Mathematica Slovaca

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Mathematica Slovaca, Vol. 51 (2001), No. 1, 45--57

Persistent URL: `http://dml.cz/dmlcz/130576`

Math. Slovaca, **51** (2001). No. 1, 45–57

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ON THE CIRCLE PROBLEM WITH GENERAL WEIGHT

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(*Communicated by Stanislav Jakubec*)

ABSTRACT. Let $R(\omega; a, b, r)$ be the number of lattice points that lie within a circle with center (a, b) and radius r, each lattice point (x, y) counted with *weight* $\omega(x, y)$. In this article, which continues earlier research, an asymptotic evaluation of $R(\omega; a, b, r)$ is given where the error estimation is uniform in the three circle parameters a, b, r and the weight function ω.

1. Introduction and statement of results

For $a, b, r \in \mathbb{R}$, $r \geq 1$, consider the disc $D = \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 \leq r^2\}$, and for a real valued function ω with $D(a, b; r) := D \subset \operatorname{dom}\omega \subset \mathbb{R}^2$ let

$$R(\omega; a, b; r) = \sum_{(x,y)\in D\cap\mathbb{Z}^2} \omega(x, y).$$

In the present paper we are going to study the asymptotic behaviour of the function $R(\omega; a, b; r)$. The error estimates of the expansion of $R(\omega; a, b; r)$ shall be uniform on the parameter domain $\{(a, b; r) \in \mathbb{R}^2 \times [1, \infty[: D(a, b; r) \subset \operatorname{dom}\omega\}$.

In previous articles [5], [6] we have studied the special case that the function ω is a polynomial. Now we are going to present a general result and apply it to special weight functions. A first result is the following proposition.

PROPOSITION. *Let $\omega\colon D \to \mathbb{R}$ be continuous and let $\omega(\cdot, y)$ be piecewise monotonic on $I_y := \{x : (x, y) \in D\}$ for all $y \in [b-r, b+r] \cap \mathbb{Z}$. Furthermore assume that the function $F(y) := \int_{I_y} \omega(\xi, y)\, \mathrm{d}\xi$ is piecewise monotonic*

2000 Mathematics Subject Classification: Primary 11P21; Secondary 11N37.
Key words: lattice point, asymptotic evaluation.

on $b-r \le y \le b+r$. *Then*

$$R(\omega;a,b;r) =$$
$$= \iint\limits_D \omega(x,y)\ \mathrm{d}(x,y) + O\Big(r^{\frac{46}{73}}(\log r)^{\frac{315}{146}}|\min_D \omega|\Big) + O\Big(r|\max_D \omega - \min_D \omega|\Big).$$

The O*-constants are absolute.*

Remark. The number of "pieces" of $\omega(\cdot,y)$, $F(\cdot)$ and all other piecewise monotonic functions occurring in this paper is assumed to be absolutely bounded throughout.

The second monotony-condition in the Proposition is rather technical and possibly hard to check in concrete situations. This problem can be facilitated and a better error estimate can be obtained if additional assumptions on ω are made as it is done in the following theorem.

THEOREM 1. *Let* $\omega\colon D \to \mathbb{R}$ *be a* C^1*-function and assume that* $\frac{\partial\omega}{\partial x}(\cdot,y)$ *and* $\frac{\partial\omega}{\partial y}(x,\cdot)$ *are piecewise monotonic for all* $y \in [b-r,b+r] \cap \mathbb{Z}$ *and for all* $x \in [a-r,a+r]$, *respectively. Furthermore assume that the functions* $B(\theta) := \omega(a+r\cos\theta, b+r\sin\theta)$ *and* $B_1(\theta) := (\tan\theta)\big(B(\theta) - \min_D \omega\big)$ *are piecewise monotonic on* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ *and* $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. *Then (with absolute* O*-constants)*

$$R(\omega;a,b;r) = \iint\limits_D \omega(x,y)\ \mathrm{d}(x,y) + O\Big(r^{\frac{46}{73}}(\log r)^{\frac{315}{146}}|\min_D \omega|\Big)$$
$$+ O\Big(r^{\frac{2}{3}}|\max_{\partial D} \omega - \min_D \omega|\Big) + O\Big(r\Big(\max_D\Big|\frac{\partial\omega}{\partial x}\Big| + \max_D\Big|\frac{\partial\omega}{\partial y}\Big|\Big)\Big),$$

where $\partial D = \{(a+r\cos\theta, b+r\sin\theta) :\ 0 \le \theta \le 2\pi\}$ *is the boundary of* D.

In the special case that on the one hand the center (a,b) of the circle is always a lattice point and on the other hand the weight ω is *rotationally symmetric*, the error estimations of Theorem 1 can be improved in the following way.

THEOREM 2. *For* $a,b \in \mathbb{Z}$, $r \in [1,\infty[$, *let* ω *be defined on* $D(a,b;r)$ *by* $\omega(x,y) = f\big((x-a)^2 + (y-b)^2\big)$, *where* $f\colon [0,r^2] \to \mathbb{R}$ *is continuously differentiable and piecewise monotonic. Then (with an absolute* O*-constant)*

$$R(\omega;a,b;r) = \iint\limits_D \omega(x,y)\ \mathrm{d}(x,y) + O\Big(r^{\frac{46}{73}}(\log r)^{\frac{315}{146}}\max_D|\omega|\Big):$$

Remark. The assumption that f is piecewise monotonic is nearby since it follows from an assumption in Theorem 1. Actually, if $\frac{\partial\omega}{\partial x}(\cdot, y)$ is assumed to be piecewise monotonic for $y = b$, then $\frac{\partial\omega}{\partial x}(x,b) = 2(x-a)f'\big((x-a)^2\big)$ has only $O(1)$ points of zero and thus $f\big((x-a)^2\big) = f(0) + 2\int_a^x (\xi - a)f'\big((\xi-a)^2\big)\,\mathrm{d}\xi$ is piecewise monotonic on $a \le x \le a + r$, therefore $f(u)$ is is piecewise monotonic on $0 \le u \le r^2$.

2. Applications

A natural application of Theorem 1 is one to weight functions F that are *generalized polynomials*, i.e.

$$F(x,y) = \sum_{(k,l)\in M} A_{k,l} x^{\alpha_k} y^{\beta_l},$$

where $M \subset \mathbb{N}^2$ is finite and the coefficients $A_{k,l}$ and the exponents α_k, β_l are arbitrary real numbers.

For the special case that $\alpha_k, \beta_l \in \mathbb{N}_0$, i.e. that $F \in \mathbb{R}[X,Y]$, we refer to [6], where a different approach leads to better results.

Since

$$R\big(F(x,y); a, b; r\big) = \sum_{(k,l)\in M} A_{k,l} R\big(x^{\alpha_k} y^{\beta_l}; a, b; r\big),$$

it is sufficient to consider $R(f; a, b; r)$, where $f(x,y) = x^\alpha y^\beta$ with arbitrary real α, β.

Since $R(x^\alpha y^\beta; a, b; r) = R(x^\beta y^\alpha; b, a; r)$, we may assume w.l.o.g. that $\alpha \ge \beta$. Furthermore, for the sake of simplicity we consider only the case that $\alpha, \beta \ge 0$.

The following corollary deals with the case that $\beta = 0$.

Corollary 1. *For arbitrary $\alpha > 0$ we have for $r \to \infty$ (uniformly in the region $a - r \ge 0$ if $\alpha \ge 1$, and uniformly in the region $a - r \ge 1$ if $\alpha < 1$),*

$$\sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x-a)^2+(y-b)^2\le r^2}} x^\alpha \sim \iint_{(x-a)^2+(y-b)^2\le r^2} x^\alpha \,\mathrm{d}(x,y).$$

More precisely,

$$\sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x-a)^2+(y-b)^2\le r^2}} x^\alpha = \iint_{(x-a)^2+(y-b)^2\le r^2} x^\alpha \,\mathrm{d}(x,y) + \Delta(a,r),$$

where $\Delta(a,r)$ *can be estimated as follows.*

(i) *If* $\alpha \geq 1$, *then for* $a \geq r$,

$$\Delta(a,r) \ll (a-r)^{\alpha} r^{\frac{46}{73}} (\log r)^{\frac{315}{146}} + a^{\alpha-1} r^{\frac{5}{3}} \ll r^{\frac{2}{3}} a^{\alpha} .$$

(ii) *If* $0 < \alpha \leq 1/2$, *then for* $a \geq r+1$,

$$\Delta(a,r) \ll (a-r)^{\alpha} r^{\frac{46}{73}} (\log r)^{\frac{315}{146}} + r^{\frac{2}{3}} \min\left\{a^{\alpha}, \frac{r}{a^{\alpha}}\right\} + \frac{r}{(a-r)^{1-\alpha}} .$$

(iii) *If* $1/2 \leq \alpha < 1$, *then for* $a \geq r+1$,

$$\Delta(a,r) \ll (a-r)^{\alpha} r^{\frac{46}{73}} (\log r)^{\frac{315}{146}} + \frac{r^{\frac{5}{3}}}{a^{1-\alpha}} + \frac{r}{(a-r)^{1-\alpha}} .$$

Proof. Theorem 1 yields (since $a + r \asymp a$)

$$\Delta(a,r) \ll (a-r)^{\alpha} r^{\frac{46}{73}} (\log r)^{\frac{315}{146}} + r^{\frac{2}{3}} T + C ,$$

where $C = ra^{\alpha-1}$ if $\alpha \geq 1$, and $C = r(a-r)^{\alpha-1}$ if $0 < \alpha < 1$, and

$$T = (a+r)^{\alpha} - (a-r)^{\alpha} = 2\alpha r (a+\varepsilon r)^{\alpha-1} \qquad (\varepsilon = \varepsilon(a,r) \in \,]-1,1[\,) .$$

Thus for $\alpha \geq 1$, $T \leq 2\alpha r(a+r)^{\alpha-1} \ll ra^{\alpha-1} \leq a^{\alpha}$. This proves (i). If $0 < \alpha < 1$, we compute

$$T = \frac{(a+r)^{2\alpha} - (a-r)^{2\alpha}}{(a+r)^{\alpha} + (a-r)^{\alpha}} \leq 4\alpha r \frac{(a+\varepsilon r)^{2\alpha-1}}{(a+r)^{\alpha}} \qquad (\,|\varepsilon| < 1\,) .$$

Thus we obtain

$$T \ll r(a+r)^{\alpha-1} \leq ra^{\alpha-1} \qquad \text{if} \quad \alpha \geq \frac{1}{2}$$

and

$$T \ll r(a+r)^{-\alpha} \leq ra^{-\alpha} \qquad \text{if} \quad \alpha \leq \frac{1}{2} .$$

Of course, $T \ll a^{\alpha}$ is still true and this proves (ii) and (iii).

Finally, $\iint_D x^{\alpha} \, \mathrm{d}(x,y) \geq a^{\alpha} r^2 \pi/2$, and this concludes the proof of Corollary 1. □

Now we consider the case that $\alpha \geq \beta > 0$.

Corollary 2. *For $\alpha \geq \beta > 0$ we have for $r \to \infty$ (uniformly in the region $a \geq r+\delta(\alpha)$, $b \geq r+\delta(\beta)$, with $\delta(z)=0$ for $z \geq 1$ and $\delta(z)=1$ for $0<z<1$),*

$$\sum_{\substack{(x,y)\in\mathbb{Z}^2\\(x-a)^2+(y-b)^2\leq r^2}} x^\alpha y^\beta \sim \iint_{(x-a)^2+(y-b)^2\leq r^2} x^\alpha y^\beta \, \mathrm{d}(x,y).$$

More precisely,

$$\sum_{\substack{(x,y)\in\mathbb{Z}^2\\(x-a)^2+(y-b)^2\leq r^2}} x^\alpha y^\beta = \iint_{(x-a)^2+(y-b)^2\leq r^2} x^\alpha y^\beta \, \mathrm{d}(x,y) + \Delta(a,b;r),$$

where $\Delta(a,b;r)$ can be estimated as follows.

(i) *If $\alpha, \beta \geq 1$, then for $a, b \geq r$,*

$$\Delta(a,b;r) \ll r^{\frac{46}{73}}(\log r)^{\frac{315}{146}} a^\alpha b^\beta + r^{\frac{5}{3}}\left(a^\alpha b^{\beta-1} + a^{\alpha-1}b^\beta\right).$$

(ii) *If $1/2 \leq \beta \leq 1 \leq \alpha$, then for $a \geq r$ and $b \geq r+1$,*

$$\Delta(a,b;r) \ll r^{\frac{46}{73}}(\log r)^{\frac{315}{146}} a^\alpha b^\beta + r^{\frac{5}{3}}\left(a^\alpha b^{\beta-1} + a^{\alpha-1}b^\beta\right) + \frac{r a^\alpha}{(b-r)^{1-\beta}}.$$

(iii) *If $1/2 \leq \beta \leq \alpha \leq 1$, then for $a, b \geq r+1$,*

$$\Delta(a,b;r) \ll$$

$$\ll r^{\frac{46}{73}}(\log r)^{\frac{315}{146}} a^\alpha b^\beta + r^{\frac{5}{3}}\left(a^\alpha b^{\beta-1} + a^{\alpha-1}b^\beta\right) + r\left(\frac{a^\alpha}{(b-r)^{1-\beta}} + \frac{b^\beta}{(a-r)^{1-\alpha}}\right).$$

(iv) *If $0 < \beta \leq 1/2 \leq \alpha \leq 1$, then for $a, b \geq r+1$,*

$$\Delta(a,b;r) \ll$$

$$\ll r^{\frac{46}{73}}(\log r)^{\frac{315}{146}} a^\alpha b^\beta + r^{\frac{5}{3}}\left(\frac{a^\alpha}{b^\beta} + a^{\alpha-1}b^\beta\right) + r\left(\frac{a^\alpha}{(b-r)^{1-\beta}} + \frac{b^\beta}{(a-r)^{1-\alpha}}\right).$$

(v) *If $0 < \beta \leq \alpha \leq 1/2$, then for $a, b \geq r+1$,*

$$\Delta(a,b;r) \ll r^{\frac{46}{73}}(\log r)^{\frac{315}{146}} a^\alpha b^\beta + r^{\frac{5}{3}}\left(\frac{a^\alpha}{b^\beta} + \frac{b^\beta}{a^\alpha}\right) + r\left(\frac{a^\alpha}{(b-r)^{1-\beta}} + \frac{b^\beta}{(a-r)^{1-\alpha}}\right).$$

Proof. For $f(x,y) = x^\alpha y^\beta$ let

$$A = \max_{\partial D} f, \quad B = \min_{D} f, \quad C = \max_{D}\left|\frac{\partial f}{\partial x}\right|, \quad D = \max_{D}\left|\frac{\partial f}{\partial y}\right|.$$

Furthermore let $T = (a+r)^\alpha(b+r)^\beta - (a-r)^\alpha(b-r)^\beta$. We have $A \le (a+r)^\alpha(b+r)^\beta$, $(a-r)^\alpha(b-r)^\beta \le B \le (a+r)^\alpha(b+r)^\beta$, thus $A - B \le T$, and $C \le \alpha(a+r)^{\alpha-1}(b+r)^\beta$ for $\alpha \ge 1$, $C \le \alpha(a-r)^{\alpha-1}(b+r)^\beta$ for $\alpha < 1$, $D \le \beta(a+r)^\alpha(b+r)^{\beta-1}$ for $\beta \ge 1$, $D \le \beta(a+r)^\alpha(b-r)^{\beta-1}$ for $\beta < 1$. Furthermore

$$T = 2r\big(\alpha(a+\varepsilon r)^{\alpha-1}(b+\varepsilon r)^\beta + \beta(a+\varepsilon r)^\alpha(b+\varepsilon r)^{\beta-1}\big) \qquad (\,|\varepsilon| < 1\,),$$

and

$$T = \frac{(a+r)^{2\alpha}(b+r)^{2\beta} - (a-r)^{2\alpha}(b-r)^{2\beta}}{(a+r)^\alpha(b+r)^\beta + (a-r)^\alpha(b-r)^\beta}$$
$$\ll ra^{-\alpha}b^{-\beta}\big((a+\varepsilon r)^{2\alpha-1}(b+\varepsilon r)^{2\beta} + (a+\varepsilon r)^{2\alpha}(b+\varepsilon r)^{2\beta-1}\big) \qquad (\,|\varepsilon| < 1\,).$$

Now it is straightforward to check clauses (i) to (v) and this proves Corollary 2. □

We conclude this section with an example where ω is defined on $\mathbb{R}^2$, but ω is not a polynomial.

COROLLARY 3. *Let $\alpha, \beta > 0$ be fixed. Then for $r \to \infty$,*

$$\sum_{\substack{(x,y)\in\mathbb{Z}^2\\ (x-a)^2+(y-b)^2\le r^2}} |x|^\alpha|y|^\beta$$
$$= \iint\limits_{(x-a)^2+(y-b)^2\le r^2} |x|^\alpha|y|^\beta \,\mathrm{d}(x,y) + O\big(r^{\alpha+\beta+\frac{2}{3}}\big) + O\big(r^{\alpha+1}\big) + O\big(r^{\beta+1}\big)$$

uniformly in $a, b \ll r$.

Proof. Again we may assume w.l.o.g. that $\alpha \ge \beta$. For $\omega(x,y) = |x|^\alpha|y|^\beta$ there is no problem with the conditions of Theorem 1 if $\alpha, \beta > 1$. Otherwise, we consider the weight

$$\omega_1(x,y) = \begin{cases} |x|^\alpha g(y) & \text{if } \alpha > 1 \ge \beta > 0\,,\\ f(x)g(y) & \text{if } 1 > \alpha \ge \beta > 0\,,\end{cases}$$

where

$$f(x) = \begin{cases} |x|^\alpha & \text{if } |x| \ge 1\,,\\ \frac{\alpha}{2}x^2 + 1 - \frac{\alpha}{2} & \text{if } |x| \le 1\,,\end{cases} \quad \text{and} \quad g(y) = \begin{cases} |y|^\beta & \text{if } |y| \ge 1\,,\\ \frac{\beta}{2}y^2 + 1 - \frac{\beta}{2} & \text{if } |y| \le 1\,.\end{cases}$$

Then ω_1 is continuously differentiable (and fulfills all the other conditions of Theorem 1, too) and $\iint\limits_D \omega_1 = \iint\limits_D \omega + O\big(r^{\alpha+1}\big) + O\big(r^{\beta+1}\big)$ and $\sum\limits_D \omega_1 = \sum\limits_D \omega + O\big(r^{\alpha+1}\big) + O\big(r^{\beta+1}\big)$. We have $0 \le \min\limits_D \omega_1 \ll 1$, $0 \le \max\limits_D \omega \le r^\alpha r^\beta$, and

$$\max_D\left|\frac{\partial\omega_1}{\partial x}\right| + \max_D\left|\frac{\partial\omega_1}{\partial y}\right| \ll r^{\alpha+\beta-1} + r^{\alpha-1} + r^{\beta-1} + 1\,.$$

This proves Corollary 3. □

Remark. If the center coordinates a, b are not bounded by r, then results similar to Corollary 1 and Corollary 2 can be obtained. On the other hand, if we substitute $x^\alpha y^\beta$ by $\omega_1(x,y)$ from above, Corollary 1 and Corollary 2 can be adapted in a way that the error estimates are uniform in $a, b \geq r$. Specifically, the asymptotic equivalence between the double sum and the double integral is always uniform in the region $a, b \geq r$.

3. Preparation of the proof Some lemmata

Let the rounding error functions ψ and ψ_1 be defined by

$$\psi(z) = z - [z] - \frac{1}{2} \quad (z \in \mathbb{R}) \qquad \text{and} \qquad \psi_1(z) = \begin{Bmatrix} \psi(z) & \iff & z \notin \mathbb{Z} \\ 1/2 & \iff & z \in \mathbb{Z} \end{Bmatrix} \quad (z \in \mathbb{R})$$

throughout the paper. ([ ] are the Gauss brackets.)

Lemma 1. (Abelian summation) *For arbitrary* $P, Q \in \mathbb{Z}$, $P \leq Q$ *and* $g, h\colon \mathbb{Z} \to \mathbb{C}$,

$$\sum_{k=P}^{Q} g(k)h(k) = g(Q) \sum_{k=P}^{Q} h(k) + \sum_{l=P}^{Q-1} \big(g(l) - g(l+1)\big) \sum_{k=P}^{l} h(k).$$

Lemma 2. (Euler summation formula, cf. [2], [4]) *For every real valued continuous and piecewise monotonic function* f *on* $[\alpha, \beta] \subset \mathbb{R}$,

(i)

$$\sum_{\alpha \leq k \leq \beta} f(k) = \int_\alpha^\beta f(t)\, \mathrm{d}t + O\Big(\max_{\alpha \leq t \leq \beta} |f(t)| \Big).$$

If f *is continuous on* $[\alpha, \beta]$ *and continuously differentiable on* $]\alpha, \beta[$, *then*

(ii)

$$\sum_{\alpha \leq k \leq \beta} f(k) = \int_\alpha^\beta f(t)\, \mathrm{d}t + \psi_1(\alpha) f(\alpha) - \psi(\beta) f(\beta) + \int_\alpha^\beta \psi(t) f'(t)\, \mathrm{d}t.$$

Lemma 3. (van der Corput, cf. [1], [4]) *Let f be a real valued function, twice continuously differentiable on $[\alpha,\beta]\subset\mathbb{R}$. Furthermore, let f'' be monotonic and nonzero on $[\alpha,\beta]$. Then for $\varphi\in\{\psi,\psi_1\}$,*

$$\sum_{\alpha\le k\le\beta}\varphi(f(k))\ll\int_\alpha^\beta |f''(t)|^{\frac13}\,\mathrm{d}t+|f''(\alpha)|^{-\frac12}+|f''(\beta)|^{-\frac12},$$

where the $\ll$-constant is absolute.

The following lemma is an immediate consequence of the second mean value theorem.

Lemma 4. *Let f be a real valued function, continuous and piecewise monotonic on $[\alpha,\beta]\subset\mathbb{R}$. Then for $\varphi\in\{\psi,\psi_1\}$,*

$$\left|\int_\alpha^\beta \varphi(t)f(t)\,\mathrm{d}t\right|\le\frac{c}{4}\max_{\alpha\le t\le\beta}|f(t)|,$$

where c is the number of monotonic pieces of f.

4. Proof of the Proposition and Theorem 1

We write

$$R(\omega;a,b;r)=\sum_{b-r\le y\le b+r}\ \sum_{\alpha(y)\le x\le\beta(y)} g(x,y)+\Bigl(\min_D\omega\Bigr)\bigl(\#(D\cap\mathbb{Z}^2)\bigr),$$

where

$$\alpha(y)=a-\sqrt{r^2-(y-b)^2},\qquad \beta(y)=a+\sqrt{r^2-(y-b)^2},$$

and

$$g(x,y)=\omega(x,y)-\min_D\omega.$$

Note that $(\alpha(y),y),(\beta(y),y)\in\partial D$ for all $y\in[b-r,b+r]$.

It is well known that (with an absolute O-constant)

$$\#\bigl(D(a,b;r)\cap\mathbb{Z}^2\bigr)=r^2\pi+O\Bigl(r^{\frac{46}{73}}(\log r)^{\frac{315}{146}}\Bigr).$$

This deep result was proved by Huxley in 1993. A proof can be found in [3; Theorem 18.3.2].

Now we apply twice Lemma 2 (i) to the above double sum and, since $\max|g|=|\max\omega-\min\omega|$ and $\iint_D \mathrm{d}(x,y)=r^2\pi$ this proves the Proposition.

In order to prove Theorem 1 we apply Lemma 2(ii) and compute

$$\sum\sum g = \sum_{b-r\le y\le b+r}\left(\int\limits_{\alpha(y)}^{\beta(y)} g(x,y)\,\mathrm{d}x + \int\limits_{\alpha(y)}^{\beta(y)} \frac{\partial\omega}{\partial x}(\xi,y)\psi(\xi)\,\mathrm{d}\xi\right) + S_1 - S_2\,,$$

where

$$S_1 = \sum_{b-r\le y\le b+r} g(\alpha(y),y)\psi_1(\alpha(y)) \qquad \text{and} \qquad S_2 = \sum_{b-r\le y\le b+r} g(\beta(y),y)\psi(\beta(y))\,.$$

First we estimate the sums S_1 and S_2. For $l\in[b-r,b+r]\cap\mathbb{Z}$, let

$$\Psi_1(l) = \sum_{b-r\le y\le l} \psi_1\left(a-\sqrt{r^2-(y-b)^2}\,\right).$$

Then, by Lemma 1,

$$S_1 = g\big(\alpha([b+r]),[b+r]\big)\Psi_1([b+r]) + \sum_{b-r\le l\le b+r-1}\big(g(\alpha(l),l)-g(\alpha(l+1),l+1)\big)\Psi_1(l)\,.$$

Since ω is piecewise monotonic on the boundary of D, $g\big(\alpha(\cdot),\cdot\big)$ is piecewise monotonic, too, and $g\big(\alpha(l),l\big) - g\big(\alpha(l+1),l+1\big)$ changes its sign only $O(1)$ times. Thus

$$S_1 \ll \Big(\max_{\partial D}|g|\Big)\Big(\max_{b-r\le y\le b+r}|\Psi_1(l)|\Big)\,.$$

Now we apply van der Corput's Method (Lemma 3) on $\Psi_1(l)$ for every l and obtain $\Psi_1(l)\ll r^{2/3}$ uniformly in l, and this yields

$$S_1 \ll r^{\frac23}\Big(\max_{\partial D}|g|\Big) = r^{\frac23}\big|\max_{\partial D}\omega - \min_{D}\omega\big|\,.$$

The sum S_2 can be treated analogously and the same estimate is obtained. Now we concentrate on the integrals. By Lemma 4 we have

$$\sum_{b-r\le y\le b+r}\ \int\limits_{\alpha(y)}^{\beta(y)} \frac{\partial\omega}{\partial x}(\xi,y)\psi(\xi)\,\mathrm{d}\xi \ll r\max_{D}\left|\frac{\partial\omega}{\partial x}\right|.$$

By Lemma 2(ii),

$$\sum_{b-r\le y\le b+r}\ \int\limits_{\alpha(y)}^{\beta(y)} g(x,y)\,\mathrm{d}x$$

$$= \iint\limits_{D} g(x,y)\,\mathrm{d}(x,y) + \int\limits_{b-r}^{b+r}\left(\frac{\mathrm{d}}{\mathrm{d}y}\int\limits_{\alpha(y)}^{\beta(y)} g(x,y)\,\mathrm{d}x\right)\psi(y)\,\mathrm{d}y\,,$$

since $\alpha(b\pm r)=a=\beta(b\pm r)$. Now for $b-r<y<b+r$,

$$\frac{\mathrm{d}}{\mathrm{d}y}\int\limits_{\alpha(y)}^{\beta(y)} g(x,y)\;\mathrm{d}x=\int\limits_{\alpha(y)}^{\beta(y)}\frac{\partial\omega}{\partial y}(x,y)\;\mathrm{d}x-\frac{y-b}{\sqrt{r^2-(y-b)^2}}\big(g\big(\alpha(y),y\big)+g\big(\beta(y),y\big)\big)\,.$$

We have, by Lemma 4,

$$\int\limits_{b-r}^{b+r}\int\limits_{\alpha(y)}^{\beta(y)}\frac{\partial\omega}{\partial y}(x,y)\psi(y)\;\mathrm{d}x\,\mathrm{d}y$$

$$=\int\limits_{a-r}^{a+r}\int\limits_{b-\sqrt{r^2-(x-a)^2}}^{b+\sqrt{r^2-(x-a)^2}}\frac{\partial\omega}{\partial y}(x,y)\psi(y)\;\mathrm{d}y\,\mathrm{d}x\ll r\max_D\Big|\frac{\partial\omega}{\partial y}\Big|\,.$$

Furthermore we state that

$$\int\limits_{b-r}^{b+r}\frac{y-b}{\sqrt{r^2-(y-b)^2}}\big(g\big(\alpha(y),y\big)+g\big(\beta(y),y\big)\big)\psi(y)\;\mathrm{d}y\ll\sqrt{r}\max_{\partial D}|g|\,.$$

In order to verify this we divide the integral in three parts,

$$\int\limits_{b-r}^{b+r}\ldots\,\mathrm{d}y=\int\limits_{b-r}^{b-r+1}\ldots\,\mathrm{d}y+\int\limits_{b-r+1}^{b+r-1}\ldots\,\mathrm{d}y+\int\limits_{b+r-1}^{b+r}\ldots\,\mathrm{d}y\,.$$

The absolute value of the first and of the third integral, respectively, is

$$\le 2\Big(\max_{\partial D}|g|\Big)\int\limits_{b+r-1}^{b+r}\frac{y-b}{\sqrt{r^2-(y-b)^2}}\;\mathrm{d}y=2\Big(\max_{\partial D}|g|\Big)\sqrt{2r-1}\,.$$

The second integral is, by Lemma 4,

$$\ll\left(\max_{b-r+1\le y\le b+r-1}\left|\frac{y-b}{\sqrt{r^2-(y-b)^2}}\right|\right)\Big(\max_{\partial D}|g|\Big)\ll\sqrt{r}\max_{\partial D}|g|\,,$$

provided that

$$\frac{y-b}{\sqrt{r^2-(y-b)^2}}g\big(\alpha(y),y\big)\qquad\text{and}\qquad\frac{y-b}{\sqrt{r^2-(y-b)^2}}g\big(\beta(y),y\big)$$

are both piecewise monotonic on $b-r+1\le y\le b+r-1$.

But exactly this is the assumption on the function B_1 in Theorem 1, since for $\theta = \arcsin\left(\frac{y-b}{r}\right)$,

$$\frac{y-b}{\sqrt{r^2-(y-b)^2}} = \tan\theta = -\tan(2\pi-\theta)$$

and

$$\big(\beta(y), y\big) = (a + r\cos\theta, b + r\sin\theta)$$

and

$$\big(\alpha(y), y\big) = \big(a + r\cos(2\pi-\theta), b + r\sin(2\pi-\theta)\big).$$

This concludes the proof of Theorem 1.

5. Proof and application of Theorem 2

Let $\omega(x,y) = f\big((x-a)^2+(y-b)^2\big)$. Since $a, b \in \mathbb{Z}$, we can write

$$R\big(\omega(x,y), a, b; r\big) = R\big(f(x^2+y^2), 0, 0; r\big) = f(0) + \sum_{1\le n\le r^2} f(n)r(n),$$

where, as usual, $r(n) = \#\big\{(x,y)\in\mathbb{Z}^2:\ x^2+y^2=n\big\}$.

By

$$\sum_{n\le T} r(n) = \#\Big(D\big(0,0;\sqrt{T}\,\big)\cap\mathbb{Z}^2\Big) = \pi T + O\Big(T^{\frac{23}{73}}(\log T)^{\frac{315}{146}}\Big)$$

and Lemma 1, we obtain

$$\begin{aligned}
&\sum_{1\le n\le r^2} f(n)r(n)\\
&= f\big([r^2]\big)\sum_{1\le n\le r^2} r(n) + \sum_{1\le l\le r^2-1}\big(f(l)-f(l+1)\big)\sum_{k=1}^{l} r(k)\\
&= f\big([r^2]\big)[r^2]\pi + \pi\sum_{l=1}^{[r^2]-1}\big(lf(l)-(l+1)f(l+1)+f(l+1)\big)\\
&\qquad + O\Big(r^{\frac{46}{73}}(\log r)^{\frac{315}{146}} f\big([r^2]\big)\Big) + O\Big(r^{\frac{46}{73}}(\log r)^{\frac{315}{146}}\sum_{1\le l\le r^2-1}\big|f(l)-f(l+1)\big|\Big)\\
&= \pi\sum_{1\le n\le r^2} f(n) + O\Big(r^{\frac{46}{73}}(\log r)^{\frac{315}{146}}\max_{u\le r^2}\big|f(u)\big|\Big),
\end{aligned}$$

since f is piecewise monotonic.

Now, by the Euler summation formula,

$$\sum_{1\le n\le r^2} f(n) = \sum_{0<n\le r^2} f(n) = \int_0^{r^2} f(u)\,\mathrm{d}u - \frac{1}{2}f(0) + \psi(r^2)f(r^2) + \int_0^{r^2} f'(u)\psi(u)\,\mathrm{d}u\,.$$

Since f' has only $O(1)$ points of zero, we obtain

$$\int_0^{r^2} f'(u)\psi(u)\,\mathrm{d}u \ll \max_{u\le r^2}|f(u)|\,.$$

Furthermore,

$$\pi\int_0^{r^2} f(u)\,\mathrm{d}u = 2\pi\int_0^{r} f(\rho^2)\rho\,\mathrm{d}\rho = \iint_D \omega(x,y)\,\mathrm{d}(x,y)\,,$$

and this completes the proof of Theorem 2.

We conclude this section with a formula that combines the number π with the functions $\log n$, $r(n)$, and Dirichlet's divisor function $d(n)$. ($d(n)$ is the number of positive divisors of the natural number n.)

By applying Theorem 2 (with $a = b = 0$) to the weight

$$\omega(x,y) = \begin{cases} \log(x^2+y^2) & \text{if } x^2+y^2 \ge 1\,, \\ x^2+y^2-1 & \text{if } x^2+y^2 \le 1\,, \end{cases}$$

we obtain

$$\sum_{1\le n\le t} (\log n)r(n) = \pi t\log t - \pi t + O\big(t^{23/73+\varepsilon}\big) \qquad (t\to\infty)\,.$$

Furthermore, it is well known that (cf. [2])

$$\sum_{1\le n\le t} d(n) = t\log t + O(t) \qquad (t\to\infty)\,.$$

Thus the quotient of the main terms of the two expansions is exactly π and this yields a nice formula we conclude this article with.

FORMULA.

$$\lim_{t\to\infty}\frac{\sum_{1\le n\le t}(\log n)r(n)}{\sum_{1\le n\le t} d(n)} = \pi\,.$$

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Received June 17, 1999

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