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# ON UNCONDITIONAL CONVERGENCE OF SERIES IN BANACH LATTICES

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In the theory of real functions the following assertion of W. Sierpiński is known (see [3], [4] and [5] p. 89): A series $\sum_{n=1}^{\infty} f_n$ of bounded real functions is unconditionally uniformly convergent, i.e. it is uniformly convergent regardless of the ordering of its terms if and only if the series $\sum_{n=1}^{\infty} |f_n|$ is uniformly convergent. The aim of the present paper is to give a generalization of the above mentioned assertion for a class of Banach lattices.

The family $M(T)$ of all bounded real functions on $T \neq \emptyset$ with the product ordering (i.e. $x \leq y$ whenever $x(t) \leq y(t)$ for each $t \in T$) and with a norm $\|x\| = \sup_{t \in T} \{|x(t)|\}$ is a Banach lattice. The mentioned result of W. Sierpiński can be formulated as follows: The series $\sum_{n=1}^{\infty} x_n$ is (in $M(T)$) unconditionally convergent if and only if the series $\sum_{n=1}^{\infty} |x_n|$ is convergent ($|x| = x \vee (-x)$). This result raises a further problem: To give a characterization of those normed lattices in which a series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent if and only if the series $\sum_{n=1}^{\infty} |x_n|$ is convergent.

In the following we shall deal only with a Banach lattice $E$. To simplify our notation let us introduce: $S$ — the family of all series in $E$, i.e. $S = \{\Sigma x_n : x_n \in E\}$, $B = \{\Sigma x_n \in S: \Sigma x_n$ is unconditionally convergent$\}$ and $C = \{\Sigma x_n \in S: \Sigma x_n$ is convergent$\}$. Obviously $B \subset C$.

**Theorem 1.** *Let $E$ be a Banach lattice and let $B$ and $C$ have the introduced meaning. Then the following statements are equivalent:*

(a) $\Sigma |x_n| \in C$,

(b) $\Sigma x_n^+ \in B$ *and* $\Sigma x_n^- \in B$,

(c) $\Sigma x_n \in B$ *and* $\Sigma x_n^+ \in B$,

(d) $\Sigma x_n \in B$ *and* $\Sigma x_n^- \in B$,

(e) $\Sigma x_n \in B$ *and* $\Sigma x_n^+ \in C$,

(f) $\Sigma x_n \in C$ *and* $\Sigma x_n^+ \in B$,

(g) $\Sigma x_n \in B$ *and* $\Sigma x_n^- \in C$,

(h) $\Sigma x_n \in C$ *and* $\Sigma x_n^- \in B$,

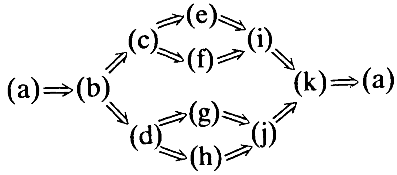
(i) $\Sigma x_n \in C$ *and* $\Sigma x_n^+ \in C$,

(j) $\Sigma x_n \in C$ *and* $\Sigma x_n \in C$,

(k) $\Sigma x_n^+ \in C$ *and* $\Sigma x_n^- \in C$,

*where* $x^+ = x \vee o (x^- = (-x) \vee o)$ *is the positive (negative) part of* $x$ ($o$ — *the additive zero element; see e.g.* [2], *p.* 230).

Proof. The statement can be proved according to the following scheme




The proof will be given for the next implications: (a) ⇒ (b) ⇒ (c) ⇒ (e) ⇒ (i) ⇒ (k) ⇒ (a). In the other cases the proof is analogical.

(a) ⇒ (b): According to the Cauchy criterion ([5], p. 86) it follows from the convergence of the series $\sum_{k=1}^{\infty} |x_k|$ that for each $\varepsilon > 0$ there is a positive integer $n_0$ such that $\left\| \sum_{k=m+1}^{n} |x_k| \right\| < \varepsilon$ holds for each $n > m \geq n_0$. If $\sum_{j-1}^{\infty} x_{k_j}^+$ is any subseries of the series $\sum_{k=1}^{\infty} x_k^+$, then

$$\left\| \sum_{m<k_j \leq n} x_{k_j}^+ \right\| \leq \left\| \sum_{m<k_j<n} x_{k_j}^+ + x_{k_j}^- \right\| = \left\| \sum_{m<k_j \leq n} |x_{k_j}| \right\| \leq \left\| \sum_{k-m+1}^{n} |x_k| \right\| < \varepsilon$$

holds, i.e. according to the Cauchy criterion the series $\sum_{j=1}^{\infty} x_{k_j}^+$ is convergent. Hence it follows from the Orlicz criterion of the unconditional convergence ([5], p. 86) that the series $\sum_{k=1}^{\infty} x_k^+$ is unconditionally convergent. Analogically we can verify that the series $\sum_{k=1}^{\infty} x_k^-$ is unconditionally convergent.

The implication (b) ⇒ (c) follows immediately from the mentioned Orlicz criterion and from the fact $x = x^+ - x^-$. The implications (c) ⇒ (e) ⇒ (i) ⇒ (k) ⇒ (a) are obvious.

**Theorem 2.** *Let E be a Banach lattice and let sets B and C have the introduced meaning. Then the following statements are equivalent:*

(i) $\Sigma |x_n| \in C$ *if and only if* $\Sigma x_n \in B$;

(ii) $\Sigma x_n \in B$ *implies* $\Sigma x_n^+ \in C$;

(iii) $\Sigma x_n \in B$ *implies* $\Sigma x_n^- \in C$.

Proof. We prove the equivalence of statements (i) and (ii): $\Sigma x_n \in B$ according to (i) implies $\Sigma|x_n| \in C$, and $\Sigma|x_n| \in C$ according to Theorem 1 (i) implies $\Sigma x_n^+ \in C$. (ii) $\Rightarrow$ (i): $\Sigma|x_n| \in C$ with respect to Theorem 1 (c) implies $\Sigma x_n \in B$. If $\Sigma x_n \in B$, then according to (ii) $\Sigma x_n^+ \in C$, and Theorem 1 (e) implies $\Sigma|x_n| \in C$.

The equivalence of the statements (i) and (iii) can be proved analogously.

Further we shall deal with Banach lattices of finite real functions defined on a set $T \neq \emptyset$ (with usual addition, scalar multiplication and the product ordering).

**Definition.** *A Banach lattice of real functions is said to be a lattice with concentrated norm whenever:*

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x \geq o} \|x\| \geq \varepsilon \Rightarrow \left(\exists_{t \in T} x(t) \geq \delta\right), \tag{1}$$

$$\forall_{\delta>0} \exists_{\eta>0} \forall_{x \geq o} \left(\exists_{t \in T} x(t) \geq \delta\right) \Rightarrow \|x\| \geq \eta. \tag{2}$$

Example 1. The Banach lattice $M(T)$ of all bounded real functions defined on the set $T \neq \emptyset$ with the norm $\|x\| = \sup_{t \in T} \{|x(t)|\}$ is a lattice with the concentrated norm.

Example 2. The Banach lattice $L$ of all finite real Lebesgue integrable functions defined on the interval $\langle 0, 1\rangle$ with the norm $\|x\| = \int_0^1 |x(t)| \, dt$ is not a lattice with the concentrated norm. Condition (1) is fulfilled, but condition (2) is not fulfilled.

**Theorem 3.** *Let $E$ be a Banach lattice of finite real functions (defined on the set $T \neq \emptyset$) with the concentrated norm and let $B$ and $C$ be sets of the introduced meaning. Then $\Sigma|x_n| \in C$ if and only if $\Sigma x_n \in B$.*

Proof. It follows from the above Theorem 2 that it is sufficient to prove that $\Sigma x_n \in B$ implies $\Sigma x_n^+ \in C$. We shall do it by a contradiction. Let $\Sigma x_n^+ \notin C$. Then, according to the Cauchy criterion, there exists $\varepsilon > 0$ such that for each positive integer number $n_0$ there are $m$ and $m_0 (m > m_0 \geq n_0)$ such that $\|x_{m_0+1}^+ + \ldots + x_m^+\| \geq \varepsilon$. It follows from the property (1) of the concentrated norm that there are $\delta > 0$ and $t \in T$ such that $x_{m_0+1}^+(t) + \ldots + x_m^+(t) \geq \delta$. Let $n_k$ be indices for which $x_{n_k}^+(t) > 0$. Then $x_{n_k}^-(t) = 0$. Obviously $\delta \leq \sum_{m_0 < n_k \leq m} x_{n_k}^+(t) = \sum_{m_0 < n_k \leq m} (x_{n_k}^+(t) - x_{n_k}^-(t)) = \sum_{m_0 < n_k \leq m} x_{n_k}(t) = \left|\sum_{m_0 < n_k \leq m} x_{n_k}(t)\right|$. It follows from the property (2) of the concentrated norm and from the fact $|y(t)| = |y|(t)$ that there exists $\eta > 0$ such that $\eta \leq \left\|\left|\sum_{m_0 < n_k \leq m} x_{n_k}\right|\right\| = \left\|\sum_{m_0 < n_k \leq m} x_{n_k}\right\|$. Hence there is a subseries $\Sigma x_{n_k}$ of the series $\Sigma x_n$

which is not convergent. Consequently, it follows from the Orlicz criterion that $\Sigma x_n \notin B$ — a contradiction.

**Theorem 4.** *Let E be any finite dimensional Banach lattice and let B and C have the introduced meaning. Then* $\Sigma |x_n| \in C$ *if and only if* $\Sigma x_n \in B$.

Proof. It is well known that each normed lattice is Archimedean (see e.g. [6], p. 129). It is also known that if a vector lattice of a finite dimension $n$ $(n \geq 1)$ is Archimedean, then it is isomorphic to the vector lattice $E_n$ of all $n$-tuples of real numbers with the product order (see [2], p. 229; [6], p. 89).

Hence it is sufficient to verify the statement of Theorem 4 for $E_n$. $E_n$ is a lattice of finite real functions on $T = \{1, \ldots, n\}$. We show that it has the concentrated norm. Let $u_1 = (\varepsilon_1, 0, \ldots, 0), \ldots, u_n = (0, \ldots, 0, \varepsilon_n)$ be a base for $E_n$. Without loss of generality we can assume $u_1 \geq o, \ldots, u_n \geq o$ and $\|u_1\| = \ldots = \|u_n\| = 1$. Let $\varepsilon > 0$ and $x = \sum_{i=1}^{n} \xi_i u_i \geq o$. From facts that $x^+ = \sum_{i \in I} \xi_i u_i$ where $I = \{i: \xi_i \geq 0\}$, and $x \geq o$ if and only if $x = x^+$, it follows $\xi_i \geq 0$ for each $i = 1, 2, \ldots, n$.

The property (1) of the concentrated norm: If $\varepsilon \leq \|x\| \leq \sum_{i=1}^{n} \|\xi_i u_i\| = \sum_{i=1}^{n} \xi_i$, then there exists $r$, $1 \leq r \leq n$, such that $\varepsilon / n \leq \xi_r$. Indeed, in the opposite case we have $\varepsilon \leq \sum_{i=1}^{n} \xi_i < n(\varepsilon/n) = \varepsilon$ — a contradiction. Hence it is sufficient to put $\delta = \varepsilon / n$. If $\delta > 0$, $x = \sum_{i=1}^{n} \xi_i u_i \geq o$ and $\xi_r \geq \delta > 0$ $(1 \leq r \leq n)$, then $\delta \leq \xi_r = \|\xi_r u_r\| \leq \left\|\sum_{i=1}^{n} \xi_i u_i\right\| = \|x\|$. Hence for $\eta = \delta$ we can verify that the property (2) of the concentrated norm is fulfilled.

In the following we use notions of $M$ and $L$ spaces, of the unit and of the spectrum of an $M$ space with unit, according to the monograph [2].

**Lemma 1.** *Each M space with unit is metrically, algebraically and lattice isomorphic to the space of all continuous real valued functions on its spectrum T with the norm* $\|x\| = \sup_{t \in T} \{|x(t)|\}$ *([2], p. 242; [6], p. 202).*

**Theorem 5.** *Let E be an M space with unit and let sets B and C have the introduced meaning. Then* $\Sigma |x_n| \in C$ *if and only if* $\Sigma x_n \in B$.

Proof. The statement of Theorem 5 is an immediate consequence of Lemma 1, of the fact that the norm $\|x\| = \sup_{t \in T} \{|x(t)|\}$ is concentrated, and of Theorem 3.

**Theorem 6.** *Let E be an infinite dimensional L space. Then there exists an unconditionally convergent series* $\Sigma x_n$ *in E such that the series* $\Sigma |x_n|$ *is not convergent.*

Proof. It is easy to verify that the characteristic property of $L$ spaces is equivalent to the condition: If $x_1 \geqslant o, \ldots, x_k \geqslant o$, then $\|x_1 + \ldots + x_k\| = \|x_1\| + \ldots + \|x_k\|$ ($k$ — a positive integer number, $k \geqslant 2$). In every infinite dimensional Banach space there exists an unconditionally convergent series $\Sigma x_n$ which is not absolutely convergent (see [1]). If $p$ is a positive integer number, then it follows that $\|x_{m+1}\| + \ldots + \|x_{m+p}\| = \| |x_{m+1}| \| + \ldots + \| |x_{m+p}| \| = \| |x_{m+1}| + \ldots + |x_{m+p}| \|$. The Cauchy condition for the divergent series $\Sigma \|x_n\|$ is not fulfilled, hence according to the last equality, it is not fulfilled for the series $\Sigma |x_n|$, i.e. $\Sigma |x_n|$ is not convergent.

**Corollary 1.** *The Banach lattice $E_1$ is the only Banach lattice which is either an $M$ space with unit and an $L$ space.*

Proof. The Banach lattice $E_1$ is obviously either an $M$ space with unit and an $L$ space. Suppose indirectly that there is a Banach lattice $E$, which is not isomorphic to $E_1$, such that it is either an $M$ space with unit and an $L$ space. If $E$ is infinite dimensional, then according to Theorem 5 $\Sigma |x_n| \in C$ if and only if $\Sigma x_n \in B$, and according to Theorem 6 there exists $\Sigma x_n \in B$ such that $\Sigma |x_n| \notin C$ — a contradiction. Hence $E$ is necessary finite dimensional.

Let $E$ be an $n$-dimensional Banach lattice. Hence it is sufficient to investigate $E_n$, $n \geqslant 2$. It follows from the assumptions of Corollary 1 and from the identity $x + y = x \vee y + x \wedge y$ that $\|x\| + \|y\| = \|x + y\| = \|x \vee y\| + \|x \wedge y\| = \|x\| \vee \|y\| + \|x \wedge y\|$ ($x \geqslant o$, $y \geqslant o$). Put $x = (\varepsilon_1, 0, 0, \ldots, 0)$, $y = (0, \varepsilon_2, 0, \ldots, 0)$ and choose numbers $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\|x\| = \|y\| = 1$. Then it follows from the last equality that $2 = 1 + \|(0, 0, \ldots, 0)\|$, i.e. $2 = 1$ — a contradiction.

In the following we shall deal with the set of series $S$. To the introduced subsets of $S$, $B$ and $C$, we add the set $A$ of all absolutely convergent series in $E$, i.e. $A = \{\Sigma x_n \in S: \Sigma \|x_n\| < +\infty\}$. It is known that $A \subset B \subset C$ ([5], p. 88). In Banach lattices it is possible to introduce the next kind of convergence, the convergence in the absolute value, in the following way: The series $\sum_{n=1}^{\infty} x_n$ converges in the absolute value if the series $\sum_{n=1}^{\infty} |x_n|$ converges. Let $A^*$ stand for the set of all series which are convergent in the absolute value, i.e. $A^* = \{\Sigma x_n \in S: \Sigma |x_n| \in C\}$.

**Theorem 7.** *Let $E$ be a Banach lattice and let $A$, $A^*$ and $B$ have the introduced meaning. Then*

(i) $A \subset A^* \subset B$,

(ii) *if $E$ is either a finite dimensional Banach lattice or an $M$ space with unit, then* $A^* = B$,

(iii) *if $E$ is an $L$ space, then* $A^* = A$.

Proof. (i): Let $p$ be a natural number. The inclusion $A \subset A^*$ follows from the Cauchy criterion and from the relations $\| |x_{m+1}| + \ldots + |x_{m+p}| \| \leq \| |x_{m+1}| \| + \ldots + \| |x_{m+p}| \| = \|x_{m+1}\| + \ldots + \|x_{m+p}\|$. The inclusion $A^* \subset B$ is a consequence of Theorem 1, (a) $\Rightarrow$ (c).

(ii): A proof of this statement is given in Theorem 4 and Theorem 5.

(iii): In the proof of Theorem 6 it is shown that the equality $\|x_{m+1}\| + \ldots + \|x_{m+p}\| = \| |x_{m+1}| + \ldots + |x_{m+p}| \|$ holds for each $L$ space. Hence it follows from the Cauchy criterion that $\Sigma |x_n| \in C$ if and only if $\Sigma \|x_n\| < +\infty$.

The set of all the series $S$ of a Banach lattice $E$ can be investigated as a metric space $(S, \varrho)$, where $\varrho(\Sigma x_n, \Sigma y_n) = \sum_{n=1}^{\infty} 2^{-n} \min \{\|x_n - y_n\|, 1\}$. It is known that the metric space $(S, \varrho)$ is a locally convex linear topological space. The sequence of the series $\{\Sigma x_n^{(r)}\}_{r=1}^{\infty}$ converges to the series $\Sigma x_n$ if and only if $\|x_n^{(r)} - x_n\| \to 0$ for each $n = 1, 2, \ldots$ ([5], p. 100, ex. 16).

**Lemma 2.** *The metric space $(S, \varrho)$ is complete.*

Proof. Let $\{\Sigma x_n^{(r)}\}_{r=1}^{\infty}$ be a Cauchy sequence of series. We show that for each $m = 1, 2, \ldots, \{x_m^{(r)}\}_{r=1}^{\infty}$ is a Cauchy sequence. Let $m$ be a natural number and let $\varepsilon$ be a positive number, $\varepsilon < 1$. There exists a natural number $r_0$ such that $\varrho(\Sigma x_n^{(r)}, \Sigma x_n^{(s)}) < \varepsilon/2^m$ holds for all natural numbers $r$ and $s$, $r_0 \leq r < s$. Then $2^{-m} \min \{\|x_m^{(r)} - x_m^{(s)}\|, 1\} < \varepsilon/2^m$ and $\|x_m^{(r)} - x_m^{(s)}\| < \varepsilon$. Hence $\{x_m^{(r)}\}_{r=1}^{\infty}$ is a Caucshy sequence and there exists a limit $\lim_{r \to \infty} x_m^{(r)} = x_m$ in the Banach lattice $E$. It follows from the above characterization of the convergence with respect to the metric $\varrho$ that $\lim_{r \to \infty} \Sigma x_n^{(r)} = \Sigma x_n$.

Further we shall deal with the sets $A$, $A^*$, $B$ and $C$ from a topological point of view.

**Theorem 8.** *The sets $A$, $A^*$, $B$ and $C$ and their complements in $S$ are dense sets in $S$.*

Proof. With respect to the inclusions $A \subset A^* \subset B \subset C$ it is sufficient to prove that the sets $A$ and $S - C$ are dense.

It is easy to verify that the set $K_1$ of all the series $\Sigma x_n$ with the property that $x_n \neq o$ holds only for a finite number of indices is a dense set in $S$. Indeed if $K(\Sigma y_n, \delta)$ is an open sphere with the center $\Sigma y_n$ and the radius $\delta > 0$, then it is sufficient to choose $m$ such that $\sum_{n=m+1}^{\infty} 2^{-n} = 2^{-m} < \delta$ and put $x_n = y_n$ for $n = 1, 2, \ldots, m$ and $x_n = o$ for $n = m+1, m+2, \ldots$,. Obviously $\Sigma x_n \in K_1 \cap K(\Sigma y_n, \delta)$ and $K_1 \subset A$.

Let $K_2$ be the set of all the series $\Sigma x_n$ for which $x_n = x \neq o$ holds with the exception of a finite number of indices. With respect to the Cauchy criterion no series in $K_2$ is convergent, hence $K_2 \subset S - C$. Analogously to the first part of the proof it can be shown that $K_2$ is dense in $S$.

**Theorem 9.** *The sets $A$, $A^*$ and $C$ are $F_{\sigma\delta}$ sets in $S$.*

Proof. We prove the statement of Theorem 9 for the set $C$. First we show that the function $\varphi_{mn}: S \to E_1(\varphi_{mn}(\Sigma x_n) = \|x_{m+1} + \ldots + x_n\|)$ defined for every pair of natural numbers $m$ and $n$ ($m < n$) is continuous. We show that for each positive $\varepsilon$, $\varepsilon < 1$, and $\Sigma x_n \in S$ there is $\delta > 0$ such that $|\varphi_{mn}(\Sigma y_n) - \varphi_{mn}(\Sigma x_n)| < \varepsilon$ holds for each series $\Sigma y_n \in K(\Sigma x_n, \delta)$. It is sufficient to put $\delta = \varepsilon/(n-m)2^n$. Then for each $k = 1, 2, \ldots$ we have $2^{-k} \min\{\|y_k - x_k\|, 1\} < \varepsilon/(n-m)2^n$ and for each $k = 1, 2, \ldots, n$ $\|y_k - x_k\| < \varepsilon/(n-m)$. Hence $|\varphi_{mn}(\Sigma y_n) - \varphi_{mn}(\Sigma x_n)| = |\|y_{m+1} + \ldots + y_n\| - \|x_{m+1} + \ldots + x_n\|| \leq \|y_{m+1} - x_{m+1}\| + \ldots + \|y_n - x_n\| < \varepsilon$.

The set $C$ can be expressed by using the Cauchy criterion in the following form ($p$, $q$, $m$ and $n$ are natural numbers):

$$C = \left\{\Sigma x_n: \underset{p\geq 1}{\forall} \underset{q\geq 1}{\exists} \underset{m\geq q}{\forall} \underset{n\geq m+1}{\forall} \|x_{m+1} + \ldots + x_n\| \leq 1/p\right\} = $$

$$= \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=q}^{\infty} \bigcap_{n=m+1}^{\infty} C_{pqmn}, \tag{3}$$

where $C_{pqmn} = \{\Sigma x_n: \|x_{m+1} + \ldots + x_n\| \leq 1/p\} = \varphi_{mn}^{-1}((-\infty, 1/p\rangle)$. The set $C_{pqmn}$ is closed because the function $\varphi_{mn}$ is continuous. The fact that the set $C$ is an $F_{\sigma\delta}$ set in $S$ is an easy consequence of the equality (3).

The statement of Theorem 9 for the set $A(A^*)$ can be proved analogously. This follows from the continuity of functions $\psi_{mn}(\Sigma x_n) = \|x_{m+1}\| + \ldots + \|x_n\| (\tau_{mn}(\Sigma x_n) = \| |x_{m+1}| + \ldots + |x_n| \|)$ which are defined for every pair of natural numbers $m$ and $n$, $m < n$, and from the expression

$$A = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=q}^{\infty} \bigcap_{n=m+1}^{\infty} \{\Sigma x_n: \|x_{m+1}\| + \ldots + \|x_n\| \leq 1/p\}$$

$$\left(A^* = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m=q}^{\infty} \bigcap_{n=m+1}^{\infty} \{\Sigma x_n: \| |x_{m+1}| + \ldots + |x_n| \| \leq 1/p\}\right).$$

**Theorem 10.** *The sets $A$, $A^*$, $B$ and $C$ are of the first category in $S$.*

Proof. Since each of the sets $A$, $A^*$ and $B$ is contained in $C$ it is sufficient to prove Theorem 10 for the set $C$. If for each $p = 1, 2, \ldots$ we put $C_p = \{\Sigma x_n: \underset{q\geq 1}{\exists} \underset{m\geq q}{\forall} \underset{n\geq m+1}{\forall} \|x_{m+1} + \ldots + x_n\| \leq 1/p\}$ ($q$, $m$ and $n$ are natural numbers), then

$$C = \bigcap_{p=1}^{\infty} C_p \text{ and } C_p = \bigcup_{q=1}^{\infty} \bigcap_{m=q}^{\infty} \bigcap_{n=m+1}^{\infty} C_{pqmn}, \tag{4}$$

where $C_{pqmn}$ are sets introduced in the proof of Theorem 9.

It follows from (4) that the set $C_p$ is an $F_\sigma$ set for each $p = 1, 2, \ldots$. For the set $K_2$ introduced in the proof of Theorem 8 there holds $K_2 \subset S - C_p$. Indeed, for each series $\Sigma x_n \in K_2$ and for each natural number $q$ there exist $m \geq q$ and $n > m$ such that $\|x_{m+1} + \ldots + x_n\| = (n-m)\|x\| > 1/p$. Hence the complement of the set $C_p$ is dense. Each $F_\sigma$ set, the complement of which is dense, is a set of the first category and from (4) it follows that $C$ is also of the first category.

**Corollary 2.** *Each of the sets $S - A$, $S - A^*$, $S - B$ and $S - C$ is residual of the second category in $S$.*

**Problem 1.** Theorem 3, Theorem 4 and Theorem 5 give only sufficient conditions for the Banach lattice to have the property (P): $\Sigma x_n \in B$ if and only if $\Sigma |x_n| \in C$. The problem to characterize Banach lattices (normed lattices) with the property (P) is open.

**Problem 2.** The method used in the proof of Theorem 9 is not applicable in general for the set $B$ to give its Borel classification in $S$. It follows from Theorem 7 (ii) and from Theorem 9 that if $E$ is either a finite dimensional Banach lattice or an $M$ space with unit, then $B$ is an $F_{\sigma\delta}$ set in $S$. Is the set $B$ an $F_{\sigma\delta}$ set in $S$ for every Banach lattice $E$?

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# О БЕЗУСЛОВНОЙ СХОДИМОСТИ РЯДОВ В СТРУКТУРАХ БАНАХА

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Резюме

В работе определены достаточные условия на структуру Банаха $E$, при которых имеет место соотношение: Ряд $\sum_{n=1}^{\infty} x_n$ безусловно сходится в $E$ тогда и только тогда, когда ряд $\sum_{n=1}^{\infty} |x_n|$ сходится в $E$ ($|x|$ — модуль элемента $x$).