0 \\ 0 & (a+b|\mathbf{e}_2'\mathbf{E}\mathbf{X}_1\boldsymbol{\beta}|)^2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & (a+b|\mathbf{e}_{n-R(\widetilde{\mathbf{X}})}'\mathbf{E}\mathbf{X}_1\boldsymbol{\beta}|)^2 \end{pmatrix}.$$

We get the model

$$(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}), \tag{1.1}$$

where $\boldsymbol{\Sigma} = \sigma^2\boldsymbol{\Sigma}(\boldsymbol{\beta})$. We assume that in this model $\mathbf{Y}$ is normally distributed.

The class of unbiased linear-quadratic estimators of the function

$$g(\cdot)\colon \mathbb{R}^k \longrightarrow \{0\}$$

in model (1.1) we denote

$$\mathcal{O}_{lq} = \left\{\mathbf{b}_{1,n}'\mathbf{Y} + \mathbf{Y}'\mathbf{B}_{n,n}\mathbf{Y} : \ \mathcal{E}_{\boldsymbol{\beta}}(\mathbf{b}'\mathbf{Y} + \mathbf{Y}'\mathbf{B}\mathbf{Y}) = 0 \quad \forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}\right\}.$$

THEOREM 1.1. *The random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE* ($\boldsymbol{\beta}_\circ$ *- locally best linear-quadratic unbiased estimator) of its mean value in* (1.1) *if and only if*

$$\forall\{\tau_\circ \in \mathcal{O}_{lq}\} \qquad \mathcal{E}_{\boldsymbol{\beta}_\circ}\big(\tau_\circ(\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y})\big) = 0.$$

P r o o f . See in [3, Theorem 3.1 and next Corollary].

Let us denote $\mathcal{D}$ the class of matrices $\mathbf{B}_{n,n}$ satisfying the next three conditions

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\boldsymbol{\beta}| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}| \end{pmatrix} = 0, \tag{1.2}$$

$$\operatorname{Tr}\mathbf{B} = 0, \tag{1.3}$$

$$\mathbf{X}'\Big(\mathbf{B} + \sigma^2 b^2 \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X} = \mathbf{O}. \tag{1.4}$$

($\operatorname{Tr}\mathbf{B}$ is the trace of $\mathbf{B}$ i.e. $\sum\limits_{i=1}^{n} \mathbf{e}_i'\mathbf{B}\mathbf{e}_i$.)

THEOREM 1.2. *In model* (1.1) *the random variable* $\mathbf{b}'\mathbf{Y} + \mathbf{Y}'\mathbf{B}\mathbf{Y}$ *belongs to* $\mathcal{O}_{lq}$ *if and only if* $\mathbf{b} \in \operatorname{Ker} \mathbf{X}' = \{\mathbf{u} \in \mathbb{R}^n : \ \mathbf{X}'\mathbf{u} = \mathbf{O}\}$ *and* $\mathbf{B} \in \mathcal{D}$.

Proof. In model (1.1) there holds

$$\begin{aligned}\mathcal{E}_{\beta}(\mathbf{b}'\mathbf{Y} + \mathbf{Y}'\mathbf{B}\mathbf{Y}) &= \mathbf{b}'\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{B}\mathbf{X}\boldsymbol{\beta} + \operatorname{Tr}\mathbf{B}\boldsymbol{\Sigma} \\ &= \mathbf{b}'\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{B}\mathbf{X}\boldsymbol{\beta} + \sigma^2 \operatorname{Tr}\mathbf{B}\boldsymbol{\Sigma}(\boldsymbol{\beta}) .\end{aligned} \tag{1.5}$$

Let $\mathbf{b}'\mathbf{Y} + \mathbf{Y}'\mathbf{B}\mathbf{Y} \in \mathcal{O}_{lq}$. We see from (1.5) that for $\boldsymbol{\beta} = \mathbf{O}$

$$\sigma^2 \operatorname{Tr}\mathbf{B}\boldsymbol{\Sigma}(\mathbf{O}) = \sigma^2 \operatorname{Tr}\mathbf{B}(a^2\mathbf{I}) = \sigma^2 a^2 \operatorname{Tr}\mathbf{B} = 0 ,$$

i.e.

$$\operatorname{Tr}\mathbf{B} = 0 . \tag{1.6}$$

If we choose $\boldsymbol{\beta} = \alpha\mathbf{e}_i$ then for $i = 1,2,\dots,n$ we get from (1.5) and (1.6)

$$\begin{aligned}&\alpha\mathbf{b}'\mathbf{X}\mathbf{e}_i + \alpha^2\mathbf{e}_i\mathbf{X}'\mathbf{B}\mathbf{X}\mathbf{e}_i + \sigma^2 a^2 \operatorname{Tr}\mathbf{B} \\ &\quad + 2\sigma^2 ab \operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\mathbf{e}_i\alpha| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\mathbf{e}_i\alpha| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\mathbf{e}_i\alpha| \end{pmatrix} \\ &\quad + \sigma^2 b^2 \operatorname{Tr}\mathbf{B}\begin{pmatrix} (\mathbf{e}_1'\mathbf{X}\mathbf{e}_i\alpha)^2 & 0 & \dots & 0 \\ 0 & (\mathbf{e}_2'\mathbf{X}\mathbf{e}_i\alpha)^2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & (\mathbf{e}_n'\mathbf{X}\mathbf{e}_i\alpha)^2 \end{pmatrix} \\ &= \alpha\mathbf{b}'\mathbf{X}\mathbf{e}_i + \alpha^2\mathbf{e}_i'\mathbf{X}'\mathbf{B}\mathbf{X}\mathbf{e}_i + 2|\alpha|\sigma^2 ab \operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\mathbf{e}_i| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\mathbf{e}_i| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\mathbf{e}_i| \end{pmatrix} \\ &\quad + \alpha^2\sigma^2 b^2 \operatorname{Tr}\mathbf{B}\begin{pmatrix} (\mathbf{e}_1'\mathbf{X}\mathbf{e}_i)^2 & 0 & \dots & 0 \\ 0 & (\mathbf{e}_2'\mathbf{X}\mathbf{e}_i)^2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & (\mathbf{e}_n'\mathbf{X}\mathbf{e}_i)^2 \end{pmatrix} = 0 \qquad \forall\{\alpha \in \mathbb{R}\} .\end{aligned}$$

That is why for $i = 1,2,\dots,k$

$$\alpha\mathbf{b}'\mathbf{X}\mathbf{e}_i + 2\alpha\sigma^2 ab \operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\mathbf{e}_i| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\mathbf{e}_i| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\mathbf{e}_i| \end{pmatrix} = 0 \qquad \forall\{\alpha > 0\} \tag{1.7a}$$

and

$$\alpha \mathbf{b}'\mathbf{X}\mathbf{e}_i - 2\alpha\sigma^2 ab \operatorname{Tr} \mathbf{B} \begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\mathbf{e}_i| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\mathbf{e}_i| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\mathbf{e}_i| \end{pmatrix} = 0 \qquad \forall\{\alpha < 0\} \tag{1.7b}$$

hold.

If for $i \in \{1, 2, \dots, k\}$

$$\operatorname{Tr} \mathbf{B} \begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\mathbf{e}_i| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\mathbf{e}_i| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\mathbf{e}_i| \end{pmatrix} \neq 0,$$

then from (1.7a)

$$\mathbf{b}'\mathbf{X}\mathbf{e}_i = -2\sigma^2 ab \operatorname{Tr} \mathbf{B} \begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\mathbf{e}_i| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\mathbf{e}_i| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\mathbf{e}_i| \end{pmatrix}$$

and from (1.7b)

$$\mathbf{b}'\mathbf{X}\mathbf{e}_i = 2\sigma^2 ab \operatorname{Tr} \mathbf{B} \begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\mathbf{e}_i| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\mathbf{e}_i| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\mathbf{e}_i| \end{pmatrix}.$$

Thus for such i

$$\mathbf{b}'\mathbf{X}\mathbf{e}_i = 0$$

holds.

If for $i \in \{1, 2, \dots, k\}$

$$\operatorname{Tr} \mathbf{B} \begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\mathbf{e}_i| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\mathbf{e}_i| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\mathbf{e}_i| \end{pmatrix} = 0,$$

then for such i for all $\alpha > 0$

$$\mathbf{b}'\mathbf{X}\mathbf{e}_i = 0\,,$$

i.e. for such i

$$\mathbf{b}'\mathbf{X}\mathbf{e}_i = 0$$

is valid.

We have obtained

$$\mathbf{b}'\mathbf{X} = \mathbf{O}\,,$$

or, equivalently,

$$\mathbf{b} \in \operatorname{Ker}\mathbf{X}'\,. \tag{1.8}$$

If we take into consideration (1.6) and (1.8), we have for $\mathbf{b}'\mathbf{Y} + \mathbf{Y}'\mathbf{B}\mathbf{Y} \in \mathcal{O}_{lq}$ according to (1.5) the relation

$$\begin{aligned}
&\mathcal{E}_{\boldsymbol\beta}(\mathbf{b}'\mathbf{Y} + \mathbf{Y}'\mathbf{B}\mathbf{Y}) \\
&= \mathbf{b}'\mathbf{X}\boldsymbol\beta + \boldsymbol\beta'\mathbf{X}'\mathbf{B}\mathbf{X}\boldsymbol\beta + 2ab\sigma^2 \operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\boldsymbol\beta| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\boldsymbol\beta| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\boldsymbol\beta| \end{pmatrix} \\
&\qquad + b^2\sigma^2 \operatorname{Tr}\mathbf{B}\begin{pmatrix} \mathbf{e}_1'\mathbf{X}\boldsymbol\beta\boldsymbol\beta'\mathbf{X}\mathbf{e}_1 & 0 & \dots & 0 \\ 0 & \mathbf{e}_2'\mathbf{X}\boldsymbol\beta\boldsymbol\beta'\mathbf{X}'\mathbf{e}_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \mathbf{e}_n'\mathbf{X}\boldsymbol\beta\boldsymbol\beta'\mathbf{X}'\mathbf{e}_n \end{pmatrix} \\
&= \boldsymbol\beta'\mathbf{X}'\Big(\mathbf{B} + \sigma^2 b^2 \sum_{i=1}^n \mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X}\boldsymbol\beta \\
&\qquad + 2ab\sigma^2 \operatorname{Tr}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\boldsymbol\beta| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\boldsymbol\beta| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\boldsymbol\beta| \end{pmatrix} = 0 \qquad \forall\{\boldsymbol\beta \in \mathbb{R}^k\}\,,
\end{aligned}$$

or, equivalently,

$$\boldsymbol\beta'\mathbf{X}'\Big(\mathbf{B} + \sigma^2 b^2 \sum_{i=1}^n \mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X}\boldsymbol\beta = 0 \qquad \forall\{\boldsymbol\beta \in \mathbb{R}^k\} \tag{1.9}$$

and

$$\operatorname{Tr}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\boldsymbol\beta| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\boldsymbol\beta| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\boldsymbol\beta| \end{pmatrix} = 0 \qquad \forall\{\boldsymbol\beta \in \mathbb{R}^k\}\,. \tag{1.10}$$

The relation (1.9) is valid if and only if

$$\mathbf{X}'\Big(\mathbf{B}+\mathbf{B}'+2\sigma^2 b^2\sum_{i=1}^{n}\mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X}=\mathbf{O}\,. \tag{1.11}$$

If $\mathbf{b}'\mathbf{Y}+\mathbf{Y}'\mathbf{B}\mathbf{Y}\in\mathcal{O}_{lq}$, then for $\mathbf{B}$ the relations (1.6), (1,10) and (1.11) and for $\mathbf{b}$ the relation (1.8) hold. It is easy to see that conditions (1.6), (1.10), (1.11) and (1.8) are also sufficient for $\mathbf{B}$ and $\mathbf{b}$ in order to $\mathbf{b}'\mathbf{Y}+\mathbf{Y}'\mathbf{B}\mathbf{Y}$ belongs to $\mathcal{O}_{lq}$.

Because of the equality

$$\mathbf{Y}'\mathbf{B}\mathbf{Y}=\mathbf{Y}\frac{\mathbf{B}+\mathbf{B}'}{2}\mathbf{Y}\,,$$

which is valid for any $n\times n$ matrix $\mathbf{B}$, we obtain that

$$\begin{aligned}
&\left\{\mathbf{b}'\mathbf{Y}+\mathbf{Y}'\mathbf{B}\mathbf{Y}:\ \operatorname{Tr}\mathbf{B}\begin{pmatrix}|\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0\\ 0 & |\mathbf{e}_2'\mathbf{X}\boldsymbol{\beta}| & \dots & 0\\ \vdots & & \ddots & \vdots\\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}|\end{pmatrix}=0\quad\forall\{\boldsymbol{\beta}\in\mathbb{R}^k\}\,,\right.\\
&\qquad\left.\operatorname{Tr}\mathbf{B}=0\,,\quad \mathbf{X}'\Big(\mathbf{B}+\mathbf{B}'+2\sigma^2 b^2\sum_{i=1}^{n}\mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X}=\mathbf{O}\,,\quad \mathbf{b}\in\operatorname{Ker}\mathbf{X}'\right\}\\
&=\left\{\mathbf{b}'\mathbf{Y}+\mathbf{Y}'\mathbf{D}\mathbf{Y}:\ \operatorname{Tr}\mathbf{D}\begin{pmatrix}|\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0\\ 0 & |\mathbf{e}_2'\mathbf{X}\boldsymbol{\beta}| & \dots & 0\\ \vdots & & \ddots & \vdots\\ 0 & & \dots & |\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}|\end{pmatrix}=0\quad\forall\{\boldsymbol{\beta}\in\mathbb{R}^k\}\,,\right.\\
&\qquad\left.\operatorname{Tr}\mathbf{D}=0\,,\quad \mathbf{X}'\Big(\mathbf{D}+2\sigma^2 b^2\sum_{i=1}^{n}\mathbf{e}_i\mathbf{e}_i'\mathbf{D}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X}=\mathbf{O}\,,\quad \mathbf{b}\in\operatorname{Ker}\mathbf{X}'\right\}\\
&=\{\mathbf{b}'\mathbf{Y}+\mathbf{Y}'\mathbf{D}\mathbf{Y}:\ \mathbf{b}\in\operatorname{Ker}\mathbf{X}',\quad \mathbf{D}\in\mathcal{D}\}\,.
\end{aligned}$$

(We only remark that here $\mathbf{D}$ need not be a symmetric matric.) The theorem is proved.

THEOREM 1.3. *In model* (1.1) *the random variable* $\mathbf{a}'\mathbf{Y}+\mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE of its mean value (in the class of linear-quadratic estimators) if and only if there exists a vector* $\mathbf{z}\in\mathbb{R}^n$ *such that*

$$\mathbf{a}=-(\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ+(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z} \tag{1.12}$$

and

$$\forall\{\mathbf{D}\in\mathcal{D}\}$$
$$\mathrm{Tr}(\mathbf{D}+\mathbf{D}')\{\sigma^2\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)+2\mathbf{X}\boldsymbol{\beta}_\circ\mathbf{z}'\mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\big]'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\}=0\,. \tag{1.13}$$

($(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}$ *is an arbitrary but fixed minimum* $\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)$*-norm g-inverse of the matrix* $\mathbf{X}'$*, i.e., a matrix satisfying the relations* $\mathbf{X}'(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'=\mathbf{X}'$ *and* $\big((\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\big)'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)=\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'$ *).*

P r o o f . According to Theorem 1.1 $\mathbf{a}'\mathbf{Y}+\mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the $\boldsymbol{\beta}_\circ$-LBLQUE of its mean value if and only if

$$\forall\{\mathbf{b}'\mathbf{Y}+\mathbf{Y}'\mathbf{D}\mathbf{Y}\in\mathcal{O}_{lq}\}$$
$$\begin{aligned}&\mathcal{E}_{\boldsymbol{\beta}_\circ}\{(\mathbf{b}'\mathbf{Y}+\mathbf{Y}'\mathbf{D}\mathbf{Y})(\mathbf{a}'\mathbf{Y}+\mathbf{Y}'\mathbf{A}\mathbf{Y})\}\\&=\sigma^2\Big\{\mathrm{Tr}(\mathbf{D}+\mathbf{D}')\Big[\frac{\sigma^2}{2}\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)+\mathbf{X}\boldsymbol{\beta}_\circ\big((\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ+\mathbf{a}\big)'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\Big]\\&\qquad+\big((\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ+\mathbf{a}\big)'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\mathbf{b}\Big\}=0\,.\end{aligned} \tag{1.14}$$

By Theorem 1.2, for all $\mathbf{b}\in\mathrm{Ker}\,\mathbf{X}'$ and $\mathbf{D}=\mathbf{O}\in\mathcal{D}$, $\mathbf{b}'\mathbf{Y}+\mathbf{Y}'\mathbf{D}\mathbf{Y}$ belongs to $\mathcal{O}_{lq}$ and we have by (1.14) that

$$\forall\{\mathbf{b}\in\mathrm{Ker}\,\mathbf{X}'\}\qquad\big((\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ+\mathbf{a}\big)'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\mathbf{b}=0\,,$$

or, equivalently,

$$\forall\{\boldsymbol{\xi}\in\mathbb{R}^n\}\quad\big((\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ+\mathbf{a}\big)'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big(\mathbf{I}-(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\big)\xi=0\,, \tag{1.15}$$

where $(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}$ is an arbitrary but fixed minimum $\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)$-norm g-inverse of the matrix $\mathbf{X}'$.

By (1.15) it follows that

$$\big((\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ+\mathbf{a}\big)'\big(\mathbf{I}-(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\big)'=\mathbf{O}$$

and therefore

$$\big((\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ+\mathbf{a}\big)\in\mathrm{Ker}\,\big(\mathbf{I}-(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\big)=\mu\big((\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\big)\,, \tag{1.16}$$

where $\mu(\mathbf{Z}_{n,k})=\{\mathbf{Z}\mathbf{u}:\ \mathbf{u}\in\mathbb{R}^k\}$.

The relation (1.16) holds if and only if there exists a vector $\mathbf{z} \in \mathbb{R}^n$ that

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ + (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z}\,.$$

The necessity of (1.12) is proved.

If in (1.14) we choose $\mathbf{b} = \mathbf{O} \in \operatorname{Ker}\mathbf{X}'$, then according to Theorem 1.2 and (1.12) we obtain (1.13). The necessary condition of the theorem is proved.

The sufficiency of (1.12) and (1.13) is easily obtained from (1.14). The theorem is proved.

For our investigations we still need the following lemma.

Lemma 1.4. *For arbitrary matrices* $\mathbf{A}$, $\mathbf{X}$ *and vector* $\boldsymbol{\beta}_\circ$ *there holds*

$$\begin{aligned} &\exists\{\mathbf{z} \in \mathbb{R}^n\} \quad -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ + (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z} \in \operatorname{Ker}\mathbf{X}' \\ \iff &\exists\{\mathbf{z} \in \mathbb{R}^n\} \quad \big(\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\big)\big(-(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ\big) \\ &\qquad = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ + (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z}\,. \end{aligned}$$

Proof.

$$\begin{aligned} &\exists\{\mathbf{z} \in \mathbb{R}^n\} \quad -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ + (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z} \in \operatorname{Ker}\mathbf{X}' \\ \iff &\exists\{\mathbf{z} \in \mathbb{R}^n\} \quad \mathbf{X}'\big(-(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ + (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z}\big) = \mathbf{O} \\ \iff &\exists\{\mathbf{z} \in \mathbb{R}^n\} \quad (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ = (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z} \\ \iff &\exists\{\mathbf{z} \in \mathbb{R}^n\} \quad -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ + (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z} \\ &\qquad = \big(\mathbf{I} - (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\big)\big(-(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ\big)\,. \end{aligned}$$

The lemma is proved.

2. Case $R(\mathbf{X}) = n \leqq k$

Lemma 2.1. *If in model* (1.1) $R(\mathbf{X}) = n \leqq k$, *then the random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is an unbiased estimator of the functional* $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ *of parameter* $\boldsymbol{\beta}$ ($j \in \{1, 2, \dots, n\}$) *if and only if*

$$\mathbf{a} \in \operatorname{Ker}\mathbf{X}'\,, \tag{2.1}$$

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \quad \textit{for all} \quad i \in \big\{\{1, 2, \dots, n\} - \{j\}\big\}\,, \qquad \mathbf{e}_j'\mathbf{A}\mathbf{e}_j = 1\,, \tag{2.2}$$

and

$$\mathbf{X}'(\mathbf{A}+\mathbf{A}')\mathbf{X}=\mathbf{O}\,. \tag{2.3}$$

P r o o f . The random variable $\mathbf{a}'\mathbf{Y}+\mathbf{Y}'\mathbf{A}\mathbf{Y}$ is an unbiased estimator of $\sigma^2\big(a+b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ if and only if

$$\begin{aligned}&\forall\{\boldsymbol{\beta}\in\mathbb{R}^k\}\\ &\mathcal{E}_{\boldsymbol{\beta}}(\mathbf{a}'\mathbf{Y}+\mathbf{Y}'\mathbf{A}\mathbf{Y})=\mathbf{a}'\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta}+\operatorname{Tr}\mathbf{A}\boldsymbol{\Sigma}=\sigma^2\big(a+b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2\,.\end{aligned} \tag{2.4}$$

It means that for all $\boldsymbol{\beta}\in\mathbb{R}^k$

$$\begin{aligned}&\mathbf{a}'\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta}+\operatorname{Tr}\mathbf{A}\boldsymbol{\Sigma}(\boldsymbol{\beta})\\ &=\mathbf{a}'\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta}+\sigma^2a^2\operatorname{Tr}\mathbf{A}+2ab\sigma^2\sum_{i=1}^n\mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}|\\ &\qquad\qquad+b^2\sigma^2\sum_{i=1}^n\mathbf{e}_i'\mathbf{A}\mathbf{e}_i(\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta})^2\\ &=\sigma^2a^2+2ab\sigma^2|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|+b^2\sigma^2(\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta})^2\,,\end{aligned}$$

which is equivalent to the next three relations

$$\operatorname{Tr}\mathbf{A}=1\,, \tag{2.5}$$

$$\forall\{\boldsymbol{\beta}\in\mathbb{R}^k\}\qquad \mathbf{a}'\mathbf{X}\boldsymbol{\beta}+2ab\sigma^2\sum_{i=1}^n\mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}|=2ab\sigma^2|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}| \tag{2.6}$$

and

$$\forall\{\boldsymbol{\beta}\in\mathbb{R}^k\}\qquad \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta}+b^2\sigma^2\sum_{i=1}^n\mathbf{e}_i'\mathbf{A}\mathbf{e}_i(\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta})^2=b^2\sigma^2(\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta})^2\,. \tag{2.7}$$

From (2.6) we obtain that for all $\boldsymbol{\beta}\in\mathbb{R}^k$

$$\mathbf{a}'\mathbf{X}\boldsymbol{\beta}+2ab\sigma^2\sum_{i=1}^n\mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}|=2ab\sigma^2|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\,,$$

but also

$$-\mathbf{a}'\mathbf{X}\boldsymbol{\beta}+2ab\sigma^2\sum_{i=1}^n\mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}|=2ab\sigma^2|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|$$

is valid. Thus

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \mathbf{a}'\mathbf{X}\boldsymbol{\beta} = 0\,, \tag{2.8}$$

which is the relation (2.1).

With respect to (2.8) it follows from (2.6) that

$$\begin{aligned}\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad & \mathbf{e}_1'\mathbf{A}\mathbf{e}_1|\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}| + \dots + \mathbf{e}_{j-1}'\mathbf{A}\mathbf{e}_{j-1}|\mathbf{e}_{j-1}'\mathbf{X}\boldsymbol{\beta}| \\ & + (\mathbf{e}_j'\mathbf{A}\mathbf{e}_j - 1)|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}| + \mathbf{e}_{j+1}'\mathbf{A}\mathbf{e}_{j+1}|\mathbf{e}_{j+1}'\mathbf{X}\boldsymbol{\beta}| + \dots + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}| = 0\,.\end{aligned} \tag{2.9}$$

As $\mathbf{X}$ is of full rank in rows, the last relation is equivalent to the next:

$$\begin{aligned}&\forall\{\mathbf{u} = (u_1, u_2, \dots, u_{R(\mathbf{X})})' \in \mathbb{R}^{R(\mathbf{X})}\} \\ &\mathbf{e}_1'\mathbf{A}\mathbf{e}_1|u_1| + \dots + \mathbf{e}_{j-1}'\mathbf{A}\mathbf{e}_{j-1}|u_{j-1}| + (\mathbf{e}_j'\mathbf{A}\mathbf{e}_j - 1)|u_j| + \dots + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n|u_n| = 0\,,\end{aligned}$$

which is again equivalent to (2.2).

From (2.7) considering (2.2) we have that

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta} = 0\,,$$

which is equivalent to (2.3). The lemma is proved.

COROLLARY 2.2. *If in model* (1.1) *is* $R(\mathbf{X}) = n \leqq k$, *then the* $\boldsymbol{\beta}_\circ$*-LBLQUE of* $\sigma^2\big(a = b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ *for any* $j \in \{1, 2, \dots, n\}$ *does not exist.*

Proof. If $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, then it is obviously an unbiased estimator of $\sigma^2(a = b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|)^2$ and according to Lemma 2.1 $\mathbf{A}$ satisfies conditions (2.2) and (2.3). As $R(\mathbf{X}) = n \leqq k$, there exists a right inverse $\mathbf{X}_R^{-1}$ of the matrix $\mathbf{X}$ and a left inverse $\mathbf{X'}_L^{-1}$ of the matrix $\mathbf{X}'$ (see e.g. in [5, p. 19]).

From (2.3) we obtain

$$\mathbf{O} = \mathbf{X'}_L^{-1}\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X}\mathbf{X}_R^{-1} = \mathbf{A} + \mathbf{A}',$$

which is a contradiction to (2.2). So the $\boldsymbol{\beta}_\circ$-LBLQUE (even any unbiased linear-quadratic estimator) of $\sigma^2\big(a + b\,\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for any $j \in \{1, 2, \dots, n\}$ does not exist.

3. Case $\mathbf{E} = \gamma \mathbf{e}'_s$; $\gamma \neq 0$, $|\gamma| \neq 1$

LEMMA 3.1. *If in model* (1.1) $\mathbf{E} = \gamma \mathbf{e}'_s$, $\gamma \neq 0$, $|\gamma| \neq 1$ *and* $s \in \{1,2,\dots,n-1\}$, *then* $\mathbf{B} \in \mathcal{D}$ *(see* (1.2)–(1.4)*) if and only if* $\mathbf{e}'_i\mathbf{B}\mathbf{e}_i = 0$ $i = 1,2,\dots,n$ *and* $\mathbf{X}'\mathbf{B}\mathbf{X} = \mathbf{O}$ *hold.*

Proof.

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}$$

$$\operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}'_1\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0 \\ 0 & |\mathbf{e}'_2\mathbf{X}\boldsymbol{\beta}| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}'_n\mathbf{X}\boldsymbol{\beta}| \end{pmatrix} = 0$$

$$\iff \forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}$$

$$\sum_{\substack{i=1\\ i\neq s}}^{n-1} \mathbf{e}'_i\mathbf{B}\mathbf{e}_i|\mathbf{e}_i\mathbf{X}_1\boldsymbol{\beta}| + \mathbf{e}'_s\mathbf{B}\mathbf{e}_s|\mathbf{e}_s\mathbf{X}_1\boldsymbol{\beta}| + \mathbf{e}'_n\mathbf{B}\mathbf{e}_n|\gamma\mathbf{e}_n\mathbf{X}_1\boldsymbol{\beta}| = 0$$

$$\iff \forall\{\mathbf{u} = (u_1,u_2,\dots,u_{n-1})' \in \mathbb{R}^{n-1}\}$$

$$\sum_{\substack{i=1\\ i\neq s}}^{n-1} \mathbf{e}'_i\mathbf{B}\mathbf{e}_i|u_i| + \big(\mathbf{e}'_s\mathbf{B}\mathbf{e}_s + |\gamma|\mathbf{e}'_n\mathbf{B}\mathbf{e}_n\big)|u_s| = 0$$

$$\iff \mathbf{e}'_i\mathbf{B}\mathbf{e}_i \qquad \text{for} \quad i \in \big\{\{1,2,\dots,n\} - \{s,n\}\big\}$$

and

$$\mathbf{e}'_s\mathbf{B}\mathbf{e}_s + |\gamma|\mathbf{e}'_n\mathbf{B}\mathbf{e}_n = 0\,.$$

Relations

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}'_1\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0 \\ 0 & |\mathbf{e}'_2\mathbf{X}\boldsymbol{\beta}| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & |\mathbf{e}'_n\mathbf{X}\boldsymbol{\beta}| \end{pmatrix} = 0$$

and

$$\operatorname{Tr}\mathbf{B} = 0$$

imply (with respect to conditions of the lemma) that

$$\mathbf{e}'_i\mathbf{B}\mathbf{e}_i = 0 \qquad i = 1,2,\dots,n\,.$$

That is why the condition

$$\mathbf{X}'\Big(\mathbf{B} + \sigma^2 b^2 \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\Big)\mathbf{X} = \mathbf{O}$$

is of the form

$$\mathbf{X}'\mathbf{B}\mathbf{X} = \mathbf{O}$$

and the lemma is proved.

LEMMA 3.2. *If in model* (1.1) $\mathbf{E} = \gamma\mathbf{e}_s'$, $\gamma \neq 0$, $|\gamma| \neq 1$ *and* $s \in \{1,2,\dots,n-1\}$, *then*

$$\mathcal{O}_{lq} = \{\mathbf{b}'\mathbf{Y} + \mathbf{YDY} :\ \mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{O},\ \mathbf{b} \in \operatorname{Ker}\mathbf{X}',\ \mathbf{e}_i'\mathbf{D}\mathbf{e}_i = 0,\quad i = 1,2\dots,n\}.$$

Proof. The lemma is easy to proof according to Theorem 1.2 and Lemma 2.1.

LEMMA 3.3. *In model* (1.1) *with* $\mathbf{E} = \gamma\mathbf{e}_s'$, $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1,2,\dots,n-1\}$ *is* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *the* $\boldsymbol{\beta}_\circ$*-LBLQUE of its mean value if and only if* $\exists\{\mathbf{z} \in \mathbb{R}^n\}$ $\exists\gamma \in \{\mathbb{R}^{k^2+n}\}$ *that*

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ + (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z} \tag{3.1}$$

and

$$\big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A} + \mathbf{A}') = \mathcal{X}\gamma, \tag{3.2}$$

where $(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}$ *is an arbitrary but fixed minimum* $\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)$*-norm g-inverse of the matrix* $\mathbf{X}'$, $\otimes$ *means the Kronecker product* (*see e.g.* [5, p. 11]),

$$\operatorname{vec}\mathbf{A}_{n,m} = (a_{11}, a_{21}, \dots, a_{n1}, a_{12}, a_{22}, \dots, a_{n2}, \dots, a_{1m}, a_{2m}, \dots, a_{nm})'$$

and

$$\mathcal{X}_{n^2,k^2+n} = (\mathbf{X} \otimes \mathbf{X}, \mathbf{e}_1 \otimes \mathbf{e}_1, \dots, \mathbf{e}_n \otimes \mathbf{e}_n).$$

Proof. According to Theorem 1.3 $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the $\boldsymbol{\beta}_\circ$-LBLQUE of its mean value if and only if (1.12) and (1.13) hold. From Lemma 3.1 we have that if $\mathbf{E} = \gamma\mathbf{e}_s'$, $\gamma \neq 0$, $|\gamma| \neq 1$ and $s \in \{1,2,\dots,n-1\}$, then

$$\mathcal{D} = \{\mathbf{B}_{n,n} :\ \mathbf{e}_i\mathbf{B}\mathbf{e}_i = 0,\ i = 1,2,\dots,n,\ \mathbf{X}'\mathbf{B}\mathbf{X} = \mathbf{O}\}.$$

Using the formula

$$\operatorname{Tr}\mathbf{AB} = (\operatorname{vec}\mathbf{B}')'\operatorname{vec}\mathbf{A}$$

and the equivalency

$$\mathbf{ABC} = \mathbf{O} \iff \operatorname{vec} \mathbf{ABC} = (\mathbf{C}' \otimes \mathbf{A}) \operatorname{vec} \mathbf{B} = \mathbf{O}$$

we obtain that

$$\{\mathbf{D}_{n,n} :\ \mathbf{D} \in \mathcal{D}\} = \{\mathbf{D}_{n,n} :\ \operatorname{vec} \mathbf{D} \in \operatorname{Ker} \mathcal{X}'\}.$$

Thus we have the next equivalences:

$$\begin{aligned}
&\forall\{\mathbf{D} \in \mathcal{D}\}\\
&\operatorname{Tr}(\mathbf{D}+\mathbf{D}')\{\sigma^2 \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)+2\mathbf{X}\boldsymbol{\beta}_\circ \mathbf{z}'\mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\big]'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\} = 0\\
\iff &\forall\{\mathbf{D} \in \mathcal{D}\} \qquad \operatorname{Tr}(\mathbf{D}+\mathbf{D}')\big(\sigma^2 \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\\
&\qquad\qquad + 2\operatorname{Tr}(\mathbf{D}+\mathbf{D}')\big(\mathbf{X}\boldsymbol{\beta}_\circ \mathbf{z}'\mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\big]'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\\
&\qquad = \operatorname{Tr}(\mathbf{D}+\mathbf{D}')\big(\sigma^2 \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big) = 0\\
&(\text{because of } \ \mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{X}'\mathbf{D}'\mathbf{X} = \mathbf{O})\\
\iff &\forall\{\mathbf{D} \in \mathcal{D}\}\\
&\sigma^2 \operatorname{Tr} \mathbf{D}\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) + \sigma^2 \operatorname{Tr} \mathbf{D}'\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\\
&= 2\sigma^2 \operatorname{Tr} \mathbf{D}\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) = 0\\
\iff &\forall\{\mathbf{D} \in \mathcal{D}\} \qquad \sigma^2 \operatorname{Tr} \mathbf{D}\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) = 0\\
\iff &\forall\{\mathbf{D} \in \mathcal{D}\} \qquad \big[\operatorname{vec} \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)(\mathbf{A}+\mathbf{A}')\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big]' \operatorname{vec} \mathbf{D} = 0\\
\iff &\exists \boldsymbol{\gamma} \in \{\mathbb{R}^{k^2+n}\} \qquad \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big) \operatorname{vec}(\mathbf{A}+\mathbf{A}') = \mathcal{X}\boldsymbol{\gamma}.
\end{aligned}$$

We see that (3.1) and (3.2) are equivalent to (1.12) and (1.13). The lemma is proved.

LEMMA 3.4. *In model* (1.1) *with* $\mathbf{E} = \gamma \mathbf{e}_s'$, $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1, 2, \dots, n-1\}$ $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is an unbiased estimator of the functional* $\sigma^2\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}|\big)^2$ *if and only if*

$$\mathbf{a} \in \operatorname{Ker} \mathbf{X}', \tag{3.3}$$

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \quad \textit{for all} \quad i \in \big\{\{1, 2, \dots, n\} - \{s\}\big\}, \qquad \mathbf{e}_s'\mathbf{A}\mathbf{e}_s = 1 \tag{3.4}$$

and

$$\mathbf{X}'(\mathbf{A}+\mathbf{A}')\mathbf{X} = \mathbf{O} \tag{3.5}$$

hold.

Proof. If we follow the proof of Lemma 2.1, we obtain that $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is an unbiased estimator of $\sigma^2\big(a + b|\mathbf{e}_s'\mathbf{X}\beta|\big)^2$ if and only if

$$\operatorname{Tr}\mathbf{A} = 1 \tag{3.6}$$

(see (2.5)),

$$\mathbf{a} \in \operatorname{Ker}\mathbf{X}', \tag{3.7}$$

$$\begin{aligned}\forall\{\beta \in \mathbb{R}^k\} \qquad & \mathbf{e}_1'\mathbf{A}\mathbf{e}_1|\mathbf{e}_1'\mathbf{X}\beta| + \cdots + \mathbf{e}_{s-1}'\mathbf{A}\mathbf{e}_{s-1}|\mathbf{e}_{s-1}'\mathbf{X}\beta| \\ &+ (\mathbf{e}_s'\mathbf{A}\mathbf{e}_s - 1)|\mathbf{e}_s'\mathbf{X}\beta| + \mathbf{e}_{s+1}'\mathbf{A}\mathbf{e}_{s+1}|\mathbf{e}_{s+1}'\mathbf{X}\beta| + \cdots + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n|\mathbf{e}_n'\mathbf{X}\beta| = 0\end{aligned} \tag{3.8}$$

(see (2.9)) and

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta} + b^2\sigma^2\sum_{i=1}^{n}\mathbf{e}_i'\mathbf{A}\mathbf{e}_i(\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta})^2 = b^2\sigma^2(\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta})^2 \tag{3.9}$$

(see (2.7)) hold.

In the same way as in the proof of Lemma 3.1 we obtain that (3.8) is equivalent to the next two conditions:

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad i = 1, 2, \dots, s-1, s+1, \dots, n-1 \tag{3.10}$$

and

$$(\mathbf{e}_s'\mathbf{A}\mathbf{e}_s - 1) + |\gamma|\mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 0\,. \tag{3.11}$$

Condition (3.6) together with (3.10) and (3.11) under the conditions of lemma are equivalent to (3.4).

Relation (3.9) is now of the form

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta} = 0\,,$$

which is equivalent to (3.5). The lemma is proved.

LEMMA 3.5. *In model* (1.1) *with* $\mathbf{E} = \gamma\mathbf{e}_s'$, $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1, 2, \dots, n-1\}$ *is* $\mathbf{a}'\mathbf{Y}+\mathbf{Y}'\mathbf{A}\mathbf{Y}$ *an unbiased estimator of the functional* $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, *where* $j \in \big\{\{1, 2, \dots, n\} - \{s\}\big\}$, *if and only if*

$$\mathbf{a} \in \operatorname{Ker}\mathbf{X}', \tag{3.12}$$

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \quad \textit{for all} \quad i \in \big\{\{1,2,\dots,n\} - \{j\}\big\}, \qquad \mathbf{e}_j'\mathbf{A}\mathbf{e}_j = 1 \tag{3.13}$$

and

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O} \tag{3.14}$$

hold.

Proof.

(i) Let $j \in \big\{\{1,2,\dots,n-1\} - \{s\}\big\}$

Following the proof of Lemma 2.1 we obtain that $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is an unbiased estimator of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ if and only if

$$\operatorname{Tr}\mathbf{A} = 1\,, \tag{3.15}$$

$$\mathbf{a} \in \operatorname{Ker}\mathbf{X}', \tag{3.16}$$

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \sum_{\substack{i=1\\ i\notin\{s,j\}}}^{n-1} \mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i\mathbf{X}_1\boldsymbol{\beta}| + (\mathbf{e}_j'\mathbf{A}\mathbf{e}_j - 1)|\mathbf{e}_j\mathbf{X}_1\boldsymbol{\beta}|$$
$$+ \big(\mathbf{e}_s'\mathbf{A}\mathbf{e}_s + |\gamma|\mathbf{e}_n\mathbf{A}\mathbf{e}_n\big)|\mathbf{e}_s'\mathbf{X}_1\boldsymbol{\beta}| = 0 \tag{3.17}$$

and

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta} + b^2\sigma^2\sum_{i=1}^{n}\mathbf{e}_i'\mathbf{A}\mathbf{e}_i(\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta})^2 = b^2\sigma^2(\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta})^2 \tag{3.18}$$

hold.

In the same way as in the proof of Lemma 3.1 we obtain that (3.17) is equivalent to the next three conditions:

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad i \in \big\{\{1,2,\dots,n\} - \{s,j,n\}\big\}, \tag{3.19}$$

$$\mathbf{e}_j'\mathbf{A}\mathbf{e}_j = 0 \tag{3.20}$$

and

$$\mathbf{e}_s'\mathbf{A}\mathbf{e}_s + |\gamma|\mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 0\,. \tag{3.21}$$

These three conditions together with (3.15) are equivalent to

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad i \in \big\{\{1,2,\dots,n\} - \{s,j,n\}\big\}\,,$$
$$\mathbf{e}_s'\mathbf{A}\mathbf{e}_s + \mathbf{e}_j'\mathbf{A}\mathbf{e}_j + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 1\,,$$
$$\mathbf{e}_j'\mathbf{A}\mathbf{e}_j = 1\,, \mathbf{e}_s'\mathbf{A}\mathbf{e}_s + |\gamma|\mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 0\,,$$

which is again equivalent to the next two conditions

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad i \in \big\{\{1,2,\dots,n\} - \{j\}\big\}, \tag{3.22}$$

$$\mathbf{e}_j'\mathbf{A}\mathbf{e}_j = 1 . \tag{3.23}$$

That is why relation (3.18) is now of the form

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta} = 0 ,$$

which is equivalent to (3.14). Case (i) is proved.

(ii) Let $j = n$.

Instead of (3.17) we have

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \sum_{\substack{i=1\\ i\neq s}}^{n-1} \mathbf{e}_i'\mathbf{A}\mathbf{e}_i|\mathbf{e}_i\mathbf{X}_1\boldsymbol{\beta}| + \big(\mathbf{e}_s'\mathbf{A}\mathbf{e}_s - |\gamma| + |\gamma|\mathbf{e}_n'\mathbf{A}\mathbf{e}_n\big)|\mathbf{e}_s\mathbf{X}_1\boldsymbol{\beta}| = 0 ,$$

which is equivalent to

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad i \in \big\{\{1,2,\dots,n\} - \{s,n\}\big\}$$

and

$$\mathbf{e}_s'\mathbf{A}\mathbf{e}_s - |\gamma| + |\gamma|\mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 0 .$$

The last equations together with (3.15) are equivalent to

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad i \in \{1,2,\dots,n-1\}$$

and

$$\mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 1 .$$

The lemma is proved.

Let us now investigate the problem of determining the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}|\big)^2$ of $\boldsymbol{\beta}$.

The random variable $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the $\boldsymbol{\beta}_\circ$-LBLQUE of its mean value if and only if (3.1) and (3.2) are valid (Lemma 3.3) and is an unbiased estimator of the functional $\sigma^2\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}|\big)^2$ if and only if (3.3), (3.4) and (3.5) hold (Lemma 3.4).

According to Lemma 1.4, relations (3.1), (3.3) and (3.5) are equivalent to $\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ$ and (3.5).

Similarly we investigate the problem of determining the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \in \big\{\{1,2,\dots,n\} - \{s\}\big\}$.

The random variable $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the $\boldsymbol{\beta}_\circ$-LBLQUE of its mean value if and only if (3.1) and (3.2) are valid (Lemma 3.3) and is an unbiased estimator of the functional $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \in \big\{\{1,2,\dots,n\} - \{s\}\big\}$ if an only if (3.12), (3.13) and (3.14) hold (Lemma 3.5).

According to Lemma 1.4, relations (3.1), (3.3) and (3.14) are equivalent to $\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ$ and (3.14).

We have outlined the proof of the next lemma.

Lemma 3.6. *In model* (1.1) *with* $\mathbf{E} = \gamma\mathbf{e}_s'$, $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1,2,\dots,n-1\}$ *is* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *the* $\boldsymbol{\beta}_\circ$*-LBLQUE of the functional* $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, *where* $j \in \{1,2,\dots,n\}$ *if and only if*

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ\,, \tag{3.25}$$

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \quad i \in \big\{\{1,2,\dots,n\} - \{j\}\big\}\,, \qquad \mathbf{e}_j'\mathbf{A}\mathbf{e}_j = 1\,, \tag{3.26}$$

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O} \tag{3.27}$$

and

$$\exists\gamma \in \{\mathbb{R}^{k^2+n}\} \qquad \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A} + \mathbf{A}') = \mathcal{X}\gamma \tag{3.28}$$

hold.

Let us denote $\mathbf{I}^\star$ the nonsingular matrix of order $n^2 \times n^2$ for which the next assertion is valid:

$$\forall\{\mathbf{A}_{n,n}\} \qquad \mathbf{I}^\star \operatorname{vec}\mathbf{A} = \operatorname{vec}\mathbf{A}'.$$

For our further investigations we need the next three lemmas.

Lemma 3.7.

$$\mu(\mathbf{I}^\star\mathcal{X}) = \mu(\mathcal{X}).$$

P r o o f. It is obvious that $\operatorname{Ker}\mathcal{X}' = \operatorname{Ker}\mathcal{X}'\mathbf{I}^\star$ and $(\mathbf{I}^\star)' = \mathbf{I}^\star$. It is equivalent to the statement of the lemma.

Lemma 3.8.

$$\mathbf{I}^\star\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big) = \big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbf{I}^\star.$$

P r o o f. It is obvious that

$$\mathbf{I}^\star(\mathbf{e}_i \otimes \mathbf{e}_j) = (\mathbf{e}_j \otimes \mathbf{e}_i)\,.$$

That is why for i, j, k and l belonging to $\{1, 2, \dots, n\}$ there is

$$
\begin{aligned}
&(\mathbf{e}_i' \otimes \mathbf{e}_j')\mathbf{I}^\star\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{e}_k \otimes \mathbf{e}_l) \\
&\quad = \frac{1}{\big(a + b|\mathbf{e}_k'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\big(a + b|\mathbf{e}_l'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2}(\mathbf{e}_i' \otimes \mathbf{e}_j')\mathbf{I}^\star(\mathbf{e}_k \otimes \mathbf{e}_l) \\
&\quad = \frac{1}{\big(a + b|\mathbf{e}_k'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\big(a + b|\mathbf{e}_l'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2}(\mathbf{e}_i' \otimes \mathbf{e}_j')(\mathbf{e}_l \otimes \mathbf{e}_k) \\
&\quad = \frac{\delta_{il}\delta_{jk}}{\big(a + b|\mathbf{e}_k'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\big(a + b|\mathbf{e}_l'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} \\
&\quad = \frac{1}{\big(a + b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2}(\mathbf{e}_i' \otimes \mathbf{e}_j')(\mathbf{e}_l \otimes \mathbf{e}_k) \\
&\quad = \frac{1}{\big(a + b|\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2}(\mathbf{e}_i' \otimes \mathbf{e}_j')\mathbf{I}^\star(\mathbf{e}_k \otimes \mathbf{e}_l) \\
&\quad = (\mathbf{e}_i' \otimes \mathbf{e}_j')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathbf{I}^\star(\mathbf{e}_k \otimes \mathbf{e}_l).
\end{aligned}
$$

(We note that $\delta_{il} = 0$ for $i \neq l$ and $\delta_{il} = 1$ for $i = l$.) The lemma is proved.

LEMMA 3.9. *Let $j \in \{1, 2, \dots, n\}$. There exists an $n \times n$ matrix $\mathbf{A}$ with the next four properties*

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad \textit{for all} \quad i \in \big\{\{1, 2, \dots, n\} - \{j\}\big\}, \tag{3.29}$$

$$\mathbf{e}_j'\mathbf{A}\mathbf{e}_j = 1, \tag{3.30}$$

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O}, \tag{3.31}$$

$$\exists \gamma \in \{\mathbb{R}^{k^2+n}\} \qquad \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A} + \mathbf{A}') = \mathcal{X}\gamma \tag{3.32}$$

if and only if $\exists \boldsymbol{\delta} \in \{\mathbb{R}^{k^2+n}\}$

$$(\mathbf{X}' \otimes \mathbf{X}')(\mathbf{I} + \mathbf{I}^\star)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta} = \mathbf{O}, \tag{3.33}$$

$$(\mathbf{e}_i' \otimes \mathbf{e}_i')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta} = 0 \qquad \textit{for all} \quad i \in \big\{\{1, 2, \dots, n\} - \{j\}\big\}, \tag{3.34}$$

$$(\mathbf{e}_j' \otimes \mathbf{e}_j')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta} = 2. \tag{3.35}$$

P r o o f . From (3.32) we obtain

$$\operatorname{vec}(\mathbf{A} + \mathbf{A}') = \big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\gamma, \tag{3.36}$$

but that implies

$$(\mathbf{I}+\mathbf{I}^{\star})\operatorname{vec}(\mathbf{A}+\mathbf{A}') = 2\operatorname{vec}(\mathbf{A}+\mathbf{A}') = (\mathbf{I}+\mathbf{I}^{\star})\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big)\mathcal{X}\boldsymbol{\gamma}\,. \tag{3.37}$$

Relations (3.31) and (3.37) imply

$$\mathbf{O} = (\mathbf{X}'\otimes\mathbf{X}')2\operatorname{vec}(\mathbf{A}+\mathbf{A}') = (\mathbf{X}'\otimes\mathbf{X}')(\mathbf{I}+\mathbf{I}^{\star})\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big)\mathcal{X}\boldsymbol{\gamma}\,,$$

i.e. (3.33).

From (3.29) and (3.36) we have for $i\in\big\{\{1,2,\dots,n\}-\{j\}\big\}$

$$(\mathbf{e}_i'\otimes\mathbf{e}_i')\operatorname{vec}(\mathbf{A}+\mathbf{A}') = (\mathbf{e}_i'\otimes\mathbf{e}_i')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big)\mathcal{X}\boldsymbol{\gamma} = 0\,,$$

which is (3.34).

From (3.30) and (3.36) we obtain

$$2 = (\mathbf{e}_j'\otimes\mathbf{e}_j')\operatorname{vec}(\mathbf{A}+\mathbf{A}') = (\mathbf{e}_j'\otimes\mathbf{e}_j')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big)\mathcal{X}\boldsymbol{\gamma}\,,$$

which is (3.35).

Now let us denote

$$\operatorname{vec}\mathbf{A} = \big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big)\mathcal{X}\frac{\boldsymbol{\delta}}{2}\,. \tag{3.38}$$

From (3.35) and (3.38) we have

$$1 = (\mathbf{e}_j'\otimes\mathbf{e}_j')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big)\mathcal{X}\frac{\boldsymbol{\delta}}{2} = (\mathbf{e}_j'\otimes\mathbf{e}_j')\operatorname{vec}\mathbf{A} = \mathbf{e}_j'\mathbf{A}\mathbf{e}_j\,,$$

which is (3.30).

Relations (3.34) and (3.38) imply for $i\in\big\{\{1,2,\dots,n\}-\{j\}\big\}$

$$0 = (\mathbf{e}_i'\otimes\mathbf{e}_i')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big)\mathcal{X}\frac{\boldsymbol{\delta}}{2} = (\mathbf{e}_i'\otimes\mathbf{e}_i')\operatorname{vec}\mathbf{A} = \mathbf{e}_i'\mathbf{A}\mathbf{e}_i\,,$$

which is (3.29).

From (3.33) and (3.38) we have

$$\begin{aligned}\mathbf{O} &= (\mathbf{X}'\otimes\mathbf{X}')(\mathbf{I}+\mathbf{I}^{\star})\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_{\circ})\big)\mathcal{X}\boldsymbol{\delta}\\ &= (\mathbf{X}'\otimes\mathbf{X}')(\mathbf{I}+\mathbf{I}^{\star})2\operatorname{vec}\mathbf{A} = (\mathbf{X}'\otimes\mathbf{X}')2\operatorname{vec}(\mathbf{A}+\mathbf{A}')\,,\end{aligned}$$

which implies

$$\mathbf{O} = (\mathbf{X}' \otimes \mathbf{X}')\,\mathrm{vec}(\mathbf{A} + \mathbf{A}'),$$

i.e.

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = 0,$$

(relation (3.31)).

Finally, according to Lemma 3.7 there exists $\boldsymbol{\psi} \in \mathbb{R}^{k^2+n}$ that

$$\mathbf{I}^{\star}\mathcal{X}\frac{\boldsymbol{\delta}}{2} = \mathcal{X}\boldsymbol{\psi}.$$

Using (3.38) and Lemma 3.8 we obtain that

$$\begin{aligned}\exists\Big\{\boldsymbol{\gamma} = \tfrac{\boldsymbol{\delta}}{2} + \boldsymbol{\psi} \in \mathbb{R}^{k^2+n}\Big\} \qquad &\big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\,\mathrm{vec}(\mathbf{A} + \mathbf{A}') \\ &= \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I} + \mathbf{I}^{\star})\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\frac{\boldsymbol{\delta}}{2} \\ &= \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I} + \mathbf{I}^{\star})\mathcal{X}\frac{\boldsymbol{\delta}}{2} = \mathcal{X}\boldsymbol{\gamma},\end{aligned}$$

which is the relation (3.32). The lemma is proved.

We finish our considerations in this section with the next theorem.

THEOREM 3.10. *In model* (1.1) *with* $\mathbf{E} = \gamma\mathbf{e}_s'$, $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1, 2, \dots, n-1\}$ *is* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *the* $\boldsymbol{\beta}_\circ$*-LBLQUE of the functional* $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, *where* $j \in \{1, 2, \dots, n\}$ *if and only if* $\exists\boldsymbol{\delta} \in \{\mathbb{R}^{k^2+n}\}$

$$\begin{gathered}(\mathbf{X}' \otimes \mathbf{X}')(\mathbf{I} + \mathbf{I}^{\star})\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta} = \mathbf{O} \\ (\mathbf{e}_i' \otimes \mathbf{e}_i')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta} = 0 \qquad \textit{for all} \quad i \in \big\{\{1, 2, \dots, n\} - \{j\}\big\}, \\ (\mathbf{e}_j' \otimes \mathbf{e}_j')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta} = 2.\end{gathered} \tag{3.39}$$

If (3.39) *is consistent, then*

$$\mathrm{vec}\,\mathbf{A} = \frac{1}{2}\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta}$$

and

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ.$$

Proof. The assertion of the theorem is a consequence of Lemma 3.6 and Lemma 3.9 and is omitted.

We have shown that the existence of the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2(a+b|\mathbf{e}'_j\mathbf{X}\boldsymbol{\beta}|)^2$, where $j \in \{1,2,\dots,n\}$ in model (1.1) with $\mathbf{E} = \gamma\mathbf{e}'_s$, $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1,2,\dots,n-1\}$ is equivalent to the consistency of the linear system (3.39).

Remark 3.11. Equations (3.39) are consistent for $j \in \{s,n\}$ and are not consistent for $j \in \big\{\{1,2,\dots,n-1\} - \{s\}\big\}$. That is why in model (1.1) with $\mathbf{E} = \gamma\mathbf{e}'_s$, $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1,2,\dots,n-1\}$ the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2(a+b|\mathbf{e}'_j\mathbf{X}\boldsymbol{\beta}|)^2$ for $j \in \{s,n\}$ exists and the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2(a+b|\mathbf{e}'_j\mathbf{X}\boldsymbol{\beta}|)^2$ for $j \in \big\{\{1,2,\dots,n-1\} - \{s\}\big\}$ does not exist. The proof of this fact will be contained in Section 6.1.

4. Case $\mathbf{E} = \gamma\mathbf{e}'_s$, $|\gamma| = 1$

LEMMA 4.1. *If in model* (1.1) $\mathbf{E} = \gamma\mathbf{e}'_s$, $|\gamma| = 1$, $s \in \{1,2,\dots,n-1\}$, *then* $\mathbf{B} \in \mathcal{D}$ (*see* (1.2)–(1.4)) *if and only if*

$$\mathbf{e}'_i\mathbf{B}\mathbf{e}_i = 0 \qquad \textit{for} \quad i \notin \{s,n\},$$
$$\mathbf{e}'_s\mathbf{B}\mathbf{e}_s + \mathbf{e}'_n\mathbf{B}\mathbf{e}_n = 0$$

and

$$\mathbf{X}'\mathbf{B}\mathbf{X} = \mathbf{O}$$

hold.

Proof.

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}'_1\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0 \\ 0 & |\mathbf{e}'_2\mathbf{X}\boldsymbol{\beta}| & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & |\mathbf{e}'_n\mathbf{X}\boldsymbol{\beta}| \end{pmatrix} = 0$$

$$\iff \forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \sum_{\substack{i=1\\ i\neq s}}^{n-1} \mathbf{e}'_i\mathbf{B}\mathbf{e}_i|\mathbf{e}'_i\mathbf{X}_1\boldsymbol{\beta}| + \mathbf{e}'_s\mathbf{B}\mathbf{e}_s|\mathbf{e}'_s\mathbf{X}_1\boldsymbol{\beta}| + \mathbf{e}'_n\mathbf{B}\mathbf{e}_n|\gamma\mathbf{e}'_n\mathbf{X}_1\boldsymbol{\beta}| = 0$$

$$\iff \forall\{\mathbf{u} = (u_1,u_2,\dots,u_{n-1})' \in \mathbb{R}^{n-1}\}$$

$$\sum_{\substack{i=1\\ i\neq s}}^{n-1} \mathbf{e}'_i\mathbf{B}\mathbf{e}_i|u_i| + (\mathbf{e}'_s\mathbf{B}\mathbf{e}_s + \mathbf{e}'_n\mathbf{B}\mathbf{e}_n)|u_s| = 0$$

$$\iff \mathbf{e}'_i\mathbf{B}\mathbf{e}_i = 0 \qquad \text{for} \quad i \in \big\{\{1,2,\dots,n\} - \{s,n\}\big\}$$

and

$$\mathbf{e}_s'\mathbf{B}\mathbf{e}_s + \mathbf{e}_n'\mathbf{B}\mathbf{e}_n = 0\,.$$

Relations

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\boldsymbol{\beta}| & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & & |\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}| \end{pmatrix} = 0$$

and

$$\operatorname{Tr}\mathbf{B} = 0$$

imply (with respect to conditions of the lemma) that

$$\mathbf{e}_i'\mathbf{B}\mathbf{e}_i = 0 \qquad \text{for} \quad i \in \big\{\{1,2,\dots,n\} - \{s,n\}\big\}$$

and

$$\mathbf{e}_s'\mathbf{B}\mathbf{e}_s + \mathbf{e}_n'\mathbf{B}\mathbf{e}_n = 0\,.$$

The third condition for $\mathbf{B}$ to belong to $\mathcal{D}$ is now of the form

$$\mathbf{X}'\big(\mathbf{B} + \sigma^2 b^2(\mathbf{e}_s\mathbf{e}_s'\mathbf{B}\mathbf{e}_s\mathbf{e}_s' + \mathbf{e}_n\mathbf{e}_n'\mathbf{B}\mathbf{e}_n\mathbf{e}_n')\big)\mathbf{X} = \mathbf{O}\,.$$

Because of the equations

$$\begin{aligned} \mathbf{X}'\mathbf{e}_n &= \gamma\mathbf{X}'\mathbf{e}_s\,, \\ \gamma^2 &= 1 \end{aligned}$$

and

$$\mathbf{e}_n'\mathbf{B}\mathbf{e}_n = -\mathbf{e}_s'\mathbf{B}\mathbf{e}_s\,,$$

we can write

$$\begin{aligned} &\mathbf{X}'\big(\mathbf{B} + \sigma^2 b^2(\mathbf{e}_s\mathbf{e}_s'\mathbf{B}\mathbf{e}_s\mathbf{e}_s' + \mathbf{e}_n\mathbf{e}_n'\mathbf{B}\mathbf{e}_n\mathbf{e}_n')\big)\mathbf{X} \\ &\qquad = \mathbf{X}'\mathbf{B}\mathbf{X} + \sigma^2 b^2\mathbf{X}'\mathbf{e}_s\mathbf{e}_s'\mathbf{B}\mathbf{e}_s\mathbf{e}_s'\mathbf{X} - \sigma^2 b^2\gamma^2\mathbf{X}'\mathbf{e}_s\mathbf{e}_s'\mathbf{B}\mathbf{e}_s\mathbf{e}_s'\mathbf{X} = \mathbf{X}'\mathbf{B}\mathbf{X}\,. \end{aligned}$$

The lemma is proved.

LEMMA 4.2. *If in model* (1.1) $\mathbf{E} = \gamma \mathbf{e}'_s$, $|\gamma| = 1$, $s \in \{1, 2, \dots, n-1\}$, *then* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE of its mean value if and only if*

$$\exists\{\mathbf{z} \in \mathbb{R}^n\} \qquad \exists \gamma \in \{\mathbb{R}^{k^2+n-1}\}$$

that

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ + (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}\mathbf{X}'\mathbf{z} \tag{4.1}$$

and

$$\big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big) \operatorname{vec}(\mathbf{A} + \mathbf{A}') = \mathcal{X}_1 \gamma, \tag{4.2}$$

where $(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}$ *is an arbitrary but fixed minimum* $\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)$*-norm g-inverse of the matrix* $\mathbf{X}'$ *and*

$$\mathcal{X}_1 = (\mathbf{X} \otimes \mathbf{X}, \mathbf{e}_1 \otimes \mathbf{e}_1, \dots, \mathbf{e}_{s-1} \otimes \mathbf{e}_{s-1}, \mathbf{e}_{s+1} \otimes \mathbf{e}_{s+1}, \dots, \mathbf{e}_{n-1} \otimes \mathbf{e}_{n-1}, \mathbf{e}_s \otimes \mathbf{e}_s + \mathbf{e}_n \otimes \mathbf{e}_n).$$

Proof. The proof follows in the same way as the proof of Lemma 3.3 (with $\mathcal{X}_1$ instead of $\mathcal{X}$) and is omitted.

LEMMA 4.3. *If in model* (1.1) $\mathbf{E} = \gamma \mathbf{e}'_s$, $|\gamma| = 1$, $s \in \{1, 2, \dots, n-1\}$, *then* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is an unbiased estimator of the functional* $\sigma^2\big(a + b|\mathbf{e}'_s\mathbf{X}\boldsymbol{\beta}|\big)^2$ *if and only if*

$$\mathbf{a} \in \operatorname{Ker} \mathbf{X}', \tag{4.3}$$

$$\mathbf{e}'_i\mathbf{A}\mathbf{e}_i = 0 \quad \textit{for all} \quad i \in \big\{\{1, 2, \dots, n\} - \{s, n\}\big\}, \qquad \mathbf{e}'_s\mathbf{A}\mathbf{e}_s + \mathbf{e}'_n\mathbf{A}\mathbf{e}_n = 1 \tag{4.4}$$

and

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O} \tag{4.5}$$

hold.

Proof. Let us again follow the proof of Lemma 2.1. We obtain that $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is an unbiased estimator of $\sigma^2\big(a + b|\mathbf{e}'_s\mathbf{X}\boldsymbol{\beta}|\big)^2$ if and only if

$$\operatorname{Tr} \mathbf{A} = 1 \tag{4.6}$$

(see (2.5)),

$$\mathbf{a} \in \operatorname{Ker} \mathbf{X}', \tag{4.7}$$

$$\begin{aligned} \forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad & \mathbf{e}'_1\mathbf{A}\mathbf{e}_1|\mathbf{e}'_1\mathbf{X}\boldsymbol{\beta}| + \dots + \mathbf{e}'_{s-1}\mathbf{A}\mathbf{e}_{s-1}|\mathbf{e}'_{s-1}\mathbf{X}\boldsymbol{\beta}| \\ & + (\mathbf{e}'_s\mathbf{A}\mathbf{e}_s - 1)|\mathbf{e}'_s\mathbf{X}\boldsymbol{\beta}| + \mathbf{e}'_{s+1}\mathbf{A}\mathbf{e}_{s+1}|\mathbf{e}'_{s+1}\mathbf{X}\boldsymbol{\beta}| + \dots + \mathbf{e}'_n\mathbf{A}\mathbf{e}_n|\mathbf{e}'_s\mathbf{X}\boldsymbol{\beta}| = 0 \end{aligned} \tag{4.8}$$

(see (2.9)) and

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta} + b^2\sigma^2 \sum_{i=1}^{n} \mathbf{e}_i'\mathbf{A}\mathbf{e}_i(\mathbf{e}_i'\mathbf{X}\boldsymbol{\beta})^2 = b^2\sigma^2(\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta})^2 \tag{4.9}$$

(see (2.7)) hold.

In the same way as in the proof of Lemma 4.1 we obtain that (4.8) is equivalent to the condition (4.4).

Because of the equality

$$\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}'\mathbf{e}_n = \gamma^2\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}'\mathbf{e}_s\,,$$

condition (4.9) is now of the form

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \boldsymbol{\beta}'\mathbf{X}'\mathbf{A}\mathbf{X}\boldsymbol{\beta} = 0\,,$$

which is equivalent to (4.5). The lemma is proved.

Similarly we can prove the next lemma.

LEMMA 4.4. *If in model* (1.1) $\mathbf{E} = \gamma\mathbf{e}_s'$, $|\gamma| = 1$, $s \in \{1,2,\dots,n-1\}$, *then* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is an unbiased estimator of the functional* $\sigma^2\big(a + b|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}|\big)^2$ *if and only if*

$$\mathbf{a} \in \operatorname{Ker}\mathbf{X}', \tag{4.10}$$

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \quad \textit{for all} \quad i \in \big\{\{1,2,\dots,n\} - \{s,n\}\big\}\,, \qquad \mathbf{e}_s'\mathbf{A}\mathbf{e}_s + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 1 \tag{4.11}$$

and

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O} \tag{4.12}$$

hold.

LEMMA 4.5. *If in model* (1.1) $\mathbf{E} = \gamma\mathbf{e}_s'$, $|\gamma| = 1$, $s \in \{1,2,\dots,n-1\}$, *then the random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is an unbiased estimator of the functional* $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, *where* $j \in \big\{\{1,2,\dots,n-1\} - \{s\}\big\}$, *if and only if*

$$\mathbf{a} \in \operatorname{Ker}\mathbf{X}', \tag{4.13}$$

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad \textit{for all} \quad i \in \big\{\{1,2,\dots,n\} - \{j,s,n\}\big\}\,,$$
$$\mathbf{e}_j'\mathbf{A}\mathbf{e}_j = 1\,, \qquad \mathbf{e}_s'\mathbf{A}\mathbf{e}_s + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 0 \tag{4.14}$$

and

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O} \tag{4.15}$$

hold.

P r o o f . As in the proof in previous lemmas. The crucial point of the proof is the condition

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \qquad \mathbf{e}_1'\mathbf{A}\mathbf{e}_1|\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}| + \cdots + \mathbf{e}_{j-1}'\mathbf{A}\mathbf{e}_{j-1}|\mathbf{e}_{j-1}'\mathbf{X}\boldsymbol{\beta}|$$
$$+ (\mathbf{e}_j'\mathbf{A}\mathbf{e}_j - 1)|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}| + \mathbf{e}_{j+1}'\mathbf{A}\mathbf{e}_{j+1}|\mathbf{e}_{j+1}'\mathbf{X}\boldsymbol{\beta}| + \cdots + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}| = 0\,,$$

which is equivalent here to the condition (4.14). Now it is easy to finish the proof.

Let us now again investigate the problem of determining the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ of $\boldsymbol{\beta}$ for $j \in \{s, n\}$.

The random variable $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the $\boldsymbol{\beta}_\circ$-LBLQUE of its mean value if and only if (4.1) and (4.2) are valid (Lemma 4.2) and is an unbiased estimator of the functional $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \in \{s, n\}$ if and only if (4.3), (4.4) and (4.5) (i.e. (4.10), (4.11) and (4.12)) hold (Lemma 4.3 and Lemma 4.4).

According to Lemma 1.4, relations (4.1), (4.3) and (4.5) are equivalent to $a = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ$ and (4.5).

We have outlined the proof of the next lemma.

LEMMA 4.6. *If in model* (1.1) $\mathbf{E} = \gamma\mathbf{e}_s'$, $|\gamma| = 1$, $s \in \{1, 2, \ldots, n-1\}$, *then the random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE of the functional* $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ *of* $\boldsymbol{\beta}$, *where* $j \in \{s, n\}$ *if and only if*

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ\,,$$
$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \quad \textit{for all} \quad i \in \big\{\{1, 2, \ldots, n\} - \{s, n\}\big\}\,, \qquad \mathbf{e}_s'\mathbf{A}\mathbf{e}_s + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 1\,,$$
$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O}$$

and

$$\exists\boldsymbol{\gamma} \in \{\mathbb{R}^{k^2+n-1}\} \quad \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A} + \mathbf{A}') = \mathcal{X}_1\boldsymbol{\gamma}$$

hold.

Similarly we investigate the problem of determination of the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ of $\boldsymbol{\beta}$ for $j \in \big\{\{1, 2, \ldots, n\} - \{s, n\}\big\}$.

The random variable $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the $\boldsymbol{\beta}_\circ$-LBLQUE of its mean value if and only if (4.1) and (4.2) are valid (Lemma 4.2) and is an unbiased estimator of the functional $\sigma^2(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|)^2$ for $j \in \{\{1,2,\dots,n\} - \{s,n\}\}$ if an only if (4.13), (4.14) and (4.15) hold (Lemma 4.5).

According to Lemma 1.4, relations (4.1), (4.13) and (4.15) are equivalent to $a = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ$ and (4.15).

We have outlined the proof of the next lemma.

LEMMA 4.7. *If in model* (1.1) $\mathbf{E} = \gamma\mathbf{e}_s'$, $|\gamma| = 1$, $s \in \{1,2,\dots,n-1\}$, *then the random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE of the functional* $\sigma^2(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|)^2$ *of* $\boldsymbol{\beta}$, *where* $j \in \{\{1,2,\dots,n\} - \{s,n\}\}$ *if and only if*

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ\,,$$

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \quad \textit{for all} \quad i \in \{\{1,2,\dots,n\} - \{s,n\}\}\,, \qquad \mathbf{e}_s'\mathbf{A}\mathbf{e}_s + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 1\,,$$

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O}$$

and

$$\exists\gamma \in \{\mathbb{R}^{k^2+n-1}\} \qquad \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A} + \mathbf{A}') = \mathcal{X}_1\gamma$$

hold.

To obtain our final result in this section we still need the next two lemmas. Their proofs are like the proof of Lemma 3.9 and are omitted.

LEMMA 4.8. *Let* $j \in \{s,n\}$. *There exists an* $n \times n$ *matrix* $\mathbf{A}$ *with the next four properties*

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad \textit{for all} \quad i \in \{\{1,2,\dots,n\} - \{s,n\}\}\,,$$

$$\mathbf{e}_s'\mathbf{A}\mathbf{e}_s + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 1\,,$$

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O}\,,$$

$$\exists\gamma \in \{\mathbb{R}^{k^2+n-1}\} \qquad \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A} + \mathbf{A}') = \mathcal{X}_1\gamma$$

if and only if $\exists\boldsymbol{\delta} \in \{\mathbb{R}^{k^2+n-1}\}$

$$(\mathbf{X}' \otimes \mathbf{X}')(\mathbf{I} + \mathbf{I}^\star)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = \mathbf{O}\,,$$

$$(\mathbf{e}_i' \otimes \mathbf{e}_i')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 0 \qquad \textit{for all} \quad i \in \{\{1,2,\dots,n\} - \{s,n\}\},$$

$$(\mathbf{e}_s' \otimes \mathbf{e}_s' + \mathbf{e}_n' \otimes \mathbf{e}_n')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 2\,.$$

LEMMA 4.9. *Let* $j \in \{\{1,2,\dots,n\} - \{s,n\}\}$ *There exists an* $n \times n$ *matrix* $\mathbf{A}$ *with the next five properties*

$$\mathbf{e}_i'\mathbf{A}\mathbf{e}_i = 0 \qquad \text{for all} \quad i \in \{\{1,2,\dots,n\} - \{j,s,n\}\},$$

$$\mathbf{e}_j'\mathbf{A}\mathbf{e}_j = 1,$$

$$\mathbf{e}_s'\mathbf{A}\mathbf{e}_s + \mathbf{e}_n'\mathbf{A}\mathbf{e}_n = 0,$$

$$\mathbf{X}'(\mathbf{A} + \mathbf{A}')\mathbf{X} = \mathbf{O},$$

$$\exists \boldsymbol{\gamma} \in \{\mathbb{R}^{k^2+n-1}\} \qquad \big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big)\operatorname{vec}(\mathbf{A} + \mathbf{A}') = \mathcal{X}_1\boldsymbol{\gamma}$$

if and only if $\exists \boldsymbol{\delta} \in \{\mathbb{R}^{k^2+n-1}\}$

$$(\mathbf{X}' \otimes \mathbf{X}')(\mathbf{I} + \mathbf{I}^\star)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = \mathbf{O},$$

$$(\mathbf{e}_i' \otimes \mathbf{e}_i')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 0 \quad \text{for all} \quad i \in \{\{1,2,\dots,n\} - \{j,s,n\}\},$$

$$(\mathbf{e}_j' \otimes \mathbf{e}_j')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 2$$

$$(\mathbf{e}_s' \otimes \mathbf{e}_s' + \mathbf{e}_n' \otimes \mathbf{e}_n')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 0.$$

We finish our considerations in this section with the next theorem.

THEOREM 4.10. *If in model* (1.1) $\mathbf{E} = \gamma\mathbf{e}_s'$, $|\gamma| = 1$, $s \in \{1,2,\dots,n-1\}$, *then the random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE of the functional* $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ *of* $\boldsymbol{\beta}$ *if and only if*

(i) *case* $j \in \{s,n\}$

$\exists \boldsymbol{\delta} \in \{\mathbb{R}^{k^2+n-1}\}$

$$\begin{gathered}(\mathbf{X}' \otimes \mathbf{X}')(\mathbf{I} + \mathbf{I}^\star)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = \mathbf{O}\\ (\mathbf{e}_i' \otimes \mathbf{e}_i')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 0 \qquad \text{for all} \quad i \in \{\{1,2,\dots,n\} - \{s,n\}\}\\ (\mathbf{e}_s' \otimes \mathbf{e}_s' + \mathbf{e}_n' \otimes \mathbf{e}_n')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 2.\end{gathered} \tag{4.16}$$

(ii) *case* $j \in \{\{1,2,\dots,n\} - \{s,n\}\}$

$\exists \boldsymbol{\delta} \in \{\mathbb{R}^{k^2+n-1}\}$

$$\begin{gathered}(\mathbf{X}' \otimes \mathbf{X}')(\mathbf{I} + \mathbf{I}^\star)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = \mathbf{O}\\ (\mathbf{e}_i' \otimes \mathbf{e}_i')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 0 \quad \text{for all } i \in \{\{1,2,\dots,n\} - \{j,s,n\}\}\\ (\mathbf{e}_j' \otimes \mathbf{e}_j')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 2\\ (\mathbf{e}_s' \otimes \mathbf{e}_s' + \mathbf{e}_n' \otimes \mathbf{e}_n')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta} = 0.\end{gathered} \tag{4.17}$$

If (4.16) *or* (4.17) *are consistent, then*

$$\operatorname{vec}\mathbf{A} = \frac{1}{2}\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}_1\boldsymbol{\delta}$$

and

$$\mathbf{a} = -(\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ\,.$$

Proof. The assertions of the theorem are consequences of Lemma 4.6 and Lemma 4.8 (in case (i)) and of Lemma 4.7 and Lemma 4.9 (in case (ii)). They are omitted.

We have shown that the existence of the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2\big(a+b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, where $j\in\{s,n\}$ in model (1.1) with $\mathbf{E}=\gamma\mathbf{e}_s'$, $|\gamma|=1$, $s\in\{1,2,\dots,n-1\}$ is equivalent to the consistency of the linear system (4.16).

The existence of the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2\big(a+b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, where $j\in\big\{\{1,2,\dots,n\}-\{s,n\}\big\}$ in model (1.1) with $\mathbf{E}=\gamma\mathbf{e}_s'$, $|\gamma|=1$, $s\in\{1,2,\dots,n-1\}$ is equivalent to the consistency of the linear system (4.17).

Remark 4.11. Equations (4.16) are consistent and (4.17) are not consistent. That is why in model (1.1) with $\mathbf{E}=\gamma\mathbf{e}_s'$, $|\gamma|=1$, $s\in\{1,2,\dots,n-1\}$ the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2\big(a+b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j\in\{s,n\}$ exists and for $j\in\big\{\{1,2,\dots,n\}-\{s,n\}\big\}$ does not exist. The proof of this fact will be in Section 6.2.

5. Case $\mathbf{E}=\sum\limits_{i=1}^{t}\gamma_i\mathbf{e}_{s_i}'$

Lemma 5.1. *If in model* (1.1) $\mathbf{E}=\sum\limits_{i=1}^{t}\gamma_i\mathbf{e}_{s_i}'$, $\gamma_i\neq 0$, $s_i\in\{1,2,\dots,n-1\}$ *for* $i=1,2,\dots,t$, $t\geqq 2$, *then* $\mathbf{B}\in\mathcal{D}$ *(see* (1.2)-(1.4)*) if and only if* $\mathbf{e}_i'\mathbf{B}\mathbf{e}_i=0$ $i=1,2,\dots,n$ *and* $\mathbf{X}'\mathbf{B}\mathbf{X}=\mathbf{O}$ *are valid.*

Proof. First let us analyse the condition

$$\forall\{\boldsymbol{\beta}\in\mathbb{R}^k\}\qquad \operatorname{Tr}\mathbf{B}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}| & 0 & \dots & 0\\ 0 & |\mathbf{e}_2'\mathbf{X}\boldsymbol{\beta}| & \dots & 0\\ \vdots & & \ddots & \vdots\\ 0 & \dots & & |\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}|\end{pmatrix}=0\,. \tag{5.1}$$

If we denote $\mathcal{A}$ the class

$$\mathcal{A}=\{s_i:\ i=1,2,\dots,t\}\,,$$

then condition (5.1) (similarly see in the proof of Lemma 3.1) is equivalent to

$$\forall\{\mathbf{u}=(u_1,u_2,\dots,u_{n-1})'\in\mathbb{R}^{n-1}\}$$
$$\sum_{\substack{i=1\\ i\notin\mathcal{A}}}^{n-1}\mathbf{e}_i'\mathbf{B}\mathbf{e}_i|u_i|+\sum_{j=1}^{t}\mathbf{e}_{s_j}'\mathbf{B}\mathbf{e}_{s_j}|u_{s_j}|+\mathbf{e}_n'\mathbf{B}\mathbf{e}_n\Big|\sum_{j=1}^{t}\gamma_j u_{s_j}\Big|=0\,,$$

which is again equivalent to conditions

$$\mathbf{e}_i'\mathbf{B}\mathbf{e}_i\,,\qquad i\notin\mathcal{A} \tag{5.2}$$

and

$$\forall\{(u_{s_1},u_{s_2},\dots,u_{s_t})'\in\mathbb{R}^t:\ s_i\in\mathcal{A}\quad i=1,2,\dots,t\}$$
$$\sum_{j=1}^{t}\mathbf{e}_{s_j}'\mathbf{B}\mathbf{e}_{s_j}|u_{s_j}|+\mathbf{e}_n'\mathbf{B}\mathbf{e}_n\Big|\sum_{j=1}^{t}\gamma_j u_{s_j}\Big|=0\,. \tag{5.3}$$

If we choose $u_{s_2}=u_{s_3}=\dots=u_{s_t}=0$, we get from (5.3)

$$\forall\{u_{s_1}\in\mathbb{R}\}\qquad \mathbf{e}_{s_1}'\mathbf{B}\mathbf{e}_{s_1}|u_{s_1}|+\mathbf{e}_n'\mathbf{B}\mathbf{e}_n|\gamma_1||u_{s_1}|=0$$

i.e.

$$\mathbf{e}_{s_1}'\mathbf{B}\mathbf{e}_{s_1}=-\mathbf{e}_n'\mathbf{B}\mathbf{e}_n|\gamma_1|\,.$$

In the same way we obtain

$$\mathbf{e}_{s_j}'\mathbf{B}\mathbf{e}_{s_j}=-\mathbf{e}_n'\mathbf{B}\mathbf{e}_n|\gamma_j|\qquad j=1,2,\dots,t\,. \tag{5.4}$$

Because of $t\geqq 2$, we can in (5.3) take $u_{s_3}=u_{s_4}=\dots=u_{s_t}=0$ and we obtain

$$\begin{aligned}&\forall\{u_{s_i}\in\mathbb{R}:\ i=1,2\}\\ &\mathbf{e}_{s_1}'\mathbf{B}\mathbf{e}_{s_1}|u_{s_1}|+\mathbf{e}_{s_2}'\mathbf{B}\mathbf{e}_{s_2}|u_{s_2}|+\mathbf{e}_n'\mathbf{B}\mathbf{e}_n|\gamma_1u_{s_1}+\gamma_2u_{s_2}|=0\,.\end{aligned} \tag{5.5}$$

According to (5.4) the last condition is of the form

$$\begin{aligned}&\forall\{u_{s_i}\in\mathbb{R}:\ i=1,2\}\\ &-\mathbf{e}_n'\mathbf{B}\mathbf{e}_n|\gamma_1||u_{s_1}|-\mathbf{e}_n'\mathbf{B}\mathbf{e}_n|\gamma_2||u_{s_2}|+\mathbf{e}_n'\mathbf{B}\mathbf{e}_n|\gamma_1u_{s_1}+\gamma_2u_{s_2}|=0\,.\end{aligned} \tag{5.6}$$

Condition (5.6) yields

$$\mathbf{e}_n'\mathbf{B}\mathbf{e}_n=0\,, \tag{5.7}$$

because for $u_{s_2} \neq 0$ and $u_{s_1} = -\frac{\gamma_2}{\gamma_1} u_{s_1} \neq 0$ is

$$-|\gamma_1|\left|-\frac{\gamma_2}{\gamma_1} u_{s_2}\right| - |\gamma_2||u_{s_2}| = -2|\gamma_2||u_{s_2}| \neq 0 .$$

If (5.7) is valid, we easily obtain from (5.4) that

$$\mathbf{e}'_{s_j} \mathbf{B} \mathbf{e}_{s_j} = 0 , \qquad j = 1, 2, \dots, t . \tag{5.8}$$

Conditions (5.2), (5.7) and (5.8) are necessary for (5.1) to be true. It is easily to see that they are also sufficient.

We have proved that (5.1) is equivalent to

$$\mathbf{e}'_i \mathbf{B} \mathbf{e}_i = 0 , i = 1, 2, \dots, n . \tag{5.9}$$

According to (1.2), (1.3) and (1.4) we can easily finish the proof.

LEMMA 5.2. *If in model* (1.1) $\mathbf{E} = \sum_{i=1}^{t} \gamma_i \mathbf{e}'_{s_i}$, $\gamma_i \neq 0$, $s_i \in \{1, 2, \dots, n-1\}$ *for* $i = 1, 2, \dots, t$, $t \geqq 2$, *then the random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE of its mean value if and only if* $\exists \{\mathbf{z} \in \mathbb{R}^n\}$ $\exists \gamma \in \{\mathbb{R}^{k^2+n}\}$ *that*

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ + (\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))} \mathbf{X}'\mathbf{z}$$

and

$$\big(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)\big) \operatorname{vec}(\mathbf{A} + \mathbf{A}') = \mathcal{X}\gamma ,$$

where $(\mathbf{X}')^-_{m(\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ))}$ *is an arbitrary but fixed minimum* $\boldsymbol{\Sigma}(\boldsymbol{\beta}_\circ)$*-norm g-inverse of the matrix* $\mathbf{X}'$.

Proof. The proof is the same as the proof of Lemma 3.3 and is omitted.

LEMMA 5.3. *If in model* (1.1) $\mathbf{E} = \sum_{i=1}^{t} \gamma_i \mathbf{e}'_{s_i}$, $\gamma_i \neq 0$, $s_i \in \{1, 2, \dots, n-1\}$ *for* $i = 1, 2, \dots, t$, $t \geqq 2$, *then the random variable* $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is an unbiased estimator of the functional* $\sigma^2\big(a + b|\mathbf{e}'_j\mathbf{X}\boldsymbol{\beta}|\big)^2$, *where* $j \in \{1, 2, \dots, n\}$, *if and only if*

$$\mathbf{a} \in \operatorname{Ker} \mathbf{X}', \tag{5.10}$$

$$\mathbf{e}'_i \mathbf{A} \mathbf{e}_i = 0 \quad \text{for all} \quad i \in \big\{\{1, 2, \dots, n\} - \{j\}\big\} , \qquad \mathbf{e}'_j \mathbf{A} \mathbf{e}_j = 1 \tag{5.11}$$

and

$$\mathbf{X}'(\mathbf{A}+\mathbf{A}')\mathbf{X}=\mathbf{O} \tag{5.12}$$

hold.

P r o o f . The proof follows in the same way as the proof of Lemma 2.1. The crucial point is the relation

$$\forall\{\boldsymbol{\beta}\in\mathbb{R}^k\} \qquad \mathbf{e}_1'\mathbf{A}\mathbf{e}_1|\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}|+\cdots+\mathbf{e}_{j-1}'\mathbf{A}\mathbf{e}_{j-1}|\mathbf{e}_{j-1}'\mathbf{X}\boldsymbol{\beta}|$$
$$+(\mathbf{e}_j'\mathbf{A}\mathbf{e}_j-1)|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|+\mathbf{e}_{j+1}'\mathbf{A}\mathbf{e}_{j+1}|\mathbf{e}_{j+1}'\mathbf{X}\boldsymbol{\beta}|+\cdots+\mathbf{e}_n'\mathbf{A}\mathbf{e}_n|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}|=0,$$

which according to the proof of Lemma 5.1 is under our conditions equivalent to the relation (5.11) (because (5.1) is equivalent to (5.9)).

Let us now investigate the problem of the determination of the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a+b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$. In the same way as in Section 3 or Section 4 we obtain the next theorem:

THEOREM 5.4. *If in model* (1.1) $\mathbf{E}=\sum\limits_{i=1}^{t}\gamma_i\mathbf{e}_{s_i}'$, $\gamma_i\neq 0$, $s_i\in\{1,2,\dots,n-1\}$ *for* $i=1,2,\dots,t$, $t\geqq 2$, *then the random variable* $\mathbf{a}'\mathbf{Y}+\mathbf{Y}'\mathbf{A}\mathbf{Y}$ *is the* $\boldsymbol{\beta}_\circ$*-LBLQUE of the functional* $\sigma^2\big(a+b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, *where* $j\in\{1,2,\dots,n\}$, *if and only if*

$\exists\boldsymbol{\delta}\in\{\mathbb{R}^{k^2+n}\}$

$$\begin{gathered}(\mathbf{X}'\otimes\mathbf{X}')(\mathbf{I}+\mathbf{I}^\star)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta}=\mathbf{O}\\ (\mathbf{e}_i'\otimes\mathbf{e}_i')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta}=0 \qquad \text{for all} \quad i\in\big\{\{1,2,\dots,n\}-\{j\}\big\},\\ (\mathbf{e}_j'\otimes\mathbf{e}_j')\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta}=2.\end{gathered} \tag{5.13}$$

If (5.13) *is consistent, then*

$$\operatorname{vec}\mathbf{A}=\frac{1}{2}\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)\mathcal{X}\boldsymbol{\delta}$$

and

$$\mathbf{a}=-(\mathbf{A}+\mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ.$$

We have shown that the existence of the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2\big(a+b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, where $j\in\{1,2,\dots,n\}$ in model (1.1) with $\mathbf{E}=\sum\limits_{i=1}^{t}\gamma_i\mathbf{e}_{s_i}'$, $\gamma_i\neq 0$, $s_i\in\{1,2,\dots,n-1\}$ for $i=1,2,\dots,t$, $t\geqq 2$, is equivalent to the consistency of the linear system (5.13).

R e m a r k 5.5. Equations (5.13) are consistent for $j \in \{s_1, s_2, \dots, s_t\}$ and are not consistent for $j \notin \{s_1, s_2, \dots, s_t\}$. That is why in model (1.1) with $\mathbf{E} = \sum_{i=1}^{n} \gamma_i \mathbf{e}'_{s_i}$, $\gamma_i \neq 0$, $s_i \in \{1, 2, \dots, n-1\}$ for $i = 1, 2, \dots, t$, $t \geqq 2$, the $\boldsymbol{\beta}_\circ$-LBLQUE of the functional $\sigma^2\big(a + b|\mathbf{e}'_j\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \in \{s_1, s_2, \dots, s_t\}$ exists and for $j \notin \{s_1, s_2, \dots, s_t\}$ does not exist. The proof of this fact will be in Section 6.3.

6. Existence of the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}'_j\mathbf{X}\boldsymbol{\beta}|\big)^2$

It is shown in Section 2 that in the case $R(\mathbf{X}) = n \leqq k$ the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}'_j\mathbf{X}\boldsymbol{\beta}|\big)^2$ does not exist for any $j \in \{1, 2, \dots, n\}$ (Corollary 2.2). In other investigated cases is the existence of the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}'_j\mathbf{X}\boldsymbol{\beta}|\big)^2$ equivalent to the consistency of linear systems (3.39), (4.16), (4.17) and (5.13) (see Theorem 3.10, Theorem 4.10 and Theorem 5.4). Let us now investigate the consistency of the above mentioned systems in each case separately. First let us write some lemmas needed in the sequel.

LEMMA 6.1. *Let us denote*

$$\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) = \begin{pmatrix} (a + b|\mathbf{e}'_1\mathbf{X}\boldsymbol{\beta}_\circ|)^{-1} & 0 & \dots & 0 \\ 0 & (a + b|\mathbf{e}'_2\mathbf{X}\boldsymbol{\beta}_\circ|)^{-1} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & (a + b|\mathbf{e}'_n\mathbf{X}\boldsymbol{\beta}_\circ|)^{-1} \end{pmatrix}.$$

For $i = 1, 2, \dots, n$ there hold

$$\begin{aligned} &4(\mathbf{e}'_i \otimes \mathbf{e}'_i)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big) \\ &\quad = (\mathbf{e}_i \otimes \mathbf{e}_i)\big(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I} + \mathbf{I}^\star)(\mathbf{I} + \mathbf{I}^\star)\big(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ)\big), \end{aligned} \tag{6.1}$$

$$\begin{aligned} &2(\mathbf{I} + \mathbf{I}^\star)\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big) \\ &\quad = \big(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ)\big)(\mathbf{I} + \mathbf{I}^\star)(\mathbf{I} + \mathbf{I}^\star)\big(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ)\big). \end{aligned} \tag{6.2}$$

P r o o f . The proof is based on similar considerations as in Lemma 3.8 with $\big(\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\beta}_\circ)\big)$ instead of $\big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\big)$. It is easy and is omitted.

LEMMA 6.1.2.1. *The vector* $\mathbf{d}_o \in \mathbb{R}^{n^2}$ *with components*

$$d_o^{ss} = 4\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\,,$$
$$d_o^{ns} = -\frac{4}{\gamma}\big(a + b|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\,,$$
$$d_o^{pq} = 0\,, \qquad \textit{for} \quad (p,q) \neq (s,s) \quad \textit{and} \quad (p,q) \neq (n,s)$$

is a solution to (6.14), (6.15) *and* (6.16) *for* $j = s$.

Proof. Using (6.5) it is easy to see that $\mathbf{d}_o$ satisfies (6.15) and (6.16).

(i) If $p = 1,2,\dots,s-1,s+1,\dots,n-1$, $q = 1,2,\dots,s-1,s+1,\dots,n-1$ and $p \neq q$, then according to (6.6)

$$\begin{aligned}(\mathbf{e}_p' \otimes \mathbf{e}_q')&\big((\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o \\ &= (\mathbf{e}_p' \otimes \mathbf{e}_q')\mathcal{B}\mathbf{d}_o = \frac{1}{\big(a + b|\mathbf{e}_p'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}_q'\mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{pq} + d_o^{qp}) = 0\,,\end{aligned}$$

(ii) if $p = 1,2,\dots,s-1,s+1,\dots,n-1$, $q = p$, then

$$\begin{aligned}(\mathbf{e}_p' \otimes \mathbf{e}_q')&\big((\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o \\ &= (\mathbf{e}_p' \otimes \mathbf{e}_p')\mathcal{B}\mathbf{d}_o = \frac{2}{\big(a + b|\mathbf{e}_p'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2}d_o^{pp} = 0\,,\end{aligned}$$

(iii) if $p = 1,2,\dots,s-1,s+1,\dots,n-1$, $q = s$, then

$$\begin{aligned}(\mathbf{e}_p' \otimes \mathbf{e}_q')\big((\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o &= \big(\mathbf{e}_p' \otimes (\mathbf{e}_s + \gamma\mathbf{e}_n)'\big)\mathcal{B}\mathbf{d}_o \\ = \frac{1}{\big(a + b|\mathbf{e}_p'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{ps} + d_o^{sp}) & \\ + \frac{\gamma}{\big(a + b|\mathbf{e}_p'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}_\circ|\big)}&(d_o^{pn} + d_o^{np}) = 0\,,\end{aligned}$$

(iv) if $p = s$, $q = 1,2,\dots,s-1,s+1,\dots,n-1$, then

$$\begin{aligned}(\mathbf{e}_p' \otimes \mathbf{e}_q')\big((\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o &= \big((\mathbf{e}_s + \gamma\mathbf{e}_n)' \otimes \mathbf{e}_q'\big)\mathcal{B}\mathbf{d}_o \\ = \frac{1}{\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}_q'\mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{sq} + d_o^{qs}) & \\ + \frac{\gamma}{\big(a + b|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}_q'\mathbf{X}\boldsymbol{\beta}_\circ|\big)}&(d_o^{nq} + d_o^{qn}) = 0\,,\end{aligned}$$

(v) if $p = s$, $q = s$, then

$$(\mathbf{e}'_p \otimes \mathbf{e}'_q)\big((\mathbf{I}_{n-1,n-1}, \gamma \mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma \mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o = \big((\mathbf{e}_s + \gamma \mathbf{e}_n)' \otimes (\mathbf{e}_s + \gamma \mathbf{e}_n)'\big)\mathcal{B}\mathbf{d}_o$$
$$= \frac{2}{\big(a + b|\mathbf{e}'_s \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{ss} + \frac{2\gamma}{\big(a + b|\mathbf{e}'_s \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)} (d_o^{sn} + d_o^{ns})$$
$$+ \frac{2\gamma^2}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{nn} = 8 - 8 = 0 .$$

We see that $\mathbf{d}_o$ is also a solution to (6.14). The lemma is proved.

LEMMA 6.1.2.2. *The vector* $\mathbf{d}_o \in \mathbb{R}^{n^2}$ *with components*

$$d_o^{nn} = 4\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2 ,$$
$$d_o^{ns} = -4\gamma\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_s \mathbf{X}\boldsymbol{\beta}_\circ|\big) ,$$
$$d_o^{pq} = 0 , \qquad for \quad (p,q) \neq (n,n) \quad and \quad (p,q) \neq (n,s)$$

is a solution to (6.14), (6.15) *and* (6.16) *for* $j = n$.

P r o o f . Proof is the same as the proof of Lemma 6.1.2.1 and is omitted.

It is evident now that equations (6.14), (6.15) and (6.16) are for $j \in \{s,n\}$ soluble and (6.13) is satisfied. That is why equations (3.39) are for $j \in \{s,n\}$ consistent and in the case $\mathbf{E} = \gamma \mathbf{e}'_s$; $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1,2, \dots, n-1\}$ there exists the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}'_s \mathbf{X}\boldsymbol{\beta}|\big)^2$ and of $\sigma^2\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}|\big)^2$.

6.2. Case $\mathbf{E} = \gamma \mathbf{e}'_s$, $|\gamma| = 1$.

6.2.1. Estimability of $\sigma^2\big(a + b|\mathbf{e}'_j \mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \in \big\{\{1,2,\dots,n-1\} - \{s\}\big\}$.

According to Theorem 4.10 in model (1.1) with $\mathbf{E} = \gamma \mathbf{e}'_s$, $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1,2,\dots,n-1\}$ the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}'_j \mathbf{X}\boldsymbol{\beta}|\big)^2$, where $j \in \big\{\{1,2,\dots,n-1\} - \{s\}\big\}$ exists if and only if equations (4.17) are consistent.

It is easy to see that these equations are equivalent to

$$\mathcal{X}'_1 \mathcal{B}\mathcal{B}' \mathcal{X}_1 \boldsymbol{\delta} = \boldsymbol{\eta} , \tag{6.17}$$

where $\boldsymbol{\eta}_{k^2+n-1,1} = (\mathbf{O}'_{k^2,1}, \mathbf{f}'_j)'$, $\mathbf{f}_j = (f_1^{(j)}, f_2^{(j)}, \dots, f_{(n-1)}^{(j)})'$ is a vector with components $f_j^{(j)} = 1$ and $f_k^{(j)} = 0$ for $k \in \big\{\{1,2,\dots,n-1\} - \{j\}\big\}$ if $1 \leqq j < s$ or $f_{j-1}^{(j)} = 1$ and $f_k^{(j)} = 0$ $k \in \big\{\{1,2,\dots,n-1\} - \{j-1\}\big\}$ if $s < j \leqq n-1$.

Similarly as in Section 6.1 we have that (6.17) is consistent if and only if

$$\exists\{\mathbf{d}_o \in \mathbb{R}^{n^2}\} \qquad \mathcal{X}_1'\mathcal{B}\mathbf{d}_o = \boldsymbol{\eta}. \tag{6.18}$$

We can write equations (6.18) as

$$(\mathbf{X}' \otimes \mathbf{X}')\mathcal{B}\mathbf{d}_o = \mathbf{O}_{k^2,1}, \tag{6.19}$$

$$\begin{pmatrix} \mathbf{e}_1' \otimes \mathbf{e}_1' \\ \mathbf{e}_2' \otimes \mathbf{e}_2' \\ \vdots \\ \mathbf{e}_{s-1}' \otimes \mathbf{e}_{s-1}' \\ \mathbf{e}_{s+1}' \otimes \mathbf{e}_{s+1}' \\ \vdots \\ \mathbf{e}_{n-1}' \otimes \mathbf{e}_{n-1}' \\ \mathbf{e}_s' \otimes \mathbf{e}_s' + \mathbf{e}_n' \otimes \mathbf{e}_n' \end{pmatrix} \mathcal{B}\mathbf{d}_o = \mathbf{f}_j. \tag{6.20}$$

In the same way as in Section 6.1 we can write equations (6.19) as

$$(\mathbf{X}_1' \otimes \mathbf{X}_1')\big((\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o = \mathbf{O},$$

which is again satisfied if and only if

$$\big((\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o = \mathbf{O}. \tag{6.21}$$

But for $p \in \big\{\{1, 2, \dots, n-1\} - \{s\}\big\}$ according to (6.5)

$$\begin{aligned}(\mathbf{e}_p' \otimes \mathbf{e}_p')&\big((\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o \\ &= (\mathbf{e}_p' \otimes \mathbf{e}_p')\mathcal{B}\mathbf{d}_o = \frac{2}{\big(a + b|\mathbf{e}_p'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{pp} = 0,\end{aligned}$$

which is a contradiction to (6.20). That is why (6.18) and backward (6.17) are not consistent. The $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ in the case $\mathbf{E} = \gamma\mathbf{e}_s'$, $|\gamma| = 1$ does not exist for every $j \in \big\{\{1, 2, \dots, n-1\} - \{s\}\big\}$.

6.2.2. Estimability of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \in \{s, n\}$.

According to Theorem 4.10 in model (1.1) with $\mathbf{E} = \gamma\mathbf{e}_s'$, $\gamma \neq 0$, $|\gamma| \neq 1$, $s \in \{1, 2, \dots, n-1\}$ the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \in \{s, n\}$ exists if and only if equations (4.16) are consistent. As formerly these equations are consistent if and only if

$\exists\{\mathbf{d}_o \in \mathbb{R}^{n^2}\}$

$$(\mathbf{X}_1' \otimes \mathbf{X}_1')\big((\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o = \mathbf{O}\,, \tag{6.22}$$

$$(\mathbf{e}_i' \otimes \mathbf{e}_i')\mathcal{B}\mathbf{d}_o = 0 \qquad \text{for} \quad i \in \big\{\{1,2,\dots,n\} - \{s,n\}\big\}, \tag{6.23}$$

$$(\mathbf{e}_s' \otimes \mathbf{e}_s' + \mathbf{e}_n' \otimes \mathbf{e}_n')\mathcal{B}\mathbf{d}_o = 8\,. \tag{6.24}$$

The matrix $(\mathbf{X}_1' \otimes \mathbf{X}_1')$ is of full rank in columns and that is why (6.22) is satisfied if and only if

$$\big((\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s) \otimes (\mathbf{I}_{n-1,n-1}, \gamma\mathbf{e}_s)\big)\mathcal{B}\mathbf{d}_o = \mathbf{O}\,. \tag{6.25}$$

LEMMA 6.2.2.1. *The vector $\mathbf{d}_o \in \mathbb{R}^{n^2}$ with components*

$$d_o^{ss} = 4\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\,,$$
$$d_o^{sn} = -\frac{4}{\gamma}\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2\,,$$
$$d_o^{pq} = 0\,, \qquad \text{for} \quad (p,q) \neq (s,s) \quad \text{and} \quad (p,q) \neq (s,n)$$

is a solution to (6.25), (6.23) *and* (6.24).

P r o o f . The proof is the same as the proof of Lemma 6.1.2.1 and is omitted. We only note that case (v) of the proof holds because of the equalities

$$|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}_\circ| = |\mathbf{E}\mathbf{X}\boldsymbol{\beta}_\circ| = |\gamma\mathbf{e}_s'\mathbf{X}_1\boldsymbol{\beta}_\circ| = |\gamma||\mathbf{e}_s'\mathbf{X}_1\boldsymbol{\beta}_\circ| = |\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\,.$$

Equations (6.22), (6.23), (6.24) and backward (4.16) are consistent and so exists the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \in \{s,n\}$ in (1.1) in the considered case $\mathbf{E} = \gamma\mathbf{e}_s'$, $|\gamma| = 1$.

6.3. Case $\mathbf{E} = \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}'$.

According to Theorem 5.4 in the case $\mathbf{E} = \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}'$, $\gamma_i \neq 0$, $s_i \in \{1,2,\dots,n-1\}$ for $i = 1,2,\dots,t$, $t \geqq 2$, the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$, where $j \in \{1,2,\dots,n\}$, exists if and only if equations (5.13) are consistent. As formerly these equations are consistent if and only if

$\exists\{\mathbf{d}_o \in \mathbb{R}^{n^2}\}$

$$(\mathbf{X}_1' \otimes \mathbf{X}_1')\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o = \mathbf{O}\,, \tag{6.26}$$

$$\begin{pmatrix} \mathbf{e}_1' \otimes \mathbf{e}_1' \\ \mathbf{e}_2' \otimes \mathbf{e}_2' \\ \vdots \\ \mathbf{e}_n' \otimes \mathbf{e}_n' \end{pmatrix} \mathcal{B}\mathbf{d}_o = 8\mathbf{e}_j\,. \tag{6.27}$$

As $\mathbf{X}_1' \otimes \mathbf{X}_1'$ is of full rank in columns, we have that (6.26) is satisfied if and only if

$$\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o = \mathbf{O}\,. \tag{6.28}$$

6.3.1. Estimability of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \notin \{s_1, s_2, \dots, s_t\}$.

If $j \notin \{s_1, s_2, \dots, s_t\}$, then according to (6.5) and (6.28) there is

$$\begin{aligned}(\mathbf{e}_j' \otimes \mathbf{e}_j')\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o \\ = (\mathbf{e}_j' \otimes \mathbf{e}_j')\mathcal{B}\mathbf{d}_o = \frac{2}{\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{jj} = 0\,,\end{aligned}$$

which is a contradiction to (6.27). That is why (5.13) is for $j \notin \{s_1, s_2, \dots, s_t\}$ not consistent and the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \notin \{s_1, s_2, \dots, s_t\}$ does not exist.

6.3.2. Estimability of $\sigma^2\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}|\big)^2$ for $j \in \{s_1, s_2, \dots, s_t\}$.

Lemma 6.3.2.1. *For $j \in \{s_1, s_2, \dots, s_t\}$ the vector $\mathbf{d}_o \in \mathbb{R}^{n^2}$ with components*

$$d_o^{jj} = 4\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)^2,$$

$$d_o^{jw} = 2\frac{\gamma_{[w]}}{\gamma_{[j]}}\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}_w'\mathbf{X}\boldsymbol{\beta}_\circ|\big) \quad \textit{for} \quad w \in \big\{\{s_1, s_2, \dots, s_t\} - \{j\}\big\}\,,$$

$$d_o^{jn} = -\frac{2}{\gamma_{[j]}}\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}_n'\mathbf{X}\boldsymbol{\beta}_\circ|\big) = d_o^{nj}\,,$$

$$d_o^{sj} = 2\frac{\gamma_{[s]}}{\gamma_{[j]}}\big(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}_j'\mathbf{X}\boldsymbol{\beta}_\circ|\big) \quad \textit{for} \quad s \in \big\{\{s_1, s_2, \dots, s_t\} - \{j\}\big\},$$

and the other components are zero, is a solution to (6.27) *and* (6.28). (*Here* $\gamma_{[w]}$ *means the term belonging to index* $w \in \{s_1, s_2, \dots, s_t\}$ *in the sum* $\sum_{i=1}^{t} \gamma_i \mathbf{e}'_{s_i}$.)

P r o o f. Using (6.5) it is easy to see that $\mathbf{d}_o$ satisfies (6.27).

(1) If $p \in \{\{1,2,\dots,n-1\} - \{s_1, s_2, \dots, s_t\}\}$, $q \in \{s_1, s_2, \dots, s_t\}$, then

$$\begin{aligned}
&(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o \\
&= \big(\mathbf{e}'_p \otimes (\mathbf{e}_q + \gamma_{[q]}\mathbf{e}_n)'\big)\mathcal{B}\mathbf{d}_o \\
&= \frac{1}{\big(a + b|\mathbf{e}'_p \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_q \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{pq} + d_o^{qp}) \\
&\qquad + \frac{\gamma_{[q]}}{\big(a + b|\mathbf{e}'_p \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{pn} + d_o^{np}) = 0 ,
\end{aligned}$$

(2) if $p \in \{\{1,2,\dots,n-1\} - \{s_1, s_2, \dots, s_t\}\}$, $q \notin \{s_1, s_2, \dots, s_t\}$, then

$$(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o = (\mathbf{e}'_p \otimes \mathbf{e}'_q)\mathcal{B}\mathbf{d}_o = 0$$

in both cases $p = q$ and $p \neq q$,

(3) if $p \in \Big\{\{s_1, s_2, \dots, s_t\} - \{j\}\Big\}$, $q \in \{\{s_1, s_2, \dots, s_t\} - \{j\}\}$, $p \neq q$, then

$$\begin{aligned}
&(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o \\
&= \big((\mathbf{e}_p + \gamma_{[p]}\mathbf{e}_n)' \otimes (\mathbf{e}_q + \gamma_{[q]}\mathbf{e}_n)'\big)\mathcal{B}\mathbf{d}_o \\
&= \frac{1}{\big(a + b|\mathbf{e}'_p \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_q \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{pq} + d_o^{qp}) \\
&\qquad + \frac{\gamma_{[p]}}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_q \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{nq} + d_o^{qn}) \\
&\qquad + \frac{\gamma_{[q]}}{\big(a + b|\mathbf{e}'_p \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{pn} + d_o^{np}) \\
&\qquad + \frac{2\gamma_{[p]}\gamma_{[q]}}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{nn} = 0 ,
\end{aligned}$$

(4) if $p \in \{\{s_1, s_2, \dots, s_t\} - \{j\}\}$, $q \notin \{s_1, s_2, \dots, s_t\}$, i.e. also $q \neq j$ and $q \neq p$ is valid, then

$$
\begin{aligned}
&(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o \\
&= \big((\mathbf{e}_p + \gamma_{[p]}\mathbf{e}_n)' \otimes \mathbf{e}'_q\big)\mathcal{B}\mathbf{d}_o \\
&= \frac{1}{\big(a + b|\mathbf{e}'_p \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_q \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{pq} + d_o^{qp}) \\
&\qquad + \frac{\gamma_{[p]}}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_q \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{nq} + d_o^{qn}) = 0\,,
\end{aligned}
$$

(5) if $p \in \{\{s_1, s_2, \dots, s_t\} - \{j\}\}$, $q = p$, then

$$
\begin{aligned}
&(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o \\
&= \big((\mathbf{e}_p + \gamma_{[p]}\mathbf{e}_n)' \otimes (\mathbf{e}_p + \gamma_{[p]}\mathbf{e}_n)'\big)\mathcal{B}\mathbf{d}_o \\
&= \frac{2}{\big(a + b|\mathbf{e}'_p \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{pp} + \frac{2\gamma_{[p]}}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_p \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{np} + d_o^{pn}) \\
&\qquad + \frac{2\gamma_{[p]}^2}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{nn} = 0\,,
\end{aligned}
$$

(6) if $p \in \{\{s_1, s_2, \dots, s_t\} - \{j\}\}$, $q = j$, then

$$
\begin{aligned}
&(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o \\
&= \big((\mathbf{e}_p + \gamma_{[p]}\mathbf{e}_n)' \otimes (\mathbf{e}_j + \gamma_{[j]}\mathbf{e}_n)'\big)\mathcal{B}\mathbf{d}_o \\
&= \frac{1}{\big(a + b|\mathbf{e}'_p \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_j \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{pj} + d_o^{jp}) \\
&\qquad + \frac{\gamma_{[p]}}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_j \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{nj} + d_o^{jn}) \\
&\qquad + \frac{\gamma_{[j]}}{\big(a + b|\mathbf{e}'_p \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{pn} + d_o^{np}) + \frac{2\gamma_{[p]}^2}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{nn} \\
&= 4\frac{\gamma_{[p]}}{\gamma_{[j]}} - \gamma_{[p]}\frac{4}{\gamma_{[j]}} = 0\,,
\end{aligned}
$$

(7) if $p = j$, $q \in \{\{s_1, s_2, \ldots, s_t\} - \{j\}\}$, then

$$(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o$$

$$= \big((\mathbf{e}_j + \gamma_{[j]}\mathbf{e}_n)' \otimes (\mathbf{e}_q + \gamma_{[q]}\mathbf{e}_n)'\big)\mathcal{B}\mathbf{d}_o$$

$$= \frac{1}{\big(a + b|\mathbf{e}'_j \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_q \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{jq} + d_o^{qj})$$

$$+ \frac{\gamma_{[j]}}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_q \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{nq} + d_o^{qn})$$

$$+ \frac{\gamma_{[q]}}{\big(a + b|\mathbf{e}'_j \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{jn} + d_o^{nj}) + \frac{2\gamma_{[j]}\gamma_{[q]}}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{nn}.$$

$$= 4\frac{\gamma_{[q]}}{\gamma_{[j]}} - \gamma_{[q]}\frac{4}{\gamma_{[j]}} = 0,$$

(8) if $p = j$, $q \notin \{s_1, s_2, \ldots, s_t\}$, then

$$(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o$$

$$= \big((\mathbf{e}_j + \gamma_{[j]}\mathbf{e}_n)' \otimes \mathbf{e}'_q\big)\mathcal{B}\mathbf{d}_o$$

$$= \frac{1}{\big(a + b|\mathbf{e}'_j \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_q \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{jq} + d_o^{qj})$$

$$+ \frac{\gamma_{[j]}}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_q \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{nq} + d_o^{qn}) = 0,$$

(9) if $p = j$, $q = j$, then

$$(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o$$

$$= \big((\mathbf{e}_j + \gamma_{[j]}\mathbf{e}_n)' \otimes (\mathbf{e}_j + \gamma_{[j]}\mathbf{e}_n)'\big)\mathcal{B}\mathbf{d}_o$$

$$= \frac{2}{\big(a + b|\mathbf{e}'_j \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{jj} + \frac{2\gamma_{[j]}}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)\big(a + b|\mathbf{e}'_j \mathbf{X}\boldsymbol{\beta}_\circ|\big)}(d_o^{nj} + d_o^{jn})$$

$$+ \frac{2\gamma_{[j]}^2}{\big(a + b|\mathbf{e}'_n \mathbf{X}\boldsymbol{\beta}_\circ|\big)^2} d_o^{nn}$$

$$= 8 - \gamma_{[j]}\frac{8}{\gamma_{[j]}} = 0.$$

We see that for $p = 1, 2, \ldots, n-1$, $q = 1, 2, \ldots, n-1$,

$$(\mathbf{e}'_p \otimes \mathbf{e}'_q)\Big(\Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big) \otimes \Big(\mathbf{I}_{n-1,n-1}, \sum_{i=1}^{t} \gamma_i \mathbf{e}_{s_i}\Big)\Big)\mathcal{B}\mathbf{d}_o = 0\,,$$

i.e. the vector $\mathbf{d}_o$ is also a solution to (6.28). The lemma is proved.

For $j \in \{s_1, s_2, \ldots, s_t\}$ equations (6.26) and (6.27) are consistent, that means also (5.13) is soluble and in this case there exists the $\boldsymbol{\beta}_o$-LBLQUE of $\sigma^2\big(a + b|\mathbf{e}'_j\mathbf{X}\boldsymbol{\beta}|\big)^2$.

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