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INDEPENDENCE IN KLEIN SPACES

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A notation of independence in an universal algebra $\mathfrak{A} = (A, F)$ has been introduced and later also generalized by E. Marczewski ([4] and [5], see also [2]). In this concept the family $Ind(A)$ of all independent subsets of A is *of finite character;* i. e. a set I belongs to $Ind(A)$ whenever each finite subset of I belongs to $Ind(A)$.

This pâper deals with a class of all *Klein spaces,* i. e. unary algebras $\mathfrak{A} = (A, G)$ in which G is a group under the superposition. Using a special closure operation c seven mutually different definitions of independence are described and compared. These concepts are not covered by those of Marczewski, since neither of our definitions of independence is "of finite character".

1. Preliminaries

By a transformation of a set $A \neq \emptyset$ we shall mean every bijective map $f: A \to A$. The set $\mathcal{T}(A)$ of all transformations of a set A with respect to the superposition $(f, g) \mapsto f \circ g$, $(f \circ g)(x) = f(g(x))$ form a group. By a Klein space (see [3]), or shortly a *k-space,* we mean any couple $\mathfrak{A} = (A, G)$, where G is a subgroup of $\mathcal{T}(A)$. The *k-closure operation* on A is a map $c: 2^A \to 2^A$ given via: $c(X)$ consists of exactly those points $x \in A$ which are invariant under each of the transformations $g \in G$ with the property $g(a) = a$ for all $a \in X$. It is easy to verify that the *k-closure* operation c is a closure operation in the sense of Birkhoff (see [1]), i. e. it is extensive, monotone and idempotent. We recall (see [1])

$$c(X \cup Y) \supset cX \cup cY \quad \text{for all} \quad X, Y \in 2^A.$$

2. Independence

Definition 1. *Let $\mathfrak{A} = (A, G)$ be a k-space and c its k-closure operation. A set $I \subset A$ is said to be (i)-independent if the following condition (i) is fulfilled; $i = 1, \ldots, 7$.*

For all $X, Y \in 2^A$, $Y \neq \emptyset$ *it holds*

(1) $X \subset I,\ cX = cI \Rightarrow X = I$

(2) $X \subset I,\ cX = cI \Rightarrow |X| = |I|$

(3) $cX \supset cI \Rightarrow |X| \geqq |I|$

(4) $X \cup Y = I,\ X \cap Y = \emptyset \Rightarrow cX \cap cY = c\emptyset \neq cY$

(5) $x \subset I,\ Y \subset I \Rightarrow cX \cap cY = c(x \cap Y),\ I \cap c\emptyset = \emptyset$

(6) $cX \supset cI \Rightarrow$ *there exists* $g \in G$ *such that* $gI \subset X$

(7) $cX = cI \Rightarrow$ *there exists* $g \in G$ *such that* $gI \subset X$.

The family of all (i)-independent sets of a k-space $\mathfrak{A} = (A, G)$ will be denoted by $Ind_i(A)$ or $Ind_i(A)$.

Example 1. Let A be a vector space and $G = Aut\ A$ its group of all automorphisms. Let $Ind\ (A)$ be the set of all independent (in usual sense) sets of A. If A is of a finite dimension then $Ind\ (A) = Ind_i\ (A)$ for each $i = 1, \ldots, 7$.

It is obvious that $\emptyset \in Ind_i\ (A)$ for each $i = 1, \ldots, 7$ and arbitrary k-space A. The assertion "family $Ind_i\ (A)$ is not (in general) of finite character", mentioned in the Introduction, is proved by the following example.

Example 2. Let $A = R$ be the set of all real numbers endowed with the usual topology and G its group of all homeomorphisms. For the k-space $\mathfrak{A} = (R, G)$ the family $Ind_i\ (A)$ is not of finite character since for each i = 1, ..., 7

(a)	$R \notin Ind_i(A)$	but
(b)	$I \in Ind_i(A)$	for all finite $I \subset R$.

Proof. It is easy to verify that in the k-space $\mathfrak{A} = (R, G)$ the k-closure operation c coincides with the topological closure operation. To show (a) let us set $X = Q$ (the set of all rational numbers) and $Y = R - Q$. the assertion (b) is obvious.

Theorem 1. *Let A be a k-space. The family* $Ind_i(A)$ *is hereditary for* $i = 1, 4, 5$ *and is not hereditary for* $i = 2, 3$.

Proof is obvious for i = 4 and i = 5. From the Example 3 it follows immediately that neither $Ind_2(A)$ nor $Ind_3(A)$ is hereditary. To prove the hereditarity of $Ind_1(A)$ let us assume $I \subset J \subset A$, $I \notin Ind_1(A)$. Therefore there exists $X \subset I$, $X \neq I$ such that $cX = cI$. Now, for the set $Y = X \cup (J - I)$ we have $cY = c(X \cup (J - I)) = c^2(X \cup (J - I)) \supset c(cX \cup c(J - I)) \supset c(cI \cup (J - I)) = cJ$, and $Y \subset J$, $Y \neq J$. Hence $J \notin Ind_i(A)$.

Example 3. Let $A = R$ be the set of all real numbers and G the group of all transformations $f\colon R \to R$ preserving the set $B = \{0,1\}$. Although $R \in Ind_2(A)$ and $R \in Ind_3(A)$, the subset B of R is neither (2)- nor (3)-independent since $c(\{1\}) = cB$

The autor does not know whether $Ind_6(A)$ and $Ind_7(A)$ are in general hereditary.

3. Comparison of (i)-independences

Theorem 2. *Let be given a k-space* $\mathfrak{A}=(A, G)$. *Then*

$$\begin{array}{ccccccccc} & & & & 3 & \leftarrow & 6 & & \\ & & & & \downarrow & & \downarrow & & \\ 5 & \rightarrow & 4 & \rightarrow & 1 & \rightarrow & 2 & \leftarrow & 7 \end{array}$$

Where "$i \rightarrow j$" *means* "$Ind_i(A) \subset Ind_j(A)$".

Proof. The assertions $1\rightarrow 2$, $3\rightarrow 2$, $6\rightarrow 3$ and $6\rightarrow 7$ are obvious.

($5\rightarrow 4$). Suppose $X\cap Y=\emptyset$, $Y\neq\emptyset$, $X\cup Y=I \in Ind_5(A)$. Then $cX\cap cY=c(X\cap Y)=c\emptyset$. Moreover $Y\cap c\emptyset\subset I\cap c\emptyset=\emptyset$ implies $cY\neq c\emptyset$.

($4\rightarrow 1$) by contradiction. Let $X\subset I$, $cX=cI$, $Y=I-X\neq\emptyset$. Since $I \in Ind_4(A)$ it is $cX\cap cY=c\emptyset\neq cY$ and therefore $cY=cY\cap cI=cY\cap cX=c\emptyset\neq cY$, a contradiction.

($7\rightarrow 2$). Let be $X\subset I \in Ind_7(A)$, $cX=cI$. Now $gI\subset X$ yields $|gI|\leqq|X|\leqq|I|=|gI|$, hence $|X|=|I|$.

Theorem 3. *There is no other relation of the type* $i\rightarrow j$, $i\neq j$, $i, j \in \{1, \ldots, 7\}$ *except those seven given in Theorem 2, and their four consequences.*

Proof of the Theorem 3 consists of five examples which are summarized in the table

1	1	2	3	4	5	6	7
1	–	+	4	8	4	4	4
2	6	–	4	6	4	4	4
3	6	+	–	6	6	6	6
4	+	+	4	–	4	4	4
5	+	+	5	+	–	5	7
6	7	+	+	8	8	–	+
7	7	+	5	8	8	5	–

The symbol + in the i-row and j-column means that $i\rightarrow j$ is true. The number n in this place means that $i\rightarrow j$ fails to be true, as follows from Example n. More precisely, in Example n there is given a k-space A and a set $I\subset A$ such that I is i-independent but is not j-independent.

The next two examples deal with the 3-dimensional real projective k-space $\mathfrak{P}^3=(RP^3, GP^3)$. We recall that the support RP^3 of $\mathfrak{P}^3$ is the set of all 1-dimensional linear subspaces of R^4; the subspace given by a vector $(x^0, x^1, x^2, x^3) \in R^4-\{0\}$ will be denoted by $(x^0\colon x^1\colon x^2\colon x^3)$. The group $GL(4, R)$ of all automorphisms of a vector space R^4 can be regarded as an action on RP^3. Since the kernel of this action is the centre $C=\{\lambda E;\ \lambda \in R-\{0\}\}$ of the group $GL(4, R)$, the group GP^3 is isomorphic to the quotient group $GL(4, R)/C$.

Example 4. Let $\mathfrak{P}^3=(RP^3, GP^3)$ be the 3-dimensional real projective k-space. For the set $I\subset RP^3$ consisting of six points $A_0=(1:0:0:0)$, $A_1=(0:1:0:0)$, $A_2=(0:0:1:0)$, $A_3=(0:0:0:1)$. $J_0=(0:1:1:1)$ and $J_1=(1:0:1:1)$ we have

$$I\in Ind_i(RP^3) \quad \text{for} \quad i=1,2,4$$

and

$$I\notin Ind_i(RP^3) \quad \text{for} \quad i=3,5,6,7.$$

Proof. Let us denote $U_1=\{A_0, A_2, A_3, J_1\}$, $U_2=\{A_1, A_2, A_3, J_0\}$ and $U_3=\{A_0, A_1, J_0, J_1\}$. Then $cU_1=\{X=(x^0: x^1: x^2: x^3);\ x^1=0\}$, $cU_2=\{X;\ x^0=0\}$ and $cU_3=\{X;\ x^2=x^3\}$. Moreover

$$c(U_i\cup\{X\})=cU_i\cup\{X\} \text{ for each point } X\in RP^3, i=1,2,3 \text{ and}$$
$$cU=U \text{ for each subset } U\subset I \text{ for which } |U|<4.$$

(i=4). Suppose there is given a disjoint decomposition $X\cup Y$ of I, $|X|\leqq|Y|$. Since $|X|\leqq 3$, we have $cX=X$ and also $cY=Y$ with the exception of the following ten cases:

$$Y=U_i,\quad i=1,2,3,\quad Y=I-\{A_j\},\quad j=0,1,2,3,$$
$$Y=I-\{j_k\},\quad k=0,1,\quad Y=I.$$

Regarding all these cases one by one we always get $cX\cap cY=\emptyset$. Hence $I\in Ind_4(RP^3)$.

(i=5). Since $cU_1\cap cU_2$ is the straight line A_2A_3 and $c(U_1\cap U_2)=U_1\cap U_2=\{A_2, A_3\}$, we have $I\notin Ind_5(RP^3)$.

(i=3, 7). For the set $X=\{A_0, A_1, A_2, A_3, J\}$, $J=(1:1:1:1)$ it holds $cX=RP^3=cI$ but $|X|<|I|$. Therefore $I\notin Ind_i(RP^3)$ for i=3, 7.

Now the last three assertions (i=1, 2, 6) are a direct consequence of Theorem 2.

Example 5. For the set $I=\{A_0, A_1, A_2, A_3, J_{01}, J_{23}\}\subset RP^3$, where $J_{01}=(0:0:1:1)$ and $J_{23}=(1:1:0:0)$ we have

$$I\in Ind_i(RP^3) \quad \text{for} \quad i=1,2,4,5,7$$

and

$$I\notin Ind_i(RP^3) \quad \text{for} \quad i=3,6.$$

Proof. Owing to Theorem 2 it is sufficient to prove only the cases i=3, 5, 7. The case i=3 is proved by the same argument as in Example 4.

(i=7). Suppose $X\subset RP^3$, $cX=cI$. Then X can be written in the form $X=X_1\cup X_2$, where $X_i\subset cU_i$, $|X_i|\geqq 3$, i=1, 2 and $U_1=\{A_0, A_1, J_{23}\}$, $U_2=\{A_2, A_3, J_{01}\}$. It is easy to find a transformation $g\in GP^3$ such that $gU_i\subset X_i$ for i=1, 2.

(i=5). If $X\cap Y=I$ then $cX\cap cY=I=c(X\cap Y)$. If $U_1\subset X\cap Y\neq I$ then

$cX\cap cY = cU_1\cup(X\cap Y) = c(X\cap Y)$; similarly for $U_2\subset X\cap Y\neq I$. Otherwise $cX\cap cY = X\cap Y = c(X\cap Y)$.

Example 6. Let (Q, G) be a k-space, where Q is the set of all rational numbers endowed with the usual topology, and G the group of all homeomorphisms $Q\to Q$. For the set $I=Q$ we have

$$I\in Ind_i(Q) \quad \text{for} \quad i=2, 3$$

and

$$I\notin Ind_i(Q) \quad \text{for} \quad i=1, 4, 5, 6, 7.$$

Proof. The first two assertions are obvious. For the proof of the second row (the last five assertions) take $X=Q-\{0\}$, $Y=\{0\}$.

Example 7. Let (R, G) be a k-space, where R is the set of all real numbers endowed with the usual topology, and G the group of all homeomorphisms $R\to R$. Let Z be the set of all integers and N the set of all positive integers. For the set $I=\{p\cdot 2^{-n};\ p\in Z,\ n\in N\}$ of all dyadic numbers we have

$$I\in Ind_i(R) \quad \text{for} \quad i=2, 3, 6, 7$$

and

$$I\notin Ind_i(R) \quad \text{for} \quad i=1, 4, 5.$$

Proof. Owing to Theorem 2, only two assertions have to be proved, namely i=1 and i=6; the case i=1 is trivial. The last statement is a consequence of the following.

Lemma. *Let X be dense in R. Then there exists a homeomorphism $g\colon R\to R$ such that $gI\subset X$.*

Proof. We start with the construction of a subset $V\{x_a^i;\ i\in Z,\ a\in N\}$ of X such that for each $(i, a)\in Z\times N$ it holds

(i) $2^{-a} < x_a^{i+1} - x_a^i < 3\cdot 2^{-a}$ and

(ii) $x_a^i = x_{a+1}^{2i}$.

Construction. Since X is dense in R, there exists a sequence $V_1=\{x_1^i\}$, $i\in Z$ such that (i) holds for all $(i, a)\in Z\times\{1\}$. Suppose we have already defined the set $V_k=\{x_a^i;\ (i, a)\in Z\times\{1, \ldots, k\}\}$ such that (i) and (ii) are fulfilled for all $(i, a)\in Z\times\{1, \ldots, k\}$. The sequence x_{k+1}^i $i\in Z$ is defined as follows:

$$\text{set} \quad x_{k+1}^{2i} = x_k^i \quad \text{and}$$
$$\text{choose} \quad x_{k+1}^{2i+1}\in D_k^i\cap X \quad \text{where}$$

$D_k^i = (x_k^i + 2^{-(k+1)}, x_k^i + 3\cdot 2^{-(k+1)})\cap(x_k^{i+1} - 3\cdot 2^{-(k+1)}, x_k^{i+1} - 2^{-(k+1)})$. It is not difficult to show that $V=\bigcup_{a=1}^{\infty} V_a$ is the required set. Moreover V is dense in R. Since the map

$$g'\colon I\to V, \quad p\cdot 2^{-n}\mapsto x_n^p$$

is continuous (as a map on I) and isotonic, there exists its extension to a homeomorphism $g: R \to R$.

Example 8. Let (M, G) be a k-space defined as follows: $M = \{a_1, b_1, c_1, a_2, b_2, c_2\}$ and G is generated by three involutions f, g and h given via

$$f(a_1) = b_1, f(b_1) = a_1 \text{ and } f(x) = x \text{ for all other } x \in M,$$
$$g(a_2) = b_2\ g(b_2) = a_2 \text{ and } f(x) = x \text{ for all other } x \in M,$$
$$h(x_1) = x_2, h(x_2) = x_1 \text{ for each } x \in \{a, b, c\}.$$

For the set $I = \{a_1, a_2\}$ we have

$$I \in Ind_i(M) \quad \text{for} \quad i = 1, 2, 3, 6, 7$$

and

$$I \notin Ind_i(M) \quad \text{for} \quad i = 4, 5.$$

Proof. Because of Theorem 2 it is enough to give the proof for i = 1, 4 and 6. To show $I \notin Ind_4(M)$, set $X = \{a_1\}$, $Y = \{a_2\}$. The rest of the proof is an easy consequence of the equality $cI = M$ and the fact: if $cX = M$, then X meets each of two subsets $\{a_1, b_1\}$, $\{a_2, b_2\}$ in at least one point.

Finally let us discuss the case of finite i-independent subsets of a k-space $\mathfrak{A} \doteq (A, G)$.

Theorem 4. *Let* $\mathfrak{A} = (A, G)$ *be a k-space, F the set of all finite subsets of A. Then*

$$(*) \qquad F \cap ind_1(A) = F \cap ind_2(A)$$

and

$$\begin{array}{ccccc} & & 3 & \leftarrow & 6 \\ & & \downarrow & & \downarrow \\ 5 \to & 4 & \to & 1 & \leftarrow 7 \end{array}$$

where an arrow "$i \to j$" means "$F \cap Ind_i(A) \subset F \cap Ind_j(A)$". Moreover the diagramm is complete, i. e. there is no another relation of the type "$i \to j$", $i \neq j$, $i, j \in \{1, 3, 4, 5, 6, 7\}$ except those six given above, and their two consequences.

Proof. The equality (*) is obvious and the diagram in question is a direct consequence of that in Theorem 2. The completness of the diagramm follows from the table (notation as in the proof of Theorem 3) and Example 9.

	1	3	4	5	6	7
1	–	4	8	4	4	4
3	+	–	8	8	9	9
4	+	4	–	4	4	4
5	+	5	+	–	5	9
6	+	+	8	8	–	+
7	+	5	8	8	5	–

Example 9. Let $\mathfrak{E}^1 = (R, GE)$ be the Euclidean line regarded as a k-space; i. e. R is the set of all reals and GE the group of all isometries $f_{a,b}: R \to R$, $x \mapsto ax + b$, $a \in \{-1, +1\}$, $b \in R$. For the set $I = \{0, 1\}$ we have

$$I \in Ind_i(\mathfrak{E}^1) \quad \text{for} \quad i = 1, 3, 4, 5$$

and

$$I \notin Ind_i(\mathfrak{E}^1) \quad \text{for} \quad i = 6, 7.$$

Proof. Since $cI = R$, $c\{0\} = \{0\}$ and $c\{1\} = \{1\}$, the proof of the first row is obvious. On the other hand $c\{0, 2\} = R$ but there is no transformation $f_{a,b} \in GE$ which carries I into $\{0, 2\}$.

4. Frame

Definition 2. *Let $\mathfrak{A} = (A, G)$ be a k-space and c its k-closure operation. A set $F \in 2^A$ is said to be an i-frame of A if*

(i) $F \in Ind_i(A)$ *and*
(ii) $cF = A$,

$i = 1, \ldots, 7$. *The set of all i-frames in A will be denoted by $Frm_i(A)$. If $Frm_i(A) \neq \emptyset$ then A is called i-frameable and i-unframeable in the opposite case $Frm_i(A) = \emptyset$.*

Theorem 5. *Each k-space is both 2- and 3-frameable.*

Proof. Let be given a k-space $\mathfrak{A} = (A, G)$ and let $\mathcal{P}$ be the subset of 2^A consisting of exactly those $X \subset A$ for which $cX = A$. Since $A \in \mathcal{P}$, $\mathcal{P}$ is not empty. Therefore there exists such a set $F \in \mathcal{P}$ that $|F| \leqq |X|$ for all $X \in \mathcal{P}$. Thus $F \in Frm_i(A)$ for both $i = 2$ and $i = 3$.

Theorem 6. *To each $i \in \{1, 4, 5, 6, 7\}$ there exists a k-space A_i such that $Frm_i(A_i) = \emptyset$.*

Proof. The assertion for $i = 1, 4$ and 5 follows from the k-space (R, G) described in Example 7. In fact, if $I \subset R$ is dense in R (i. e. $cI = R$), and $a \in I$ is a point, then $X = I - \{a\}$ is dense in R as well, hence $cX = R$. Therefore I is not 1-independent, thus $Frm_1(R) = \emptyset$. Now, because of Theorem 2, it is $Frm_5(R) \subset Frm_4(R) \subset Frm_1(R) = \emptyset$.

The assertion for $i = 6$ and 7 follows from the obvious fact $Frm_6(\mathfrak{E}^1) = Frm_7(\mathfrak{E}^1) = \emptyset$ (see Example 9.).

The end of this section is devoted to one homogenous k-space with some surprising properties. For example, if a finite subset I is i-independent then $|I| = 1$.

Theorem 7. *There exists a homogenous k-space $\mathfrak{A} = (A, G)$ such that to each point $x \in A$ there is a sequence $x_i \in A$; $i \in Z$ with the properties*

(i) $x_0 = x$,

(ii) *if* $i<j$ *then* $\varphi(x_i) \lhd \varphi(x_j)$ *and*

(iii) *if* $i<j$ *then* $c\{x_i\} \supsetneqq c\{x_j\}$,

where $\varphi(z) = \{g \in G;\, g(z) = z\}$ *is the stationary subgroup of G with respect to a point* $z \in A$ *and c is the k-closure operation on A.*

Moreover the sequence in question can be defined by $x_i = f^i(x)$, *where* $f \in G$ *does not depend upon x.*

Proof. See Example 10.

Throughout the remainder of this paper the brackets symbol [] or {} will have always the meaning of the *integer* or *fractional part* of a real numer, respectively. Hence

$$R \to Z \times [0,1),\ x \mapsto ([x], \{x\})$$

is the uniquely defined decomposition of reals. We notice three relations, which will be useful below,

(a) $[[a] + \{b\}] = [[a]] = [a]$,

(b) $\{[a] + b\} = \{\{b\}\} = \{b\}$ and

(c) $\{\{a\} - b\} = \{a - b\}$

for all $a, b \in R$.

Example 10. The required k-space $\mathfrak{A} = (A, G)$ will be obtained as one orbit of the k-space $\mathfrak{B} = (R, H)$ where H is generated by

$$f\colon R \to R,\ x \mapsto x + 1$$

and

$$g\colon R \to R,\ x \mapsto [x] + \{x + \beta[x]\};$$

where the map $\beta\colon Z \to R$ will be specified later.

The construction, namely the definition of g, is justified by.

Lemma 1. *The map*

$$h\colon R \to R,\ y \mapsto [y] + \{y - \beta[y]\}$$

is inverse to the map g given in Example 10. Hence $h, g \in \mathcal{T}(R)$.

Proof. Choose $x \in R$ and denote $y = gx$. From (a) it follows $[(hg)x] = [hy] = [y] = [gx] = [x]$, therefore $[(hg)x] = [x]$. On the other hand (b), (c) and the already proved relation $[x] = [y]$ yields

$$\{(hg)x\} = \{hy\} = \{y - \beta[y]\} = \{gx - \beta[x]\} =$$
$$= \{[x] + \{x + \beta[x] - \beta[x]\} = \{\{x + \beta[x]\} - \beta[x]\} = \{x\}.$$

Since $(hg)x = [(hg)x] + \{(hg)x\} = [x] + \{x\} = x$, it is $hg = 1_R$. Similarly $gh = 1_R$.

Our next task is to describe the stationary subgroup $\varphi(x) \subset H$ for any $x \in R$. Lemma 2 gives a tool and Lemma 3 an important result.

Lemma 2. *Let $k \geqq 0$ be an integer and let $(c, a_1, \ldots, a_k, b_1, \ldots, b_k)$ be an ordered $(2k+1)$-tuplesof integers with $b_i \neq 0$ for all $i = 1, \ldots, k$. Then the map*

$$t: R \to R, \ x \mapsto [x] + c + \{x + b_1\beta([x] + a_1) + \\ + \ldots + b_k\beta([x] + a_k)\}$$

belongs to the group H. Moreover each $t \in H$ can be written in such a form.

Proof. Let us denote

(*) $t_a = f^{-a} \circ g \circ f^a : x \mapsto [x] + \{x + \beta([x] + a)\}$.

A straightforward computation shows that

(**) $t = f^c \circ (t_{a_k})^{b_k} \circ \ldots \circ (t_{a_1})^{b_1}$,

hence $t \in H$. Conversely each $t \in H$ is of the form

$$t = f^{c_k} \circ g^{b_k} \circ \ldots \circ f^{c_1} \circ g^{b_1} \circ f^{c_0},$$
$$c_i, b_i \in Z, b_i \neq 0 \quad \text{for all } i.$$

Under the substitution $a_1 = c_0$, $a_2 = c_0 + c_1, \ldots, a_k = c_0 + \ldots + c_{k-1}$, $c = c_0 + \ldots + c_k$, the transformation t becomes that in (**).

Lemma 3. *The subgroup H' of H generated by all transformations t_a, $a \in Z$, is Abelian. If a transformation $t \in H$ has at least one invariant point, i. e. if $t \in \varphi(z)$ for at least one $z \in R$, then $t \in H'$.*

Proof. It is not difficult to show that

$$(t_b \circ t_a)(x) = [x] + \{\beta([x] + a) + \beta([x] + b)\},$$

hence H' is Abelian. A transformation $t \in H$, written in the form (**) belongs to the group H' if and only if $c = 0$. If $t \in \varphi(z)$, then $[t(z)] = [z]$ hence $c = 0$.

Remark. H is Abelian if and only if β is constant.

To finish the example we shall specify the function β putting

$$\beta: Z \to R, \quad n \mapsto 2^n.$$

Now, according to Lemma 2, each $t \in H$ is of the form

$$t: R \to R, \quad x \mapsto [x] + c + \{x + 2^{[x]}d\}$$

where the dyadic number

$$d = b_1 2^{a_1} + \ldots + b_k 2^{a_k}$$

can be expressed in the canonical form

$$d = 2^{p_1} + \ldots + 2^{p_n}, \quad p_1 < \ldots < p_n \in Z.$$

Lemma 4. *If $z \in R$ is a point then the stationary subgroup $\varphi(z)$ of H is Abelian and consists of exactly those transformations*

$$t\colon R \to R, \quad x \mapsto [x] + \{x + 2^{|x|}(2^{p_1} + \ldots + 2^{p_n})\}$$

for which $[z] + p_1 \geqq 0$.

Proof. By Lemma 3, $\varphi(z)$ is Abelian and $t \in \varphi(z)$ imply $c=0$ and $[t(z)] = [z]$. Therefore $t(z) = z$ if and only if $\{t(z)\} = \{z\}$, i. e.

$$2^{|x|}(2^{p_1} + \ldots + 2^{p_n}) \in Z.$$

The last condition is equivalent to $[z] + p_1 \geqq 0$.

Corollary. *It holds*

$$[x] = [y] \Leftrightarrow \varphi(x) = \varphi(y) \Leftrightarrow c(x) = c(y),$$
$$x \leqq y \Rightarrow \varphi(x) \subset \varphi(y), \quad c(x) \supset c(y),$$
$$c(x) = [[x], +\infty).$$

The proof f Theorem 7 is finished.

REFERENCES

[1] BIRKHOFF, G.: Lattice theory. 3. ed. Providence 1967.

[2] GLAZEK, K.: Independence with respect to family of mappings in abstract algebras. Dissertationes Mathematicae, Warszawa 1971.

[3] KLEIN, F.: Vergleichende Betrachtungen über neuere geometrische Forschungen („Erlanger Programm"). Ges. Abh. Bd. 1. Berlin 1921, 400—497.

[4] MARCZEWSKI, E.: Independence and homomorphisms in abstract algebras. Fundamenta Math., 50, 1961, 45—61.

[5] MARCZEWSKI, E.: Independence with respect to a family of mappings. Colloq. Math., 20, 1968, 11—17.

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НЕЗАВИСИМОСТЬ В ПРОСТРАНСТВЕ КЛЕЙНА

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Резюме

Пусть G группа перестановок множества A. Для любого множества $X \subset A$ определяется его замыкание cX как множество всех точек $y \in A$ неподвижных относительно любой перестановки $f \in G$ для которой $f(x) = x$, $\forall x \in X$. Подмножество $I \subset A$ называется (i)-независимым если выполняется условие (i), $i = 1, \ldots, 7$ (Определение 1.). Вопрос взаимного отношения этих определении решают Теоремы 2. и 3. Подмножество $F \subset A$ называется (i)-репером если $cF = A$ и F (i)-независимо (Определение 2.). Существование (i)-репера решают теоремы 5. и 6.